The Malliavin Derivative of Generalized Random Variables

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Abstract

We extend the Malliavin derivative to a space of generalized random variables which have a (formal) chaos expansion with kernels from the space of tempered Schwartz distributions. The extended derivative has to be interpreted in the sense of distributions. Many of the properties of the standard Malliavin derivative are proved to hold for the extension as well. In addition, we derive a representation formula for the extended Malliavin derivative involving the Wick product and a centered Gaussian random variable. We apply our results to calculate the Malliavin derivative of a class of stochastic differential equations of Wick type.

1 Introduction

The object of this paper is to extend the Malliavin derivative to generalized random variables which do not admit a chaos expansion with regular kernel functions. The space of Kondratiev distributions is considered, where the generalized random variables have singular kernel functions in their (formal) chaos expansion. In fact, the kernels are tempered (Schwartz) distributions (see [AKS, KLS] and section 2 below). We define the Malliavin derivative for Kondratiev distributions.

Ustunel, [U], has extended the Malliavin derivative to the space of Meyer-Watanabe distributions. In this case the derivative is understood in a distributional sense. Recently, Aase, Øksendal and Ubøe, [AaØU] defined the

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Malliavin derivative to a slightly bigger space $\mathcal{G}^*$ of generalized random variables, where the elements admit square integrable kernels in their chaos expansion (see [PT] for a definition). The Malliavin derivative is introduced by adapting the definition for square integrable random variables.

The starting point of our extension is the duality relation between Skorohod integration and Malliavin differentiation (see (2.13) below). We use this duality relation to find a representation of the Lebesgue integral of the Malliavin derivative of a square integrable random variable F. It can be written as the difference of two random variables. The former variable in the difference is F times a centered Gaussian random variable. The latter being the Wick product of F with the same Gaussian variable. The representation is calculated using the $\mathcal{S}$-transform, which maps random variables into functionals on the Schwartz space of rapidly decreasing functions. We utilize this representation for the generalization of the Malliavin derivative to Kondratiev distributions. The derivative will be a continuous linear functional on the product space of Schwartz functions and Kondratiev test functions. Hence, by our definition it is no longer a regular stochastic process as in the case of square integrable random variables (see e.g. [N]). However, the extension coincides with the standard Malliavin derivative on its domain of definition.

We give an outline of the paper. In section 2 the basic concepts of the White Noise Analysis is introduced. Section 3 defines the Malliavin derivative for Kondratiev distributions and prove that the extension coincides with the standard Malliavin derivative for square integrable random variables. Several properties of the extended Malliavin derivative which generalize classical results are proved. We continue with proving a product formula for the Malliavin derivative of the Wick product of two Kondratiev distributions in section 4. Such a result enables us to calculate the extended Malliavin derivative of a Skorohod integral. Finally in section 5 we apply our results to a class of stochastic differential equations of Wick-type. Such equations are known to have solutions in the Kondratiev space (see e.g. [BDP, V]).

2 Mathematical preliminaries

We give the necessary background from White Noise Analysis. The interested reader is refered to [HKPS] and [AKS, KLS] for a complete account on the theory presented below.

Let $\mathcal{S}(\mathbb{R}^n)$ denote the space of Schwartz functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ its topological dual, the space of tempered (Schwartz) distributions. Introduce the norms

$$|f|_{2,p} := |(A^{\otimes n})^p f|_{2,0}$$

on the space $\mathcal{S}(\mathbb{R}^n)$, where $p \in \mathbb{N}_0$ and $|\cdot|_{2,0} := |\cdot|_2$ is norm of $L^2(\mathbb{R}^n)$. A is a second order differential operator given by

$$A = -\frac{d^2}{dx^2} + (1 + x^2)$$

Define the Hilbert spaces $\mathcal{S}_p(\mathbb{R}^n)$ as the completion of $\mathcal{S}(\mathbb{R}^n)$ in the norm $|\cdot|_{2,p}$. $\mathcal{S}_{-p}(\mathbb{R}^n)$ denotes its dual. It is well-known that the Schwartz space is the projective limit of $\mathcal{S}_p(\mathbb{R}^n)$ and the tempered distributions the inductive limit of $\mathcal{S}_{-p}(\mathbb{R}^n)$. We will use the notation $\langle \cdot, \cdot \rangle$ for the dual pairing. For $F \in \mathcal{S}'(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$ we shall also make use of the notation $F(f)$ for the action of F on f. Introduce the probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{F}, \mu)$ where $\mathcal{F}$ is the σ-algebra induced by the weak topology. The Gaussian probability measure μ is defined by the Bochner-Minlos theorem by

$$\int_\Omega \exp\left(\langle \omega, f \rangle\right) d\mu(\omega) = \exp(-\frac{1}{2}|f|_2^2)$$

From this we observe that the coordinate processes $\omega \rightarrow \langle \omega, f \rangle$ with $f \in \mathcal{S}(\mathbb{R})$ and $\omega \in \mathcal{S}'(\mathbb{R})$ are centered Gaussian variables with variance $|f|_2^2$. By a limiting argument one can define Brownian motion as the pairing

$$B_t(\omega) := \langle \omega, \mathbf{1}_{[0,t]} \rangle \tag{2.1}$$

(the limit taken in $(L^2) := L^2(\mathcal{S}'(\mathbb{R}), \mathcal{F}, \mu)$) The coordinate processes will sometimes be denoted $W(f)(\omega) := \langle \omega, f \rangle$. If $\Phi \in (L^2)$, then it has a chaos expansion

$$\Phi = \sum_{n=0}^{\infty} I_n(f^{(n)})$$

where $f^{(n)}$ are symmetric elements of $L^2(\mathbb{R}^n)$ and I_n the n-fold Wiener-Itô integral on $\mathbb{R}^n$. The norm is given by

$$\|\Phi\|_{(L^2)}^2 = \sum_{n=0}^{\infty} n! |f^{(n)}|_{2,0}^2$$

We define the Kondratiev test functions and distributions, which will constitute our spaces of smooth and generalized random variables, respectively: Denote by $(\mathcal{S})_p^1$ the Hilbert space of random variables $\phi \in (L^2)$ with chaos expansion

$$\phi = \sum_{n=0}^{\infty} I_n(f^{(n)})$$

so that

$$\|\phi\|_{2,p,1}^2 := \sum_{n=0}^{\infty} (n!)^2 |f^{(n)}|_{2,p}^2$$

The dual space is denoted $(\mathcal{S})_{-p}^{-1}$. Define the space of Kondratiev test functions to be the projective limit of $(\mathcal{S})_p^1$. Its dual is the inductive limit of $(\mathcal{S})_{-p}^{-1}$ and is denoted $(\mathcal{S})^{-1}$. $(\mathcal{S})^{-1}$ is called the space of Kondratiev distributions, where the elements have formal chaos expansions with kernels given as symmetric tempered distributions. Moreover, if $\Phi \in (\mathcal{S})^{-1}$, then

$$\Phi = \sum_{n=0}^{\infty} I_n(F^{(n)})$$

where the $F^{(n)}$ are symmetric elements of $\mathcal{S}'(\mathbb{R}^n)$. For a $p \in \mathbb{N}_0$,

$$\|\Phi\|_{2,-p,-1}^2 := \sum_{n=0}^{\infty} |F^{(n)}|_{2,-p}^2$$

By $\langle\langle \cdot, \cdot \rangle\rangle$ we shall denote the dual pairing between the Kondratiev distributions and test functions. Moreover, if Φ and ϕ are defined as above,

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F^{(n)}, f^{(n)} \rangle \tag{2.2}$$

Note that we can define the product of a Kondratiev distribution Φ with a Kondratiev test function ψ, $\Phi \cdot \psi$. From [HKPS, AKS] this is again a Kondratiev distribution, with action on $(\mathcal{S})^1$,

$$\langle\langle \psi \cdot \Phi, \phi \rangle\rangle = \langle\langle \Phi, \psi \cdot \phi \rangle\rangle \tag{2.3}$$

Expectation can be generalized to the Kondratiev distributions as the pairing

$$\mathrm{E}[\Phi] = \langle\langle \Phi, 1 \rangle\rangle \tag{2.4}$$

Consider the $\mathcal{S}$-transform on $(\mathcal{S})^{-1}$ defined by

$$\mathcal{S}\Phi(\xi) := \langle\langle \Phi, \exp(\langle \cdot, \xi \rangle - 1/2|\xi|_{2,0}^2) \rangle\rangle \tag{2.5}$$

where $\xi \in \mathcal{S}(\mathbb{R})$ with norms $|\xi|_{2,p} < 1$. The $\mathcal{S}$-transform is a bijection onto a space of so-called $\mathcal{U}$-functionals (see e.g. [HKPS, AKS, KLS] for more information on this). In the sequel we will use the following short-hand notation for the normalized exponential $\exp(\langle \cdot, \xi \rangle - 1/2|\xi|_{2,0}^2)$:

$$\mathrm{Exp}(\langle \cdot, \xi \rangle) := \exp(\langle \cdot, \xi \rangle - 1/2|\xi|_{2,0}^2)$$

The Wick product $\diamond$ of two Kondratiev distributions Φ, Ψ is defined via the $\mathcal{S}$-transform:

$$\Phi \diamond \Psi = \mathcal{S}^{-1}\left(\mathcal{S}\Phi \cdot \mathcal{S}\Psi\right) \tag{2.6}$$

In terms of chaos expansions, we have

$$\Phi \diamond \Psi = \sum_{n,m} I_{n+m}\left(F^{(n)} \hat{\otimes} G^{(m)}\right) \tag{2.7}$$

where $\Phi = \sum I_n(F^{(n)})$ and $\Psi = \sum I_n(G^{(n)})$. $\hat{\otimes}$ stands for the symmetrized tensor products of tempered distributions. The following translation formula for Wick products with $\mathrm{Exp}W(f)$ is useful (see [B, prop. 13] for a proof). If $\Phi \in (L^2)$ and $f \in \mathcal{S}(I\!\!R)$,

$$\Phi(\omega) \diamond \mathrm{Exp}W(f)(\omega) = \Phi(\omega - f) \cdot \mathrm{Exp}W(f)(\omega) \tag{2.8}$$

Integration of a time parametrized Kondratiev distribution Φ_t is interpreted in the sense of Pettis, i.e.

$$\langle\langle \int_{I\!\!R} \Phi_t \, dt, \phi \rangle\rangle = \int_{I\!\!R} \langle\langle \Phi_t, \phi \rangle\rangle \, dt$$

whenever the right-hand side exists. The Skorohod integral can be generalized to Kondratiev distributions by the relation

$$\int_{I\!\!R} \Phi_t \delta B_t := \int_{I\!\!R} \Phi_t \diamond W_t \, dt \tag{2.9}$$

where $W_t = I_1(\delta_t)$, δ_t is the Dirac-δ function. W_t is called the *white noise*, i.e. the time derivative of Brownian motion B_t. It is well-known that $W_t \in (\mathcal{S})^{-1}$. The Skorohod integral factorizes constants in the following way:

$$\int_{I\!\!R} \Phi \diamond \Psi_t \delta B_t = \Phi \diamond \int_{I\!\!R} \Psi_t \delta B_t \tag{2.10}$$

We end this section recalling some notation and results for the Malliavin derivative of square integrable random variables (a nice and complete account can be found in Nualart, [N], or Ustunel, [U]). For $X \in (L^2)$, let $D_t X$ denote the Malliavin derivative of X. The domain for the operator D is denoted by $\mathcal{D}_{1,2}$ where we have the following characterization of this subspace of (L^2) (see e.g. [N]):

$$X \in \mathcal{D}_{1,2} \Leftrightarrow \sum_{n=1}^{\infty} n \cdot n! |f^{(n)}|_{2,0}^2 < \infty \tag{2.11}$$

with $X = \sum I_n(f^{(n)})$. Moreover, if $X \in \mathcal{D}_{1,2}$, the Malliavin derivative has chaos expansion

$$D_t X = \sum_{n=1}^{\infty} n I_{n-1}(f^{(n)}(\cdot, t)) \tag{2.12}$$

The duality relation between the Malliavin derivative and Skorohod integration can be stated as

$$\mathrm{E}\left[\Phi \cdot \int_{\mathbb{R}} \Psi_s \delta B_s\right] = \int_{\mathbb{R}} \mathrm{E}\left[D_s \Phi \cdot \Psi_s\right] ds \tag{2.13}$$

whenever $D_t\Phi$ and $\int_{\mathbb{R}} \Psi_s \delta B_s$ are well-defined and elements of (L^2).

3 The Malliavin derivative of generalized random variables

We define the extended Malliavin derivative of a Kondratiev distribution:

Definition 3.1. *For $\Phi \in (\mathcal{S})^{-1}$ define the extended Malliavin derivative of Φ, denoted $\mathcal{D}\Phi$, to be*

$$\mathcal{D}\Phi = \Phi \cdot W - \Phi \diamond W \tag{3.1}$$

where $\Phi \cdot W - \Phi \diamond W$ is the functional on the product space $\mathcal{S}(\mathbb{R}) \times (\mathcal{S})^1$ (equipped with the product topology) given by

$$(\Phi \cdot W - \Phi \diamond W)(f, \phi) := \langle\langle \Phi \cdot W(f) - \Phi \diamond W(f), \phi \rangle\rangle \tag{3.2}$$

The following proposition says that the Malliavin derivative of a Kondratiev distribution is a continuous linear functional on $\mathcal{S}(\mathbb{R}) \times (\mathcal{S})^1$:

Proposition 3.2. *Let $\Phi \in (\mathcal{S})^{-1}$. Then $D\Phi$ is a continuous linear functional on $\mathcal{S}(\mathbb{R}) \times (\mathcal{S})^1$ (i.e. is an element of the dual of this space). Moreover,*

a) For any $f \in \mathcal{S}(\mathbb{R})$ we have

$$\mathcal{D}\Phi(f) = \Phi \cdot W(f) - \Phi \diamond W(f) \in (\mathcal{S})^{-1}$$

defined by

$$\langle\langle \mathcal{D}\Phi(f), \phi \rangle\rangle = \langle\langle \Phi \cdot W(f) - \Phi \diamond W(f), \phi \rangle\rangle$$

for every $\phi \in (\mathcal{S})^1$.

b) For any $\phi \in (\mathcal{S})^1$ *we have*

$$\langle\langle \mathcal{D}\Phi, \phi\rangle\rangle = \langle\langle \Phi \cdot W - \Phi \diamond W, \phi\rangle\rangle \in \mathcal{S}'(I\!R)$$

defined by

$$\langle\langle \mathcal{D}\Phi, \phi\rangle\rangle(f) = \langle\langle \Phi \cdot W(f) - \Phi \diamond W(f), \phi\rangle\rangle$$

for every $f \in \mathcal{S}(I\!R)$

Proof. The linearity of $\mathcal{D}F$ is straightforward since $W(f+g) = W(f)+W(g)$. The proof of the continuity of the functionals goes by a norm estimation on $\langle\langle \Phi \cdot W(f) - \Phi \diamond W(f), \phi\rangle\rangle$ for $f \in \mathcal{S}(I\!R)$ and $\phi \in (\mathcal{S})^1$: Consider first $\langle\langle \Phi \cdot W(f), \phi\rangle\rangle$. Using Cauchy-Schwarz together with norm estimates for the product $\Phi \cdot W(f)$ (see e.g. [HKPS] for the case of Hida distributions. The estimate for Kondratiev distributions follow similarly):

$$\begin{aligned}
|\langle\langle \Phi \cdot W(f), \phi\rangle\rangle| &= |\langle\langle \Phi, W(f) \cdot \phi\rangle\rangle| \\
&\leq \|\Phi\|_{2,-q,-1} \|W(f) \cdot \phi\|_{2,q,1} \\
&\leq \|\Phi\|_{2,-q,-1} \|W(f)\|_{2,p,1} \|\phi\|_{2,p,1} \\
&= \|\Phi\|_{2,-q,-1} |f|_{2,p} \|\phi\|_{2,p,1}
\end{aligned}$$

where $p > k + q$ for a constant k (see e.g. [HKPS, remark, p. 89]). Consider now $\langle\langle \Phi \diamond W(f), \phi\rangle\rangle$: From norm estimates on the Wick product (see e.g. [KLS, prop. 11]), we have

$$\begin{aligned}
|\langle\langle \Phi \diamond W(f), \phi\rangle\rangle| &\leq \|\Phi \diamond W(f)\|_{2,-q,-1} \|\phi\|_{2,q,1} \\
&\leq \|\Phi\|_{2,-q,-1} |f|_{2,p} \|\phi\|_{2,q,1}
\end{aligned}$$

By the triangle inequality and compatibility of the norms we have the continuity of $\mathcal{D}\Phi$.

Results a and b follow immediately from the estimates above. □

The next theorem shows that our extension of the Malliavin derivative to the space of Kondratiev distributions is a true generalization of the Malliavin derivative:

Theorem 3.3. *Assume* $\Phi \in \mathcal{D}_{1,2}$. *Then*

$$\mathcal{D}\Phi = D\Phi$$

Thus, the Malliavin derivative coincides with the extended Malliavin derivative on the domain $\mathcal{D}_{1,2}$.

Proof. Consider the duality relation (2.13) with $\Psi_s := f(s)\mathrm{Exp}W(\xi)$ for $f, \xi \in \mathcal{S}(\mathbb{R})$: The left-hand side of (2.13) gives (using (2.10) and (2.8)),

$$\begin{aligned}
\mathrm{E}\left[\Phi \int_{\mathbb{R}} f(s)\mathrm{Exp}W(\xi)\delta B_s\right] &= \mathrm{E}\left[\Phi\left(W(f) \diamond \mathrm{Exp}W(\xi)\right)\right] \\
&= \mathrm{E}\left[\Phi\left(W(f) - \int_{\mathbb{R}} f(s)\xi(s)ds\right)\mathrm{Exp}W(\xi)\right] \\
&= \mathcal{S}\left(\Phi \cdot W(f) - \Phi \diamond W(f)\right)(\xi)
\end{aligned}$$

The right-hand side of (2.13) is equal to

$$\int_{\mathbb{R}} \mathrm{E}\left[D_s\Phi \cdot f(s)\mathrm{Exp}W(\xi)\right]ds = \int_{\mathbb{R}} f(s)\mathcal{S}(D_s\Phi)(\xi)ds$$

The result follows by uniqueness of the $\mathcal{S}$-transform. □

We consider some examples:

Example: Appealing to theorem (3.3), the extended Malliavin derivative of Brownian motion at time t, B_t, is simply equal to $\mathbf{1}_{[0,t]}(\cdot)$. We can calculate this using the definition of $\mathcal{D}$ instead:

$$\begin{aligned}
\mathcal{D}B_t(f) &= B_t \cdot W(f) - B_t \diamond W(f) \\
&= B_t \diamond W(f) + \int_0^t f(s)ds - B_t \diamond W(f) \\
&= \langle \mathbf{1}_{[0,t]}, f \rangle
\end{aligned}$$

thus reproving the theorem for the case of Brownian motion.

The next example calculates the Malliavin derivative of white noise, W_t, which is known to be a Kondratiev distribution (even a Hida distribution, see [HKPS]),

Example: From section 2 we have the chaos representation of W_t,

$$W_t(\omega) = \langle \omega, \delta_t \rangle$$

where δ_t is the Dirac-δ function. We find the Malliavin derivative of white noise, $\mathcal{D}W_t$:

$$\begin{aligned}
\mathcal{D}W_t(f) &= W_t \cdot W(f) - W_t \diamond W(f) \\
&= W_t \diamond W(f) + f(t) - W_t \diamond W(f) \\
&= f(t) \\
&= \langle \delta_t, f \rangle
\end{aligned}$$

Hence, we have $\mathcal{D}W_t = \delta_t$. An informal calculation using the standard Malliavin derivative D_s would give $D_s W_t = \delta_t(s)$. However, this calculation is now justified in the extended sense.

From our definition of the extended Malliavin derivative we are able to prove that D_t can be understood as differentiation with respect to time. However, the derivative must be interpreted in a weak sense since we essentially differentiate Brownian motion. The next proposition proves the result using the duality relation and the $\mathcal{S}$-transform.

Proposition 3.4. *Let $F \in \mathcal{D}_{1,2}$. Then*

$$D_t F = \frac{d}{dt}\left(F \cdot B_t - F \diamond B_t\right) \tag{3.3}$$

where B_t is the standard Brownian motion.

Proof. Use $\Psi_s = \mathbf{1}_{[0,t]}(s)\mathrm{Exp}W(\xi)$ for $\xi \in \mathcal{S}(\mathbb{R})$ in the duality relation (2.13) to obtain a left-hand side equal to:

$$\begin{aligned} \mathrm{E}\left[F\int_0^t \mathrm{Exp}W(\xi)\delta B_s\right] &= \mathrm{E}\left[F\left(B_t \diamond \mathrm{Exp}W(\xi)\right)\right] \\ &= \mathrm{E}\left[F\left(B_t - \int_0^t \xi(s)ds\right)\mathrm{Exp}W(\xi)\right] \\ &= \mathcal{S}\left(F \cdot B_t - F \diamond B_t\right)(\xi) \end{aligned}$$

Here again we have used (2.10) and (2.8). The right-hand side is seen to be

$$\begin{aligned} \int_0^t \mathrm{E}\left[D_s F \cdot \mathrm{Exp}W(\xi)\right] ds &= \int_0^t \mathcal{S}\left(D_s F\right)(\xi) ds \\ &= \mathcal{S}\left(\int_0^t D_s F ds\right)(\xi) \end{aligned}$$

By uniqueness of the $\mathcal{S}$-transform we have proved the proposition. □

We proceed with discussing some properties of the operator $\mathcal{D}$. From (2.12) we know that the Malliavin derivative of a random variable $X \in \mathcal{D}_{1,2}$ with chaos expansion $X = \sum I_n(f^{(n)})$ can be written

$$D_t X = \sum_{n=1}^{\infty} n I_{n-1}(f^{(n)}(\cdot, t))$$

We will show an analogous relation for the extended Malliavin derivative. But first we need some notation: Let $F^{(n)} \in \widehat{\mathcal{S}'(\mathbb{R}^n)}$ (i.e a symmetric tempered distribution on $\mathbb{R}^n$). For a $g \in \mathcal{S}(\mathbb{R})$, define $F^{(n)}(\cdot, g)$ to be an element of $\mathcal{S}'(\mathbb{R}^{n-1})$ by

$$\langle F^{(n)}(\cdot, g), f^{(n-1)} \rangle := \langle F^{(n)}, f^{(n-1)} \widehat{\otimes} g \rangle \tag{3.4}$$

for any $f^{(n-1)} \in \mathcal{S}(\mathbb{R}^{n-1})$. Thus we can introduce the notation $F^{(n)}(\cdot, \circ)$ where we "fix the last argument of $F^{(n)}$", with the interpretation

$$\langle F^{(n)}(\cdot, \circ), f^{(n-1)} \rangle \in \mathcal{S}'(\mathbb{R})$$

This tempered distribution has the action on $\mathcal{S}(\mathbb{R})$ defined by (3.4). The next result shows that the extended Malliavin derivative operates on chaos much in the same fashion as D_t.

Proposition 3.5. *Let the Kondratiev distribution Φ have the chaos expansion $\Phi = \sum I_n(F^{(n)})$. Then*

$$\mathcal{D}\Phi(\circ) = \sum_{n=1}^{\infty} n I_{n-1} \left(F^{(n)}(\cdot, \circ) \right) \tag{3.5}$$

Hence, for $\phi = \sum I_n(f^{(n)}) \in (\mathcal{S})^1$ and $g \in \mathcal{S}(\mathbb{R})$

$$\langle\langle \mathcal{D}\Phi(g), \phi \rangle\rangle = \sum_{n=1}^{\infty} n! \langle F^{(n)}, f^{(n-1)} \widehat{\otimes} g \rangle \tag{3.6}$$

Proof. Consider $\Phi = I_n(F^{(n)})$: We calculate $\mathcal{D}\Phi(g)$ in terms of chaos. Let $\phi = \sum I_n(f^{(n)})$, then, using the formula for the product of two random variables (see e.g. [HKPS]),

$$\begin{aligned}
\langle\langle \Phi \cdot W(g), \phi \rangle\rangle &= \langle\langle \Phi, W(g) \cdot \phi \rangle\rangle \\
&= n! \langle F^{(n)}, f^{(n-1)} \widehat{\otimes} g \rangle \\
&\quad + n! \langle F^{(n)}, (n+1) \int_{\mathbb{R}} f^{(n+1)}(\cdot, s) g(s)\, ds \rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle\langle \Phi \diamond W(g), \phi \rangle\rangle &= (n+1)! \langle F^{(n)} \widehat{\otimes} g, f^{(n+1)} \rangle \\
&= (n+1)! \langle F^{(n)}, \int_{\mathbb{R}} f^{(n+1)}(\cdot, s) g(s)\, ds \rangle
\end{aligned}$$

Thus we see that

$$\begin{aligned}
\mathcal{D}\Phi(g) &= n! \langle F^{(n)}, f^{(n-1)} \widehat{\otimes} g \rangle \\
&= n! \langle F^{(n)}(\cdot, g), f^{(n-1)} \rangle
\end{aligned}$$

By summing over all chaos we get the desired result. □

Even though the above result demonstrates the nice property of reducing chaos and "fixing one argument in the kernel", we can formulate a simple duality property for the Malliavin derivative using the Wick product instead.

Proposition 3.6. *Let $\Phi \in (\mathcal{S})^{-1}$ and $g \in \mathcal{S}(I\!R)$. Then*

$$\langle\langle \mathcal{D}\Phi(g), \phi \rangle\rangle = \langle\langle \Phi, \phi \diamond W(g) \rangle\rangle \tag{3.7}$$

Proof. Consider $\Phi = I_n(F^{(n)})$. From prop. (3.5) we have

$$\mathcal{D}\Phi(g) = nI_{n-1}(F^{(n)}(\cdot, g))$$

But then,

$$\begin{aligned}\langle\langle \mathcal{D}\Phi(f), \phi \rangle\rangle &= n(n-1)!\langle F^{(n)}, f^{(n-1)}\widehat{\otimes} g\rangle \\ &= \langle\langle \Phi, I_{n-1}(f^{(n-1)}\widehat{\otimes} g)\rangle\rangle \\ &= \langle\langle \Phi, \phi \diamond W(g) \rangle\rangle\end{aligned}$$

□

Remark: Note that the duality relation (3.7) could have been used to define the extended Malliavin derivative $\mathcal{D}$.

The duality relation (3.7) can be reformulated such that we see a closer connection to (2.13):

$$\langle\langle \mathcal{D}\Phi(g), \phi \rangle\rangle = \langle\langle \Phi, \int_{I\!R} g(s)\phi\, \delta B_s \rangle\rangle \tag{3.8}$$

From [U] we know that the chaos kernel functions $f^{(n)} \in L^2(I\!R^n)$ for a random variable $X = \sum I_n(f^{(n)}) \in (L^2)$ can be characterized by the Malliavin derivative. We have the following formula

$$f^{(n)}(x_1, \ldots, x_n) = \frac{1}{n!}\mathrm{E}\left[D_{x_1} D_{x_2} \cdots D_{x_n} X\right]$$

The next proposition provides us with a similar formula for the chaos kernels of a Kondratiev distribution. However, in this case the representation will be an equality in the sense of distributions.

Proposition 3.7. *If $\Phi \in (\mathcal{S})^{-1}$ have chaos expansion $\Phi = \sum I_n(F^{(n)})$, then*

$$F^{(n)} = \frac{1}{n!}E\left[\mathcal{D}^n\Phi\right] \tag{3.9}$$

Proof. By induction on (3.7) with $\phi = 1$ we get

$$\begin{aligned}\langle\langle \frac{1}{n!}\mathcal{D}^n\Phi(g_1\widehat{\otimes}\cdots\widehat{\otimes}g_n), 1\rangle\rangle &= \langle\langle \Phi, \frac{1}{n!}I_n\left(g_1\widehat{\otimes}\cdots\widehat{\otimes}g_n\right)\rangle\rangle \\ &= \langle F^{(n)}, g_1\widehat{\otimes}\cdots\widehat{\otimes}g_n\rangle\end{aligned}$$

where $g_1, \ldots, g_n \in \mathcal{S}(I\!R)$. □

We discuss localization properties of the extended Malliavin derivative. Consider the time-parametrized Kondratiev distribution Φ_t, $t \in [0, T]$, with chaos $\Phi_t = \sum I_n(F_t^{(n)})$. In [DPV, def.5.7] adaptedness of Φ_t with respect to the σ-algebra $\mathcal{F}_t$ generated by the Brownian motion B_s, $0 \le s \le t$, is defined in terms of the $\mathcal{S}$-transform. Their definition holds for Hida distributions, but can easily extended to the case of Kondratiev distributions: Φ_t is called $\mathcal{F}_t$-adapted, if, for all $f \in \mathcal{S}(I\!R)$ and all $g \in \mathcal{S}(I\!R)$ with support in the complement of $[0, t]$ so that $|f + g|_{2,p} < 1$ for all $p \ge 0$,

$$\mathcal{S}(\Phi_t)(f + g) = \mathcal{S}(\Phi)(f) \tag{3.10}$$

This definition implies that the tempered distributions $F_t^{(n)}$ have support on those $f \in \mathcal{S}(I\!R^n)$ such that

$$\text{supp } f \subset [0, t]^n$$

From this we derive that $\mathcal{D}\Phi_t(f) = 0$ whenever f has support outside the interval $[0, t]$ since, in that case,

$$\langle F_t^{(n)}(\cdot, f), f^{(n-1)}\rangle = \langle F_t^{(n)}, f^{(n-1)}\widehat{\otimes}f\rangle = 0$$

4 The Malliavin derivative and the Wick product

In this section we will show that the product rule holds for the Malliavin derivative of a Wick product of two Kondratiev distributions. In the space of Kondratiev distributions the ordinary product is of course not well-defined. However, the Wick product of two Kondratiev distributions is again a Kondratiev distribution. We are going to apply the (Wick) product rule to calculate the extended Malliavin derivative of a Skorohod integral, which will prove to be the direct analogue of the standard case. Some results in connection with Wick Calculus (see [KLS]) will be discussed.

Theorem 4.1. *Let $\Phi, \Psi \in (\mathcal{S})^{-1}$. Then*

$$\mathcal{D}(\Phi \diamond \Psi) = \mathcal{D}\Phi \diamond \Psi + \Phi \diamond \mathcal{D}\Psi \tag{4.1}$$

Proof. Consider $\Phi = I_n(F^{(n)})$ and $\Psi = I_m(G^{(m)})$. Let $\phi \sum I_n(f^{(n)})$ be a Kondratiev test function and $g \in \mathcal{S}(I\!R)$. By the formula for the product of two random variables (see e.g [HKPS]), we have

$$\phi \cdot W(g) = \sum_{m=0}^{\infty} I_{m+1}(f^{(m)} \widehat{\otimes} g) + \sum_{m=1}^{\infty} m I_{m-1}(f^{(m)} \widehat{\otimes}_1 g)$$

where

$$f^{(m)} \widehat{\otimes}_1 g(\cdot) = \int_{I\!R} f^{(m)}(\cdot, s) g(s)\, ds$$

Thus, we have

$$\begin{aligned}
\langle\langle \Phi \cdot W(g), \phi \rangle\rangle &= \langle\langle \Phi, W(g) \cdot \phi \rangle\rangle \\
&= n! \langle F^{(n)}, f^{(n-1)} \widehat{\otimes} g \rangle + n!(n+1) \langle F^{(n)}, f^{(n+1)} \widehat{\otimes}_1 g \rangle \\
&= (n-1)! \langle n F^{(n)} \widehat{\otimes}_1 g, f^{(n-1)} \rangle + (n+1)! \langle F^{(n)} \widehat{\otimes} g, f^{(n+1)} \rangle
\end{aligned}$$

In the last equality we have used the identities

$$\begin{aligned}
\langle F^{(n)} \widehat{\otimes} g, f^{(n+1)} \rangle &= \langle F^{(n)}, f^{(n+1)} \widehat{\otimes}_1 g \rangle \\
\langle F^{(n)} \widehat{\otimes}_1 g, f^{(n-1)} \rangle &= \langle F^{(n)}, f^{(n-1)} \widehat{\otimes} g \rangle
\end{aligned}$$

These calculations justify the following representation of the chaos for $\Phi \cdot W(g)$:

$$\Phi \cdot W(g) = I_{n+1}(F^{(n)} \widehat{\otimes} g) + n I_{n-1}(F^{(n)} \widehat{\otimes}_1 g)$$

We now calculate $\Psi \diamond (\Phi \cdot W(g))$:

$$\begin{aligned}
\Psi \diamond (\Phi \cdot W(g)) &= I_{n+m+1}\left((F^{(n)} \widehat{\otimes} g) \widehat{\otimes} G^{(m)}\right) + n I_{n+m-1}\left((F^{(n)} \widehat{\otimes}_1 g) \widehat{\otimes} G^{(m)}\right) \\
&= \Phi \diamond \Psi \diamond W(g) + I_{n+m-1}\left(n G^{(m)} \widehat{\otimes} (F^{(n)} \widehat{\otimes}_1 g)\right)
\end{aligned}$$

The same calculation yields

$$\Phi \diamond (\Psi \cdot W(g)) = \Phi \diamond \Psi \diamond W(g) + I_{n+m-1}\left(m F^{(n)} \widehat{\otimes} (G^{(m)} \widehat{\otimes}_1 g)\right)$$

Again by a similar calculation we get

$$\begin{aligned}
(\Phi \diamond \Psi) \cdot W(g) &= I_{n+m}(F^{(n)} \widehat{\otimes} G^{(m)}) \cdot W(g) \\
&= I_{n+m+1}(F^{(n)} \widehat{\otimes} G^{(m)} \widehat{\otimes} g) + (n+m) I_{n+m-1}\left((F^{(n)} \widehat{\otimes} G^{(m)}) \widehat{\otimes}_1 g\right) \\
&= \Phi \diamond \Psi \diamond W(g) + I_{n+m-1}\left((n+m)(F^{(n)} \widehat{\otimes} G^{(m)}) \widehat{\otimes}_1 g\right)
\end{aligned}$$

However, it can be shown that

$$(n+m)(F^{(n)}\widehat{\otimes}G^{(m)})\widehat{\otimes}_1 g = mF^{(n)}\widehat{\otimes}(G^{(m)}\widehat{\otimes}_1 g) + nG^{(m)}\widehat{\otimes}(F^{(n)}\widehat{\otimes}_1 g)$$

Therefore

$$\begin{aligned}(\Phi \diamond \Psi)\cdot W(g) = \Phi \diamond \Psi \diamond W(g) + \Phi \diamond (\Psi \cdot W(g)) \\ + \Psi \diamond (\Phi \cdot W(g)) - 2\Phi \diamond \Psi \diamond W(g)\end{aligned}$$

From this we get

$$\begin{aligned}(\Phi \diamond \Psi)\cdot W(g) - (\Phi \diamond \Psi)\diamond W(g) = \Phi \diamond (\Psi \cdot W(g)) - \Phi \diamond (\Psi \diamond W(g)) \\ + \Psi \diamond (\Phi \cdot W(g)) - \Psi \diamond (\Phi \diamond W(g))\end{aligned}$$

The theorem now follows by summing over all chaos. □

An immediate consequence of the product rule is the following: For $n \in I\!N$ we have

$$\mathcal{D}\left(\Phi^{\diamond n}\right) = n\Phi^{\diamond(n-1)} \diamond \mathcal{D}\Phi \tag{4.2}$$

In [KLS] so called Wick analytic functions are introduced. Let $\sigma(z)$ be an analytic function on the complex plane with power series expansion

$$\sigma(z) = \sum_{n=0}^{\infty} a_n z^n$$

Then we can define the Wick version of it by

$$\sigma^{\diamond}(\Phi) = \sum_{n=0}^{\infty} a_n \Phi^{\diamond n} \tag{4.3}$$

From theorem 12 in [KLS], $\sigma^{\diamond}(\Phi) \in (\mathcal{S})^{-1}$. Relation (4.2) gives

$$\mathcal{D}\sigma^{\diamond}(\Phi) = (\sigma')^{\diamond}(\Phi) \diamond \mathcal{D}\Phi \tag{4.4}$$

where σ' is the derivative of σ. We are going to consider Wick analytic functions in section 4.

We proceed with calculating the Malliavin derivative of a Skorohod integral:

Proposition 4.2. *Assume that $\Phi_s \in (\mathcal{S})^{-1}$ for every $s \in I\!R$ and that $\Phi_s \diamond W_s$ is integrable on $I\!R$ in the sense of Pettis. Furthermore, assume for every $g \in \mathcal{S}(I\!R)$ that $\mathcal{D}\Phi_s(g) \diamond W_s$ and $\Phi_s g(s)$ are integrable in the sense of Pettis. Then*

$$\mathcal{D}\left(\int_{I\!R} \Phi_s \,\delta B_s\right)(\circ) = \int_{I\!R} \mathcal{D}\Phi_s(\circ)\,\delta B_s + \int_{I\!R} \Phi_s \cdot \circ \, ds \tag{4.5}$$

Proof. Let $g \in \mathcal{S}(\mathbb{R})$. Observe that

$$\begin{aligned}
\mathcal{D}\left(\int_{\mathbb{R}} \Phi_s \,\delta B_s\right)(g) &= \mathcal{D}\left(\int_{\mathbb{R}} \Phi_s \diamond W_s \,ds\right)(g) \\
&= \left(\int_{\mathbb{R}} \Phi_s \diamond W_s \,ds\right) \cdot W(g) - \left(\int_{\mathbb{R}} \Phi_s \diamond W_s \,ds\right) \diamond W(g) \\
&= \int_{\mathbb{R}} \left((\Phi_s \diamond W_s) \cdot W(g) - (\Phi_s \diamond W_s) \diamond W(g)\right) ds \\
&= \int_{\mathbb{R}} \mathcal{D}\left(\Phi_s \diamond W_s\right)(g)\, ds
\end{aligned}$$

The product rule (4.1) gives

$$\begin{aligned}
\mathcal{D}\left(\int_{\mathbb{R}} \Phi_s \,\delta B_s\right)(g) &= \int_{\mathbb{R}} \mathcal{D}\Phi_s(g) \diamond W_s \,ds + \int_{\mathbb{R}} \Phi_s \diamond \mathcal{D}W_s(g)\, ds \\
&= \int_{\mathbb{R}} \mathcal{D}\Phi_s(g)\, \delta B_s + \int_{\mathbb{R}} \Phi_s \cdot \delta_s(g)\, ds \\
&= \int_{\mathbb{R}} \mathcal{D}\Phi_s(g)\, \delta B_s + \int_{\mathbb{R}} \Phi_s g(s)\, ds
\end{aligned}$$

Hence, the proposition is proved. □

5 Application to stochastic differential equations of Wick type

As an application of our results we will calculate the extended Malliavin derivative of the solution to a class of stochastic differential equations of Wick-type. We restrict ourselves to the equation

$$\Phi_t = x + \int_0^t \sigma^{\diamond}(\Phi_s)\delta B_s \tag{5.1}$$

where σ is a Wick analytic function introduced above. From [BDP, V] we know that under certain conditions on σ a unique solution Φ_t exists in the space of Kondratiev distributions. Our concern here will be to find an expression for the extended Malliavin derivative.

We first show that the solution Φ_t of (5.1) is adapted in the general sense defined earlier: Let $f, g \in \mathcal{S}(\mathbb{R})$ such that $|f + g|_{2,p} < 1$ for all $p \geq 0$. An application of the $\mathcal{S}$-transform to (5.1) gives

$$\mathcal{S}\Phi_t(f+g) = x + \int_0^t \sigma\left(\mathcal{S}\Phi_t(f+g)\right)(f(s) + g(s))\, ds$$

But if g has support in the complement of $[0,t]$ we have

$$\mathcal{S}\Phi_t(f+g) = x + \int_0^t \sigma\left(\mathcal{S}\Phi_t(f+g)\right) f(s)\, ds \tag{5.2}$$

Adaptedness of Φ_t then follows since

$$\mathcal{S}\Phi_t(f+g) = \mathcal{S}\Phi_t(f)$$

by uniqueness of (5.2). This implies that $\mathcal{D}\Phi_t(g) = 0$ due to the localization properties discussed earlier. Note the analogue to the Malliavin derivative $D_s X_t$ of an adapted square integrable stochastic process X_t. It is known (see e.g. [N]) that $D_s X_t = 0$ when $s > t$.

We calculate $\mathcal{D}\Phi_t(f)$ for $f \in \mathcal{S}(I\!R)$ with support in $[0,t]$. Apply (4.5) and (4.4) to get

$$\begin{aligned}
\mathcal{D}\Phi_t(f) &= \mathcal{D}\int_0^t \sigma^\diamond(\Phi_s)\delta B_s(f) \\
&= \int_0^t \mathcal{D}\left(\sigma^\diamond(\Phi_s)\right)(f)\delta B_s + \int_0^t \sigma^\diamond(\Phi_s)\cdot f(s)\, ds \\
&= \int_0^t (\sigma')^\diamond(\Phi_s) \diamond \mathcal{D}\Phi_s(f)\delta B_s + \int_0^t \sigma^\diamond(\Phi_s)\cdot f(s)\, ds
\end{aligned}$$

Introduce the notation

$$\phi_t^f(\xi) := \mathcal{S}\left(\mathcal{D}\Phi_t(f)\right)(\xi) \tag{5.3}$$

where $\xi \in \mathcal{S}(I\!R)$ with norms less than 1. $\mathcal{S}$-transformation yields the following differential equation for $\phi_t^f(\xi)$

$$\phi_t^f(\xi) = \int_0^t \sigma(\mathcal{S}\Phi_s(\xi)) f(s)\, ds + \int_0^t \sigma'(\mathcal{S}\Phi_s(\xi))\phi_t^f(\xi)\xi(s)\, ds$$

But the solution to this equation is

$$\phi_t^f(\xi) = \int_0^t \sigma(\mathcal{S}\Phi_s(\xi)) f(s)\cdot \exp\left(\int_s^t \sigma'(\mathcal{S}\Phi_u(\xi))\xi(u)\, du\right)\, ds$$

By inverting the $\mathcal{S}$-transform we obtain a representation for the extended Malliavin derivative of Φ_t:

$$\mathcal{D}\Phi_t(f) = \int_0^t \sigma^\diamond(\Phi_s)\cdot f(s) \diamond \operatorname{Exp}\left(\int_s^t (\sigma')^\diamond(\Phi_u)\delta B_u\right)\, ds \tag{5.4}$$

where

$$\mathrm{Exp}\left(\int_s^t (\sigma')^\diamond(\Phi_u)\delta B_u\right) := \sum_{n=0}^\infty \frac{1}{n!}\left(\int_s^t (\sigma')^\diamond(\Phi_u)\delta B_u\right)^{\diamond n}$$

Note that the right-hand side of (5.4) makes sense in the space of Kondratiev distributions as long as $(\sigma')^\diamond(\Phi_u)$ is Skorohod integrable (in the generalized sense) and

$$\sigma^\diamond(\Phi_s)\cdot f(s)\diamond \mathrm{Exp}\left(\int_s^t (\sigma')^\diamond(\Phi_u)\delta B_u\right)$$

is integrable in the sense of Pettis on $[0,t]$. Since σ is analytic and $\langle\langle \Phi_t, \phi\rangle\rangle$ is continuous with respect to t for $\phi \in (\mathcal{S})^1$, this will be true.

To validate our result, consider $\sigma(z) = z$. The equation (5.1) is then the familiar stochastic differential equation

$$\Phi_t = x + \int_0^t \Phi_s \, dB_s$$

(5.4) gives us the expression

$$\begin{aligned}\mathcal{D}\Phi_t(f) &= \int_0^t f(s)\Phi_s \diamond \mathrm{Exp}\left(\int_s^t \delta B_u\right) ds \\ &= \int_0^t f(s)\Phi_s \diamond \exp\left(B_t - B_s - 1/2(t-s)\right) ds\end{aligned}$$

for the extended Malliavin derivative. However, by strong independence (see [BP, lemma 3]),

$$\mathcal{D}\Phi_t(f) = \int_0^t f(s)\Phi_s \exp\left(B_t - B_s - 1/2(t-s)\right) ds$$

which implies that

$$D_s\Phi_t = \Phi_s \exp(B_t - B_s - 1/2(t-s))$$

when $s \leq t$. This expression is known from e.g. [N].

Remark: In light of prop. 3.7, expression (5.4) may be useful if one wants to study the chaos expansion of the solution Φ_t to (5.1).

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