Lectures on Multiscale and Multiplicative Processes in Fluid Flows

Thursday 23 - Tuesday 28 August, 2001
Department of Mathemathical Sciences
University of Aarhus

Contents

Preface **v**

E. Waymire **1**
Multiscale and Multiplicative Processes in Fluid Flows
1 Introduction and Preliminaries 1
2 Diffusion Models 2
3 Tree Graphs: As Network Models 8
4 Multiplicative Cascades: Statistical Rainfall and Turbulence Models . 28
5 Navier-Stokes Equations and Multiplicative Cascades 49
6 Branching Brownian Motion, Fisher/KPP Equation, Burgers Equation: Real Space Cascades 64

R. Bhattacharya **77**
Multiscale Diffusion Equations

M. Greiner **81**
Turbulent Random Multiplicative Branching Processes

V.K. Gupta **89**
Multiscale Flood Analysis on Self-similar River Networks

J.C. Orum **99**
Stochastic Cascades and 2D Fourier Navier-Stokes Equations

M. Ossiander **109**
On Estimation Theory for Multiplicative Cascades
1 Introduction 109
2 An Overview of the Theoretical Foundations 109
3 Dimension Estimates: Some Recent Results 116
4 Lognormal Versus log Poisson Generators; A Statistical Problem in Turbulence 117
5 Cascade Models and Temporal Rainfall 121

E. Thomann **127**
Partial Differential Equations and Multiplicative Processes
1 Introduction 127
2 Linear equations: Feynman-Kac Formula 128
3 Nonlinear equations: Navier-Stokes 129
4 NS-Majorizing Kernels 132
5 Successive Iterations of a Contraction Map 135
6 Conclusions and Remarks 138

B.M. Troutman **141**
River Flow Mass Exponents with Fractal Channel Networks and Rainfall

D. Winn **149**
Gradient-Directed Diffusions and River Network Models

Preface

These lecture notes were prepared for the Instructional and Research Workshop on Multiplicative Processes and Fluid Flows at MaPhySto, Aarhus University, 23-28 August 2001. Much of this is based upon collaborations within the Focussed Research Group[1] on Navier-Stokes equations at Oregon State University/Indiana University on the one hand and the turbulence research group[2] around the Max-Planck Institute for Physics of Complex Systems, Dresden, on the other. In particular, the treatment of multiplicative Fourier cascade solutions to diffusion, fractional diffusion, reaction-diffusion, Burgers, and Navier-Stokes are largely unpublished results due to the OSU/Indiana Focussed Research Group. Also included are results of long time collaborations with Vijay Gupta at University of Colorado on problems concerning hydrologic flows in ungauged river basins.

[1]OSU/Indiana Focussed Research Group: Rabi Bhattacharya, Larry Chen, Scott Dobson, Ronald Guenther, Chris Orum, Mina Ossiander, Enrique Thomann, Edward Waymire. Partially supported by a grant from the National Science Foundation.

[2]Jochen Cleve, Thomas Dziekan (now University of Upsalla), Hans Eggers (University of Stellenbosch), Martin Greiner (now Siemens München), Jürgen Schmiegel.

LECTURES ON MULTISCALE AND MULTIPLICATIVE PROCESSES IN FLUID FLOWS

EDWARD C. WAYMIRE
Department of Mathematics, Oregon State University

1 Introduction and Preliminaries

These notes are intended to provide a mathematical framework for the methods, problems, and results which will be discussed in the course of the tutorial lectures. Explanations and proofs will be filled in during the lectures. The notes will hopefully provide background for the various more specialized research topic lectures described in the extended abstracts appended to these notes.

A diverse array of disparate problem areas will be discussed within a mathematical framework which highlights certain common structure and themes. One such theme is the application of probabilistic and statistical methods to ostensibly deterministic descriptions of physical phenomena. Various perspectives on multiple scale hierarchical structure also provide a common theme for the phenomena considered. Multiplicative/branching structure related to flows will be exploited throughout these lectures.

The organization is as follows. In Section 2 we provide a deterministic derivation of the diffusion equation from principles of mass balance and Fick's (empirical) flux law. This is then connected to Brownian motion and stochastic differential equations. We indicate the role of the martingale central limit theorem in obtaining time-asymptotic solutions to diffusion equations. These lectures are intended to provide some background for the topic discussed in the research lectures by R. Bhattacharya, and by F. Mainardi. We also include Feynman-Kac formula and Duhamel's formula, as these will be used in later sections and research topics.

Branching models provide the mathematical foundations for interpreting a variety of observations and data structures which will be presented in various research lectures. In Section 3 we examine data structures associated with river network hydrology and consider several classes of deterministic and random tree graphs which arise as network models. We shall show how Brownian excursions unify and explain a number of prior results on various river network statistics. While the results may not be applicable to all models, this approach points to the essential underlying structure which relates these statistics. The material covered in this section provides background for the lectures by V. Gupta, B. Troutman, and R. Winn.

In Section 4 we introduce hierarchical models for rainfall. These include cluster random field models introduced by LeCam (1961) and multiplicative cascade models which have origins as statistical turbulence models. We indicate some standard

methods for analysis of the fine scale structure of these models in the context of data structures from rainfall and turbulence observations. These lectures provide background for the research lectures by M. Ossiander and M. Greiner.

In Section 5 we present a derivation of the Navier-Stokes equations as conservation laws. We then discuss multiplicative cascade solutions to linear, semilinear, and quasilinear partial differential equations in Fourier space. These are deterministic flows which admit representations in Fourier space as expectations of certain products over branching random walks. These lectures provide background for the research lectures by E. Thomann, and C. Orum.

In Section 6 we discuss recent connections uncovered between real space and Fourier space cascades for Fisher's equation and Burgers equation. The material here is provided as natural follow-up to the ideas presented in Section 5.

2 Diffusion Models

Within the context of fluid flow, diffusion equations describe the space-time evolution of a concentration (amount per unit volume in $\mathbf{R}^3$) of a dilute substance (solute) suspended in a fluid medium moving with a velocity $v(x,t)$. For concreteness one may imagine a suspension of ink particles (dye) in a stream of water, or dust particles suspended in air as may be observed beneath a bright lamp. Define

(**solute concentration**) $c(t,x)$ = amount of diffusing substance in an arbitrary region V of $\mathbf{R}^3$ with boundary surface ∂V.

(**solute flux**) $-\int_{\partial V} q \cdot n\sigma(dx)$ = amount of substance flowing out of V per unit time, where n is the outward unit normal to ∂V.

(**solute production rate**) $\int_V R dx$ = amount of substance created or removed in V.

Then the *principle of mass balance* may be expressed as

$$\frac{d}{dt}\int_V c(t,x)dx = -\int_{\partial V} q \cdot n\sigma(dx) + \int_V R dx \tag{2.1}$$

Let us now recall the Gauss Divergence Theorem.

Theorem 2.1 *Let V be a bounded open set in $\mathbf{R}^3$ with piecewise smooth boundary surface ∂V. Let $\overline{V} = V \cup \partial V$ and suppose that $f : \overline{V} \to \mathbf{R}^3$ is a continuous vector field which is also C^1 on V. Then*

$$\int_V div f dx = \int_{\partial V} f \cdot n d\sigma,$$

where $div f = \nabla \cdot f = \sum_j \frac{\partial f_j}{\partial x_j}, \quad f = (f_1, f_2, f_3)$.

Using the Divergence Theorem and the arbitrary choice of V, the equation (2.1) may be expressed in local form as

$$\frac{\partial c}{\partial t} = -\nabla \cdot q + R. \tag{2.2}$$

The next basic assumption is the empirical hypothesis (Fick's Law) that the mass of solute is transported by bulk movement of the fluid with velocity v, and spreads from regions of high concentration to low concentration at a (flux) rate in proportion to the magnitude of the concentration gradient $|\nabla c|$. That is

$$q = -D \cdot \nabla c + vc, \tag{2.3}$$

where the coefficient D, referred to as the dispersion or diffusion coefficient, is a non-negative definite symmetric matrix.

Finally let us assume that the local production rate is of the form

$$R = -rc + g, \quad (r \geq 0). \tag{2.4}$$

Then one obtains the *linear diffusion model*

$$\frac{\partial c}{\partial t} = \nabla \cdot (D\nabla c) - \nabla(vc) - rc + g, \tag{2.5}$$

for an *initial concentration* $c(0^+, x) = c_0(x)$.

Remark 2.1 We have suppressed explicit space-time dependence of the parameters D, r, v, g although such inhomogeneity is permitted within this general framework. The case of constant parameters is important to consider for its simplicity. The case of temporally constant parameters will be assumed for the most part without further mention.

Remark 2.2 Remark 1.2 Although the formulation of the model was presented in $n = 3$ dimensions, analagous modelling considerations, eg. for heat conduction in a bar ($n = 1$), or a price of a portfolio of n securities, provide examples of the pervasive role of such equations in all dimensions $n \geq 1$.

Remark 2.3 If D is concentration dependent or if R were a nonlinear function of concentration, then the evolution equation for concentration is in turn nonlinear. One particular illustration is that of *reaction-diffusion* where the production rate R depends on concentration through biological or chemical reaction mechanisms. A notable example was proposed by R.A. Fisher (1937) to model the spread of an advantageous gene in a biological population. Here $c(t, x)$ is the frequency of the advantageous gene in the population, the parameter s is a measure of intensity of selection, D is constant and the evolution is by the one-dimensional equation

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} + sc(1 - c), \quad c(0^+, x) = c_0(x). \tag{2.6}$$

A similar equation was also introduced by Kolmogorov, Petrovskii, and Piscunov (1937), referred to as KPP equation, for population density. The rate of increase $sc(1-c)$ is derivable under some additional simple hypothesis, and Fisher added the diffusion term by assuming that the migration of individuals is purely random, i.e. as in kicks of a suspended particle by surrounding fluid molecules, and, assuming a constant population density, the genetic flux is proportional to the flux of individuals; eg see Fife (1979).

Viewed probabilistically, the linear diffusion model arises quite naturally in connection with the Central Limit Theorem. Specifically, in Einstein's (1905) investigation of the hypothesised molecular structure of matter he considered that a suspended particle would suffer repeated random kicks by the molecules of the surrounding fluid. In a fluid with constant bulk velocity v, a suspended particle initially at x would undergo some ft kicks in time t, where f is the mean frequency of kicks. So by the Central Limit Theorem in time t its position would be displaced by a Gaussian random variable with mean vt and variance proportional to ft, say Dt. That is, the probability density of finding the particle intially at x displaced to a position y at time t is given by

$$p(t,x,y) = \frac{1}{\sqrt{2\pi Dt}} \exp\{-\frac{1}{2Dt}|y-x-vt|^2\} \tag{2.7}$$

One may easily check by differentiation that

$$\frac{\partial p}{\partial t} = \frac{1}{2}D\frac{\partial^2 p}{\partial y^2} - v\frac{\partial p}{\partial y}. \tag{2.8}$$

For a dilute suspension of solute, the particles may be assumed to move (approximately) independently of one another. Thus, by the Law of Large Numbers one may define solute concentration by

$$c(t,y) = \int_{R^3} p(t,x,y)c_0(x)dx. \tag{2.9}$$

It is immediately clear that c obeys the linear diffusion equation with constant coefficients without sources or sinks, $r = g = 0$. Note that the coefficient $\frac{1}{2}D$ appears naturally in the probabilistic model. In this context, D is referred to as the diffusion coefficient, not $\frac{1}{2}D$.

Norbert Wiener's (1923) celebrated construction, cf Paley and Wiener (1934), of the *standard Brownian motion* starting at 0, as a stochastic process $\{W(t) : t \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, P_0)$ such that

i. W(0) = 0

ii. For $0 = t_0 < t_1 < \cdots < t_k$, the increments $W(t_j) - W(t_{j-1}), 1 \leq j \leq k$, are independent

iii. For $0 \leq s < t$, $W(t) - W(s)$ is Gaussian with mean zero and variance t
iv. For $\omega \in \Omega, t \to W(\omega, t), t \geq 0$, is continuous,
is the cornerstone to the probabilistic description of linear diffusions. Specifically, Einstein's diffusion process $\{X(t) : t \geq 0\}$ can be completed mathematically by defining

$$X(t) = x + vt + \sqrt{D}W(t). \tag{2.10}$$

This process will be referred to as Brownian motion with drift v and diffusion coefficient D started at x.

In terms of the Ito calculus, this may be viewed (via stochastic integration) as an evolution according to the stochastic differential equation

$$dX(t) = vdt + \sqrt{D}dW(t), \quad X(0) = x. \tag{2.11}$$

More generally, with the Ito calculus one may define a general diffusion for given coefficients where one may view (2.11) as a local description in time t to $t + dt$ for Lipschitz continuous coefficients as an "approxmiate Brownian motion"(with $v(X(t), t)$ and $D(X(t), t)$ regarded as approximately constant over infinitesimal time intervals by continuity).

Let us also note that a natural adjoint equation to (2.8) may be obtained by integration by parts (twice). Namely,

$$\frac{\partial p}{\partial t} = \frac{1}{2}D\frac{\partial^2 p}{\partial x^2} + v\frac{\partial p}{\partial x}. \tag{2.12}$$

Thus, for a bounded sufficiently smooth function f, defining

$$u(t, x) = E_x f(X(t)) \tag{2.13}$$

one has

$$\frac{\partial u}{\partial t} = \frac{1}{2}D\frac{\partial^2 u}{\partial x^2} + v\frac{\partial u}{\partial x}, \quad u(x, 0^+) = f(x). \tag{2.14}$$

In particular by a probabilistic construction of the standard Brownian motion on a probabiility space $(\Omega, \mathcal{F}, P_0)$, together with the development of Ito calculus, one obtains a probabilistic representation of solutions to (2.14) as the expected value $u(t, x) = E_x f(X(t))$ for a suitable class of initial data f. This theme is elaborated by examples and illustrations in Bhattacharya and Waymire (1991)

Let us give a more precise mathematical description to the picture developed so far. For this let $(\Omega, \mathcal{F}, P_x)$ denote the probability space for standard Brownian motion started at x. $\{W(t) : t \geq 0\}$ will denote standard Brownian motion started at 0. Suppose given a Lipschitz continuous vector field $\mu(x) = (\mu_j(x))_{1 \leq j \leq n}$ and Lipschitz continuous matrix function $\sigma(x) = ((\sigma_{ij}))_{1 \leq i,j \leq n}$, $x \in \mathbf{R}^n$.

Theorem 2.2 *For each $x \in \mathbf{R}^n$, there is a unique (up to P_x-null events) continuous stochastic process $X(t) : t \geq 0$, nonanticipative with respect to the filtration $\mathcal{F}_t, t \geq 0$, such that $E|X(t)|^2 < \infty$, which satisfies the stochastic differential equation*

$$dX(t) = \mu(X_t)dt + \sigma(X(t))dW(t), \quad X(0) = x.$$

Moreover, $\{X(t) : t \geq 0\}$ is a strong Markov process on $\mathbf{R}^n$ with

$$E(X_j(s+t) - X_j(s))1[|X(s+t) - X(s)| \leq \epsilon | X(s) = x) = t\mu_j(x) + o(t)$$

$$E|(X_i(s+t) - X_i(s))(X_j(s+t) - X_j(s))1[|X(s+t) - X(s)| \leq \epsilon | X(s) = x) = t(\sigma^*\sigma)_{ij}(x) + o(t)$$

$$P|(X(s+t) - X(s))| > \epsilon | X(s) = x) = o(1),$$

as $t \to 0^+, i, j = 1, \ldots n$.

Definition 2.1 The continuous (strong) Markov process $\{X(t) : t \geq 0\}$ satisfying the conditions of Theorem 2.2 is referred to as the *diffusion with drift μ and diffusion matrix $D = \sigma^*\sigma$*, where * denotes matrix transpose.

Remark 2.4 It may noted that $D = \sigma^*\sigma$ is not required to be positive definite. In particular the diffusion may be supported on a lower dimensional subspace of $\mathbf{R}^n$, referred to as *singular diffusion.* However, if in addition to the Lipschitz condition, D is positive definite then it is known that the process will possess a smooth transition probability density $p(t, x, y)$; cf Stroock and Varadahn (1979).

Chief among the probabilistic tools which aid in the analysis of diffusion equations is *Ito's lemma,* which we record here for ease of reference.

Lemma 2.1 *(Ito's Lemma). Let $\mu = (\mu_1, \ldots, \mu_n)$ be a Lipschitz n-dimensional vector field, and $\sigma = ((\sigma_{ij}))_{1 \leq i,j \leq n}$ an $n \times n$ matrix of real valued Lipschitz functions on $\mathbf{R}^n$. Let $\{\mathbf{X}(t) : t \geq 0\}$ be the n-dimensional diffusion defined by (the system of n equations)*

$$d\mathbf{X} = \mu(\mathbf{X}(t))dt + \sigma(\mathbf{X}(t)) \cdot d\mathbf{W}(t), \quad \mathbf{X}(0) = \mathbf{x}.$$

Let $\varphi(t, \mathbf{x})$ be a real-valued function on $[0, T] \times \mathbf{R}^n$ which is once continuously differentiable in t and twice continuously differentiable in $\mathbf{x}$ such that

$$E \int_0^T (\partial_i \varphi)^2(s, \mathbf{X}(s))|\sigma_i(\mathbf{X}(s))|^2 ds < \infty,$$

where $\partial_i = \frac{\partial}{\partial x_i}, 1 \leq i \leq n$, $\partial_0 = \frac{\partial}{\partial t}$ $\sigma_i = (\sigma_{i1}, \ldots, \sigma_{in})$. Then

$$d\varphi(t, \mathbf{X}(t)) = \{\partial_0 \varphi(t, X(t)) + (A\varphi)(t, \mathbf{X}(t)\}dt + \sum_{i=1}^n \partial_i \varphi(t, \mathbf{X}(t))\sigma_i(\mathbf{X}(t)) \cdot d\mathbf{W}(t),$$

where

$$A = \frac{1}{2} \sum_{1 \le i,j \le n} d_{ij}(\mathbf{x}) \partial_i \partial_j + \sum_{i=1}^{n} \mu_i(\mathbf{x}) \partial_i,$$
$$D = ((d_{ij}(\mathbf{x}))) = \sigma \sigma^{tr}.$$

Let us also record the important consequence of Ito's lemma that if in addition $\partial_0 \varphi$ and $\partial_{ij} \varphi$ are bounded, then

$$\mathbf{Y}(t) = \varphi(t, \mathbf{X}(t)) - \int_0^t \{\partial_0 \varphi(s, X(s)) + (A\varphi)(t, \mathbf{X}(s)\} ds \tag{2.15}$$

is a martingale.

Remark 2.5 Under suitable conditions on the coefficients the diffusion $\{X(t) : t \ge 0\}$ will converge in distribution to a unique invariant probability π as $t \to \infty$. For such (ergodic) diffusions on a state space S the fluctuations in $\int_0^t g(\mathbf{X}(s))ds - \int_S g(\mathbf{x})\pi(d\mathbf{x})$ may be obtained for a broad class of functions g in the range of A, i.e. $g(\mathbf{x}) = A\varphi(\mathbf{x})$, by appeal to the Martingale Central Limit Theorem and (2.15). This is an essential tool in the problem discussed in the research topic lecture by R. Bhattacharya on a multiscale model of solute transport and the "growth of dispersion phenomena in saturated porous media."

We close this section with two additional basic equations of interest in applications discussed in the research topic lectures. The first may be obtained from Ito's lemma applied to the 2n-dimensional process $(\mathbf{X}(t), \int_0^t R(\mathbf{X}(s))ds)$.

Feynman-Kac Formula: Let $\mu = (\mu_1, \ldots, \mu_n)$ be a Lipschitz n-dimensional vector field, and $\sigma = ((\sigma_{ij}))_{1 \le i,j \le n}$ an $n \times n$ matrix of real valued Lipschitz functions on $\mathbf{R}^n$. Let $\{\mathbf{X}(t) : t \ge 0\}$ be the n-dimensional diffusion defined by (the system of n equations)

$$d\mathbf{X} = \mu(\mathbf{X}(t))dt + \sigma(\mathbf{X}(t)) \cdot d\mathbf{W}(t), \quad \mathbf{X}(0) = \mathbf{x}.$$

Let $u(t, \mathbf{x})$ be a real-valued function on $[0, \infty) \times \mathbf{R}^n$ which is once continuously differentiable in $t > 0$ and twice continuously differentiable in $\mathbf{x}$ such that

$$\frac{\partial u(t, \mathbf{x})}{\partial t} = Au(t, \mathbf{x}) + R(\mathbf{x})u(t, \mathbf{x}), \quad u(0, \mathbf{x}) = f(\mathbf{x}), \tag{2.16}$$

$$A = \frac{1}{2} \sum_{1 \le i,j \le n} d_{ij}(\mathbf{x}) \partial_i \partial_j + \sum_{i=1}^{n} \mu_i(\mathbf{x}) \partial_i, \quad D = ((d_{ij}(\mathbf{x}))) = \sigma^* \sigma, \tag{2.17}$$

where f is a polynomial bounded continuous function, and R is a continuous function bounded above on $\mathbf{R}^n$. If $\nabla u(t, \mathbf{x})$ and $Au(t, \mathbf{x})$ are polynomially bounded in $\mathbf{x}$ over compact sets of $t \in (0, \infty)$, then

$$u(t, \mathbf{x}) = E_{\mathbf{x}}[f(\mathbf{X}(t)) \exp\{\int_0^t R(\mathbf{X}(s))ds\}]. \tag{2.18}$$

Note that in the one-dimensional case with constant coefficients $\mu = 0, \sigma^2 = 1, R(x) = r > 0$,

$$u(t, \mathbf{x}) = E_{\mathbf{x}}[f(\mathbf{X}(t)) \exp\{-\int_0^t R(\mathbf{X}(s))ds\}] = \int_R f(y)e^{-rt}g(t, y - x), \qquad (2.19)$$

where $g(t, x - y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}$.

Duhamel Principle: Consider the one-dimensional diffusion with drift coefficient $\mu(y)$ and diffusion coefficient $\sigma^2(y)$. Assume that there is a smooth transition probability density $p(t, x, y)$. Consider the equation

$$\frac{\partial c(t,y)}{\partial t} = -\frac{\partial}{\partial y}(\mu(y)c(t,y)) + \frac{\partial^2}{\partial y^2}(\frac{1}{2}\sigma^2(y)c(t,y)) + f(t,y), \quad c(0,y) = c_0(y), \quad (2.20)$$

where $f(t, y)$ is a bounded continuous function of t, y and $\int |h(t, x)|p(t, x, y)dx < \infty$. Assume also that $c_0(y)$ is integrable. Then,

$$c(t,y) = \int_R c_0(x)p(t,x,y)dx + \int_0^t \int_R f(s,x)p(t-s,x,y)dxds. \qquad (2.21)$$

A similar result holds for the adjoint diffusion equation. In the case of constant coefficients $\mu = 0, \sigma^2 = 1$, one has $p(t, x, y) = g(t, x - y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(y-x)^2}$.

It will also be illuminating to return to the role of multiplicative branching processes for nonlinear evolution equations in this context of linear diffusion and reaction-diffusion (KPP). This topic will be revisited in a later lecture.

3 Tree Graphs: As Network Models

Branching networks play a pervasive role in the hydrologic science of river networks and other hierarchical space-time phenomena. In this section we introduce some simple branching models within this context.

To continue with the flow theme, however, let us begin with a description of flows on river networks to motivate some of the mathematical development. For this one considers a finite rooted tree graph τ to represent the drainage network. The *size* of τ, denoted $||\tau||$, is defined by the total number n of vertices (sources and junctions) $v \in \tau$. The root is a distinguished vertex, denoted θ, respresenting the *network outlet*. The root may be used to direct the tree as follows. The *height* of a vertex $v \in \tau$ is defined by the number $|v|$ of edges (links) on the unique path connecting v and θ, with $|\theta| := 0$. The edge e is directed as $e = (\overline{e}, \underline{e})$, for vertices $\overline{e}, \underline{e} \in \tau$, where $|\overline{e}| = |\underline{e}| + 1$. We identify edges and vertices according to the convention that to each edge e there is a unique vertex v such that $\overline{e} = v$. A ghost edge (stem) may be added to θ when necessary to complete the convention. We write both $e \in \tau$ and $v \in \tau$ to denote an edge e or vertex v belongs to the network τ, respectively. Also

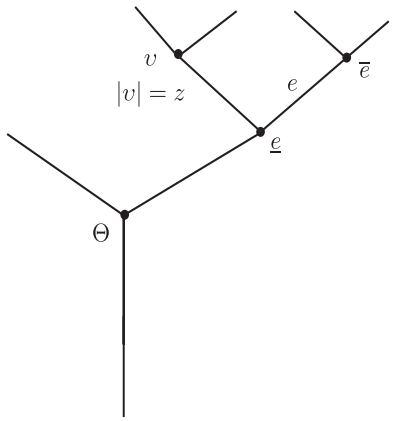


Figure 3.1: Schematic of tree graph notation

definitions for vertices are applied interchangeably to edges by this convention, e.g. the height of an edge e is $|\overline{e}|$; see Figure 3.1.

We fix a time scale on which flow in the network is observed at times $t = n\Delta t, n = 0, 1, \ldots$. As an example, one may view Δt as the time required for water to flow through a link with some mean velocity v, i.e $\Delta t = \frac{l}{v}$. We suppress Δt when clear from context and simply write $t = 0, 1, 2, \ldots$. An input (rainfall runoff) volume $R(t, e)a(e)$ per unit time arrives from two adjacent hills of area $a(e)$ at an edge e. Water will then be drained from edges $f = (\overline{f}, \underline{f})$ into the uniquely determined edge $e \in \tau$, defined by $\overline{e} = \underline{f}$. River discharges are then represented by a space-time random field $q(t, e)$ assigned to edges $e \in \tau$ at times $t \geq 0$, representing the volume of flow accross the edge $(\overline{e}, \underline{e})$, per unit time satisfying the *mass balance principle*

$$\Delta S(t, e) = -q(e, t)\Delta t + \sum_{f:\underline{f}=\overline{e}} q(t, f)\Delta t + R(t, e)a(e)\Delta t \quad t \geq 1, \tag{3.1}$$

where the left-hand side $\Delta S(t, e), e \in \tau, t \geq 0$, represents the total volume of runoff per unit length stored in the edge e.

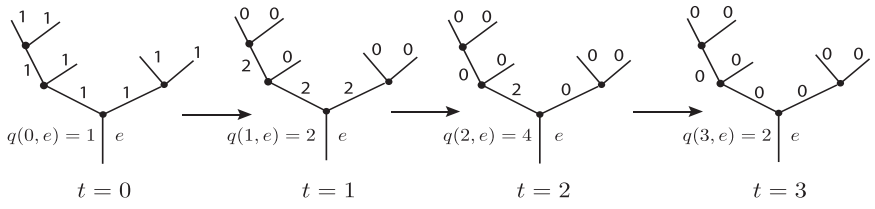


Figure 3.2: Width function hydrograph $q(t, e), t = 0, 1, 2, 3$.

Observe that in the case of unit instantaneously applied rainfall-runoff defined by $R(0, e)a(e) = 1, R(t, e) \equiv 0, t \geq 1$, one obtains for constant storage capacity, i.e.

$\Delta S(t,e) \equiv 0$, that

$$q(e,t) = Z_{\overline{e}}(t), t = 1, 2, \ldots, \tag{3.2}$$

where $\{Z_v(t) : t = 1, 2, \ldots\}$ is the number of edges located at height t in the subtree rooted at v; see Fig 3.2. Hydrologists and geomorphologists refer to the function $t \to Z_v(t), t = 1, 2, \ldots,$ as the *local width function* at v. The structure of this solution depends on the underlying network model for τ.

So the focus of this section is to provide some orientation to empirical structure of these quantities as observed in real networks and then consider corresponding mathematical versions for some simple network models. In addition to the structure of the network, for flows one must also consider the structure of the inputs into the network, eg rainfall, snowmelt. A mathematical description of inputs compatible with the branching structure of the network will be discussed in a later section. The research lectures by B.Troutman and V. Gupta will illustrate an approach to the above network flow problem which exploits self-similarities in both the inputs and the network. The network idealization incorporates self-similarities but is not spatially embedded. As a result one loses certain geometric structure of natural river basins. In the research topic lecture by R.D. Winn a singular diffusion will be introduced which is obtained as a limit distribution in a class of flow path models for river networks embedded in a geometric surface. An excellent (though somewhat dated) reference to the early empirical and theoretical results in geomorphology/hydrology of river networks is Jarvis and Waldenberg (1984).

Let us begin with a canonical representation of tree graphs. The binary tree assumption may be violated in nature but its occurence is rare enough that it is generally ignored by geomorphologists. However, for the sake of mathematical discussion we shall not always constrain the trees to binary trees when introducing more generally valid notions. Let T be the space of (possibly infinite) *labelled tree graphs rooted* at θ. An element τ of T may be coded as a *set* of finite sequences of positive integers $< i_1, i_2, \ldots, i_n > \in \tau$ such that:
(i) $\theta \in \tau$ is coded as the empty sequence
(ii) If $< i_1, \ldots, i_k > \in \tau$ then $< i_1, \ldots i_j > \in \tau \ \forall \ 1 \leq j \leq k$.
(iii) If $< i_1, i_2, \ldots, i_n > \in \tau$ then $< i_1, \ldots i_{n-1}, j > \in \tau \ \forall \ 1 \leq j \leq i_n$.

If $< i_1, \ldots, i_n > \in \tau$ then $< i_1, \ldots, i_{n-1} > \in \tau$ is referred to as the *parent* vertex to $< i_1, \ldots i_n >$. A pair of vertices are connected by an edge (adjacent) if and only if one of them is parent to the other. In this way edges are identified with the (unique) non-parental or *descendant* vertex. This specifies the graph structure of τ and makes τ a rooted connected graph without cycles. T may be viewed as a metric space with metric $\rho(\tau, \gamma) = (\sup\{n + 1 : \gamma|n = \tau|n\})^{-1}$, and $\tau|n = \{< i_1, \ldots, i_k > \in \tau : k \leq n\}$. The countable dense subset T_0 of finite labelled tree graphs rooted at θ makes T a Polish space. This fact is useful for the construction of stochastic network models as probability distributions on T.

A variety of river basin characteristics may be associated with the bifurcation or topological structure of the network in the form of positive weights $w(e), e \in \tau$.

Hydrologists interpret the weights as geomorphologic parameters such as lengths, elevation drops, drainage areas etc., associated with river networks. *The weighted height of vertex* $< i_1, i_2, \ldots, i_k >$ is defined as

$$h_w(i_1, i_2, \ldots, i_k) = \sum_{j=0}^{k} w(i_1, i_2, \ldots, i_j). \tag{3.3}$$

The weighted tree height of τ is defined as

$$h_w(\tau) = max_{<e> \in \tau} h_w(e). \tag{3.4}$$

As observed earlier, the width function associated with a tree arises naturally in the consideration of instantaneous uniform inputs with nonattenuated flows. The *width function* of the weighted tree τ at height h is defined as

$$Z(h, \tau) = \begin{cases} 1 & \text{for } 0 \leq h < w_\theta \\ \#\{< e > \in \tau : h_w(e') \leq h < h_w(e)\} & \text{else,} \end{cases}$$

where $< e' >$ is the parent of $< e >$ and $\#$ denotes the cardinality of the set. In the special case in which all $w(e) = W(i_1, i_2, \ldots, i_k) = 1$, the width function is given by

$$Z(h) = Z_k, \quad k \leq h < k + 1, \quad k = 0, 1, \ldots \tag{3.5}$$

where Z_k is the total number of vertices at height $k + 1$.

Finally we turn to a notion of *spatial resolution* associated with river networks introduced by Horton (1945) and later refined by Strahler (1957) according to the following algorithm (see Figure 3.3): First the vertices of either degree (or valence) one or two will be called "non-branching". Those of degree one are called *leaves.* All leaves and adjacent paths of adjacent non-branching vertices are assigned order one. The orders of all other vertices (or associated edges) are recursively defined as the maximum of orders of the offspring vertices when these are not all equal, else it is the common order of the offspring incremented by one. A continguous path of edges of equal order is called a *stream* of the said order. The order of the root θ defines the *order of the network* τ and is denoted $\omega(\tau)$. This scheme provides an "order or scale of resolution" in which the given tree is regarded to be at the finest scale of resolution and the next level of coarsening is obtained by removing the order 1 streams. The next level of "non-branching" vertices in the pruned tree are assigned order 2. The next level of coarsening is obtained by pruning off the (lowest) order 2 streams, etc.

Remark It may be of independent interest to note that the same algorithm for network order has been shown to provide a natural optimization parameter in binary arithmetic register allocation problems in computer science; e.g. Ershov (1958), Flajolet and Prodinger (1986). In this context one has an arithmetic expression to evaluate, say ab + (c+d)/e, which will involve successive calculations and storage

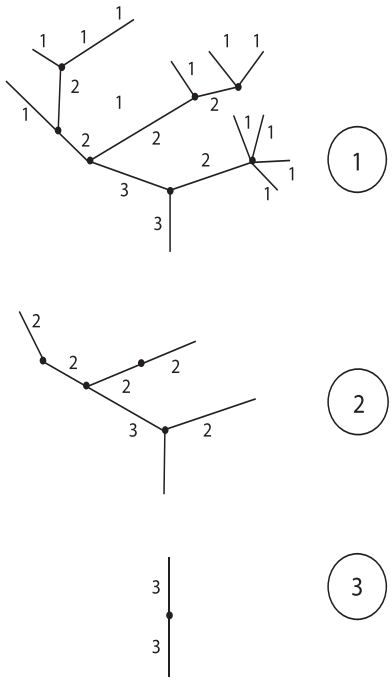


Figure 3.3: Horton order schematics

registers to get the final answer. Since these are binary operations they can be organized along a binary tree. A flow for the above example might be as follows:

a in Reg 1
b*Reg 1 in Reg 2
c in Reg 1
d + Reg 1 in Reg 1
Reg 1/e in Reg 1
Reg 1 + Reg 2

A very inefficient alternative is:

a in Reg 1
b*Reg 1 in Reg 2
c in Reg 3
d + Reg 3 in Reg 4
Reg 4/e in Reg 5
Reg 2 + Reg 5

The point is that the first one is more efficient in its use of storage registers than the second. The (two-fold) question becomes, for a given arithmetic expression, what is the smallest number of storage registers required to evaluate it and whether there an algorithm (arrangement of the sequence of operations on a tree) which will achieve this minimum number of storage registers? The idea for Horton numbers can be seen inductively on a tree as follows. Suppose that to get the end result involves a final

binary operation of two values AoB, where o is the final binary operation. Suppose it took a registers for A and b registers to evaluate B. Then the total number needed is $a+1$ if $a=b$ and $max(a,b)$ if a and b are not equal. Here is why: If $a=b$, then use the b storage registers to calculate B and then store this answer (ie $b+1$ registers). Now reuse the b registers to evaluate A. If $a \neq b$, say $a > b$, then make the calculation of A using these registers and store it in one of these a registers. Since B needs b registers and $b \leq a-1$, there is enough to calculate B. Thus requiring $a = max(a,b)$ storage registers. There is more to the problem in terms of showing the existence of an optimal tree (ie. a tree graph with minimal number of storage registers), but, given existence of the tree, the number of registers will be its Horton order.

Given the notion of (Horton) order, a finite tree graph for which the number T_{ij} of order $1 \leq j \leq i-1$ subtrees affixed to interior vertices of an order i stream, called a *stream order generator,* is *(1)* the same for each order i stream in the network and *(2)* a function of i, j only through $i-j,\ j \leq i$, is called *topologically self-similar*; ie. the matrix of stream order generators is Toeplitz; see Figure 3.4.

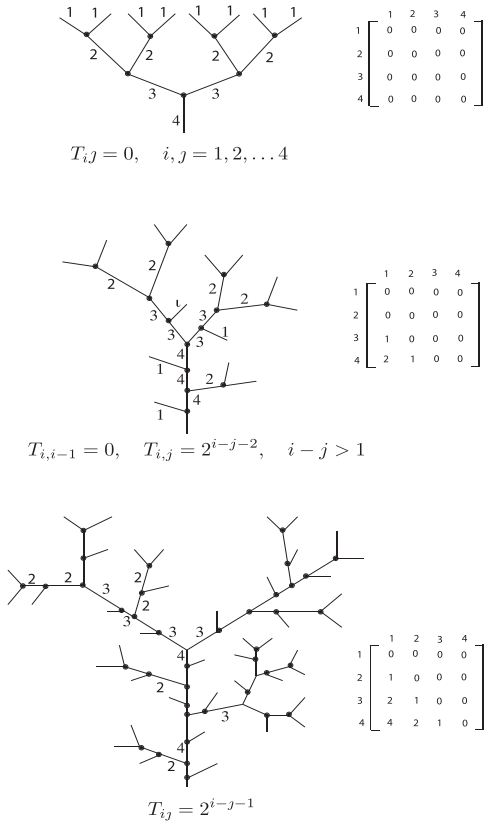


Figure 3.4: Self-similar network schematic examples

Before examining mathematical structures let us examine some observed river

network data structures.

Sample Stream Order Generators: In river network analysis one computes the sample average $\overline{T}_{ij}$ of the number of order j subtrees supported by the various streams of order i in the network. The stream order generators for the Kentucky river network have the "approximate" Toeplitz form given by the matrix in Figure 3.5.

$$\begin{array}{cccccc} 2.78 & 1.22 & 0.00 & 0.00 & 0.00 & 0.00 \\ 6.91 & 2.66 & 1.03 & 0.00 & 0.00 & 0.00 \\ 15.89 & 5.58 & 2.53 & 1.05 & 0.00 & 0.00 \\ 49.25 & 20.00 & 10.50 & 2.00 & 2.50 & 0.00 \\ 61.00 & 21.00 & 6.00 & 4.00 & 1.00 & 2.00 \end{array}$$

$$\begin{bmatrix} T_{21} & 0 & 0 & 0 & 0 & 0 \\ T_{31} & T_{32} & 0 & 0 & 0 & 0 \\ T_{41} & t_{42} & T_{43} & 0 & 0 & 0 \\ T_{51} & T_{52} & t_{53} & T_{54} & 0 & 0 \\ T_{61} & T_{62} & T_{63} & T_{64} & T_{65} & 0 \\ T_{71} & T_{72} & T_{73} & T_{74} & T_{75} & T_{76} \end{bmatrix}$$

Figure 3.5: Sample stream order generators for Kentucky River Network

Sample Width Function: The sample width function for the Kentucky River Network is given in Figure 3.6, 3.7. In one plot the length is by number of channel segments (straight line length), while in Figure 3.7, the length is measured at the finest scale of resolution; for some interesting alternative definitions of channel length see Troutman and Karlinger (1993).

Sample Mainchannel Length vs Area: The length of channel networks in relation to basin size is often recorded in log-log plots whose slope is referred to as a *Hack exponent*. Hack's law is a set of empirical observations which reports the growth of the main channel to be $O(Area)^{\alpha}$ with $\alpha \approx .6$. A sample plot for the Kentucky basin is given in Figure 3.8.

For a mathematical analysis we begin with a well-studied stochastic model in both mathematics and the hydrology/geomorphology literature which serves as a framework for viewing both agreements and departures from various empirical observations. This model was introduced into geomorphology by Shreve (1967) and is referred to as the *random model* in which all binary rooted trees of size n are assigned equal probability. The motivation was simply as an attempt to allow fluctuations which might occur in an idealized environment absent of geologic controls. For our purposes we view this model as a Bieneyme-Galton-Watson branching process conditioned on total progeny. Many of the earlier results for this model were obtained by analytic and combinatoric calculations which provided little understanding to the

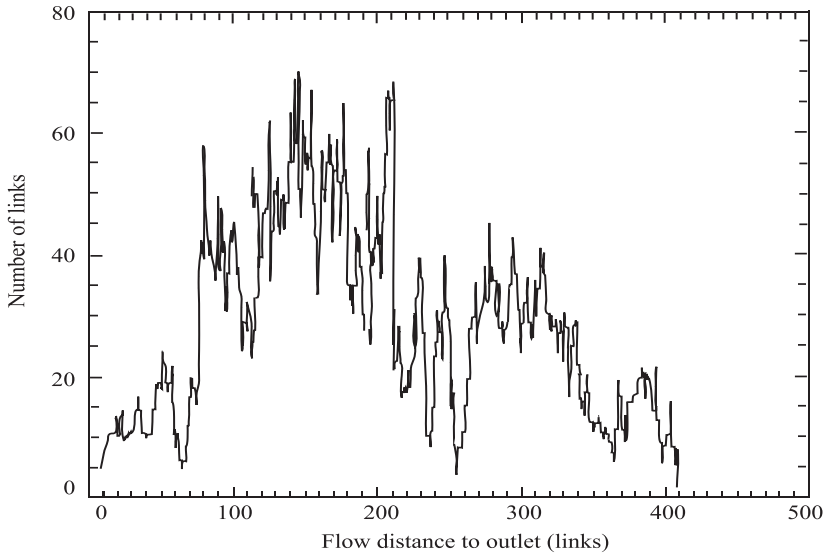


Figure 3.6: Sample width function for Kentucky river network

inter-relationships involved in the structure of these processes. The basic mathematical lesson that we wish to present is that a deeper and more unified understanding of of these quantities and their inter-relatedness may be obtained within the framework of this model by consideration of a naturally associated *Brownian excursion* process with the network. In addition to the unification, this approach also provides additional information on fluctuation laws which have not been accessible by the

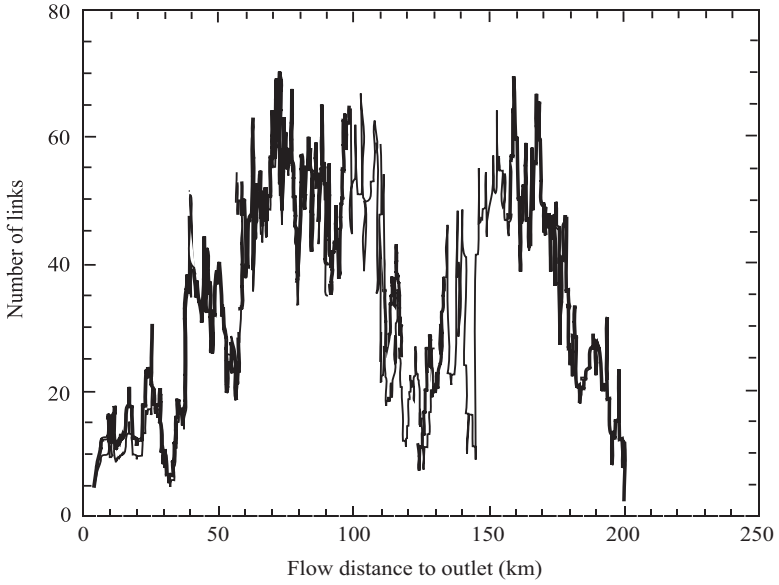


Figure 3.7: Sample width function for Kentucky river network

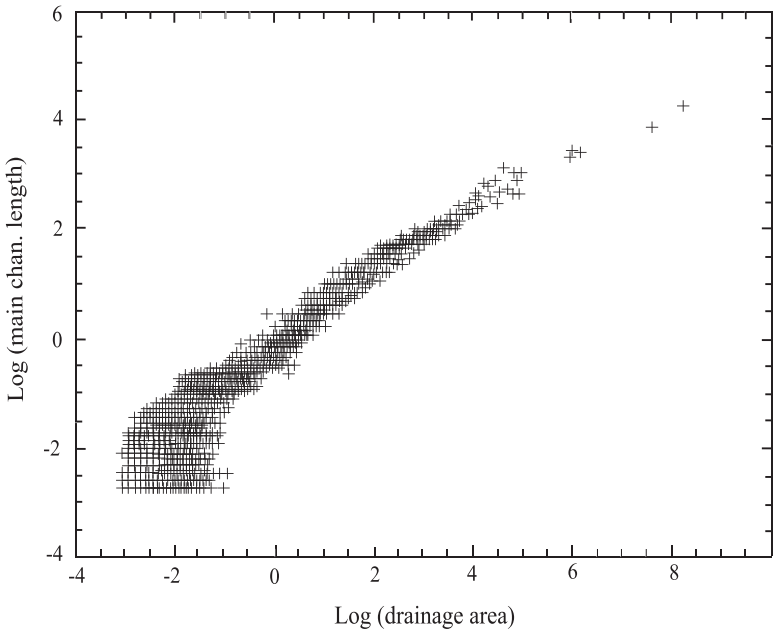


Figure 3.8: Sample mainchannel length versus area

analytic approaches.

Bienayme-Galton-Watson probability distributions (BGW) on the Borel sigma field $\mathcal{B}$ of T with single progenitor and offspring distribution $p_k, k = 0, 1, \ldots$ for which the probability assigned to a ball $B(\tau, \frac{1}{N}), \tau \in T_0, N \in \{1, 2, \ldots\}$ is

$$P(B(\tau, \frac{1}{N})) = \prod_{v \in \tau | (N-1)} p_{l(v)}, \tag{3.6}$$

where $l(v) = \#\{j :< v, j >\in \tau | N\}$. The *weighted BGW model* refers to a random field $\{W_{<e>}\}$ of positive weights independent of τ.

Here are some corresponding mathematical results which we wish to explain from the perspective of excursion theory.

Let $\mu_n(h)$ be the conditional expectation of the width function evaluated at h given total progeny $\nu = n$, i.e.:

$$\mu_n(h) = E[Z(h)|\nu = n]. \tag{3.7}$$

Let K_n be the normalization constant defined by

$$K_n = \int_0^\infty \mu_n(h) dh. \tag{3.8}$$

and define a probability measure F_n with density $K_n^{-1}\mu_n(h)$, suitably scaled. That is,

$$\frac{d}{dh} F_n(h) = a_n K_n^{-1} \mu_n(a_n h), \ h \geq 0 \tag{3.9}$$

where a_n is positive scale parameter. If we take $a_n = \sqrt{n}$ in (3.10), then

Theorem 3.1 *(Troutman and Karlinger (1984); Ossiander, Waymire, Zhang (1997)) Let τ_n be a conditioned Galton-Watson process whose offspring distribution L satisfies $EL = 1,\ 0 < VarL = \sigma^2 < \infty, gcd\{j : P(L = j) > 0\} = 1$. Suppose that the iid weights $\{W_{<e>}\}$, independent of τ_n, have mean one, variance s^2, and assume $\lim_{x\to\infty}(x \log x)^2 P(|W_\theta - 1| > x) = 0$. Then,*

$$F_n \Rightarrow F$$

where

$$F'(h) = \frac{h}{2} e^{-\frac{h^2}{4}},$$

is a Rayleigh density and " $\Rightarrow$ " denotes convergence in distribution.

For the longest channel one has the following structure.

Theorem 3.2 *(Gupta, Mesa, Waymire (1989), Durrett, Kesten, Waymire (1991)) Consider a critical weighted BGW model whose offspring distribution has finite second moment and with i.i.d. weights.*
CASE 1. If $y^2 P(W > y) \to 0$, as $y \to \infty$ then for a suitable constant $c > 0$

$$P(cn^{-\frac{1}{2}} h(\tau) \le h | \nu(\tau) = n) \to \sum_{k=-\infty}^{\infty} (1 - (kh)^2) e^{-k^2 h^2}.$$

CASE 2.If $P(W > y) \sim y^{-\alpha} L(y) \quad (y \to \infty)$ for L a slowly varying function and $0 \le \alpha < 2$ then for scaling coefficients c_n such that $n c_n^{-\alpha} L(c_n) \sim 1$ as $n \to \infty$,

$$P(c_n h(\tau) \le h | \nu(\tau) = n) \to e^{h^{-\alpha}}.$$

Remark Kesten (1994) also calculated the asymptotic distribution conditioned on both total progeny n and the unweighted height k_n, such that $\frac{k_n^2}{n}$ is bounded away from 0 and ∞ and the weights have finite fourth moment. Kesten shows that the centered weighted height rescaled by $n^{-\frac{1}{4}}$ is asymptotically Gaussian. As Kesten notes, this shows that most of the fluctuation in the weighted height is due to fluctuations in the unweighted height in this case.

For network self-similarity one has the following.

Proposition 3.1 *(Shreve (1967), Burd, Winn, Waymire (1999)) Suppose τ_n is a critical binary Galton-Watson process. Let T_{ij} denote the number of order j subnetworks in a randomly selected order i stream. Then*

$$ET_{ij} = \frac{1}{2} 2^{i-j}$$

In particular the expected BGW critical tree is self-similar. A natural extension to a form of *stochastic self-similarity* was introduced in Burd, Waymire, and Winn (1999) as follows. Define a map π on the subset T_0 of finite trees in T by $\pi(\{\theta\}) = \theta$, else $\pi(\tau)$ is the tree graph obtained by pruning the lowest order streams from τ. Also define $\overline{\pi}(\tau)$ as the tree graph obtained from $\pi(\tau)$ by identifying adjacent vertices of degree one or two with a single vertex. Then the order $\omega(\tau)$ of the tree may be expressed as

$$\omega(\tau) = \inf\{n : \pi^{(n-1)}(\tau) = \{\theta\}\} \tag{3.15}$$

Remark. We refer to the invariance under the composite map $\overline{\pi}$ of the distribution of a finite random tree τ, conditional on $Z_0 > 0$, as *stochastic self-similarity*. The following result shows that the well-studied random model is the only such model within the branching process framework.

Theorem 3.3 *(Burd et al,1999) A BGW model is stochastically self-similar if and only if it has a critical binary offspring distribution.*

In order to explain the above results within the context of Brownian excursions consider the so-called *search-depth* walk defined by the polygonal path process following the contour of the tree as follows. If τ_n is a labelled rooted tree with n vertices then define $v_0(\tau_n) = \theta$ and, given $v_k(\tau_n) =< i_1, \ldots, i_m >$, define $v_{k+1}(\tau_n) =< i_1, \ldots, i_m, j >$, where $j = \min\{i : i_1, \ldots, i_m, i > \in \tau_n$ and $< i_1, \ldots, i_m, i > \neq v_l(\tau_n) \forall l \leq k+1\}$, provided the latter set is non-empty, in which case $v_{k+1}(\tau_n) =< i_1, \ldots, i_{m-1} >$. The *unit search-depth* walk is then defined by

$$S_k(\tau_n) = |v_k(\tau_n)|, \ k = 0, 1, \ldots, 2n, \tag{3.10}$$

for the height function $|\cdot|$defined earlier, with linear interpolation between values to define $\{S_t(\tau_n) : 0 \leq t \leq 1\}$; see Figure 3.9.

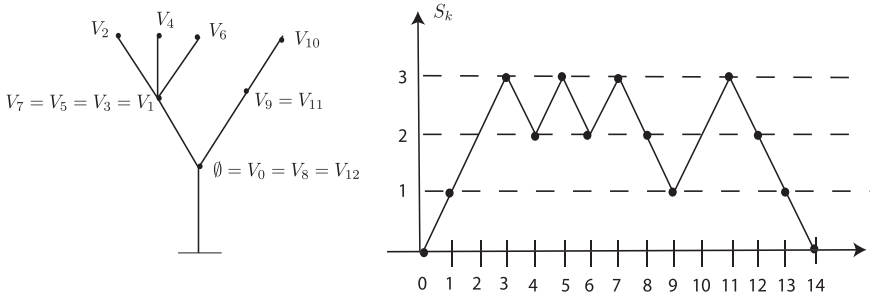


Figure 3.9: Standard search-depth walk corresponding to τ_n

Proposition 3.2 *Let τ_n be distributed as a Galton-Watson tree conditioned to have total progeny n. Then $\{S_k(\tau_n)\}_{k=0}^{2n-1}$ is distributed as a simple random walk conditioned to be positive over $k = 0, \ldots, 2n-2$ and hit zero for the first time at $k = 2n-1$ if and only if the offspring distribution is geometric.*

We refer to the conditioned random walk in Propositon 3.2 as a *random walk excursion*. It follows from results of Kaigh (1976) and/or Durrett and Iglehart (1976) that the random-walk excursion suitably scaled converges to the continuous state *Brownian excursion* $\{B^+(t) : 0 \le t \le 1\}$. The Brownian excursion may be defined, up to a reparameterization of time, as a standard Brownian motion starting at 0 on the time interval $[\tau_1, \tau_2]$, where τ_1 is the time of the last zero before time $t = 1$ and τ_2 is the time of the first zero after time $t = 1$. The following recent result of Aldous (1993) fills the gap for more general offspring distributions.

Theorem 3.4 *Let τ_n be a conditioned Galton-Watson process whose offspring distribution L satisfies $EL = 1$, $0 < Var(L) = \sigma^2 < \infty$, $gcd\{j : P(L = j) > 0\} = 1$. Let $S_t^{(n)}(\tau_n)$ denote the search-depth process in (2.2) with edges scaled by $\frac{\sigma}{\sqrt{n}}$. Then $\{S_t^{(n)}(\tau_n) : 0 \le t \le 1\} \Rightarrow \{2B^+(t) : 0 \le t \le 1\}$.*

For the applications of interest to us here we require an extension of Aldous' theorem to weighted trees given in Ossiander et al (1997) as follows.

Theorem 3.5 *Let τ_n be a conditioned Galton-Watson process whose offspring distribution L satisfies $EL = 1$, $0 < VarL = \sigma^2 < \infty, gcd\{j : P(L = j) > 0\} = 1$. Suppose that the iid weights $\{W_{<e>}\}$, independent of τ_n, have mean one, variance s^2, and assume $\lim_{x\to\infty}(x \log x)^2 P(|W_\theta - 1| > x) = 0$. Then the weighted search-depth process $\{\hat{S}_{t/2n}^{(n)}(\tau_n)\}$, defined by weights $\frac{\sigma \cdot W_{<e>}}{\sqrt{n}}$, converges in distribution to $\{2B_t^+ : 0 \le t \le 1\}$.*

Let us now consider the network statistics from the perspective of this theory.

Since clearly tree height is a continuous functional one gets weak convergence to the height of the Brownian excursion immediately from the above theorem of Ossiander et al (1997). Apart from the log factor, the conditions in Ossiander et al (1997) on the tails of the weight distribution are nearly best possible. However in the special case of this particular functional Durrett, Kesten, and Waymire (1991) is a slightly sharper result.

Next observe that the (weighted) width function may be approximated by the *local time* of the weighted search-depth process as follows. That is

$$\hat{S}_k(\tau_n) = h_w(v_k(\tau_n)), \tag{3.11}$$

with linear interpolation between values to define the weighted search depth process $\{\hat{S}_t : t \ge 0\}$; see Figure 3.10.

Now, for a Borel subset J of $\mathbf{R}$, define the (total) *search-depth occupation time* by

$$\Gamma(J) = \frac{1}{2}\int_0^1 I_J(\hat{S}_t)dt \tag{3.12}$$

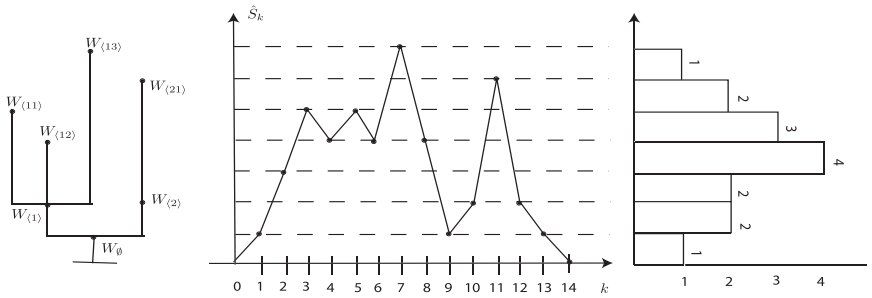


Figure 3.10: Weighted search-depth process corresponding to τ_n

The *search-depth local time* is then defined as the Radon-Nikodym derivative (with respect to Lebesgue measure λ on $\mathbf{R}$) accordingly as

$$\gamma(h) = \frac{d\Gamma}{d\lambda}(h) \tag{3.13}$$

Then observe that in the case of unit weights

$$\gamma(h) = \frac{1}{2\sqrt{n}} Z(\sqrt{n}h), \tag{3.14}$$

while under the conditions of Theorem 3.5, Ossiander et al (1997) show that $Z(h)$ and $\gamma(h)$ suitably scaled differ by a $o(1)$ term in probability.

So, to obtain the result of Ossiander et al (1997) showing that the fluctuation law for the weighted width function, viewed as a random measure, follows that of the occupation time for the Brownian excursion in the large total progeny limit one uses the fact that the Brownian excursion has a density to show that the occupation time is a.s. continuous with respect to the Brownian excursion and therefore convergence in distribution follows from Theorem 3.5 as follows.

Theorem 3.6 *Under the conditions in Theorem 3.5, let $\gamma^{(n)}(h)$ denote the local time density for the weighted search-depth process $\hat{S}_t^{(n)}$ having weights rescaled by $\frac{\sigma}{\sqrt{n}}$. Then*

$$\int_a^b \frac{\gamma^{(n)}(\sqrt{n}h)}{\sqrt{n}} dh \Rightarrow \int_0^1 I\left(\frac{a}{\sqrt{2}} \le B^+(t) < \frac{b}{\sqrt{2}}\right) dt$$

for any $0 < a < b$, where $\{B^+(t) : 0 \le t \le 1\}$ is Brownian excursion.

Remark. Theorem 3.6 also provides a weak form of a conjecture of Aldous (Conjecture IV, 1991). The more difficult problem is that of proving weak convergence of the local time processes was recently obtained by Dromta and Gittenberger (1996) using generating function methods and by Kersting (personal communication) using random walk transformations, but the extension to weighted trees is open.

The expected value results noted above of Troutman and Karlinger (1984) will now follow as a corollary from the fluctuation law calculations using a computation by Chung (1976) of the expected occupation time of $\{B^+(t)\}$ contained in the following result.

Theorem 3.7 *Let*

$$S([a,b)) = \int_0^1 I(a \le B^+(t) < b)dt$$

for $0 < a \le b < \infty$. *Then*

$$ES([a,b)) = \int_a^b 4he^{-2h^2}dh.$$

Let us now turn to self-similarity properties of the random model. As suggested by Neveu (1986), these results may be anticipated by corresponding results on Brownian excursions of Neveu and Pitman (1989) as follows. Consider the excursions of a Brownian motion conditioned to reach a level a, referred to as an $a-$excursion from 0, defined as follows by Ito (1970). Fix $a > 0$ and let $\{B_t^+ : 0 \le t < \xi\}$ be the portion of the Brownian motion started at 0 after the last zero before the first time to reach a, until it returns to zero; see Figure 3.11.

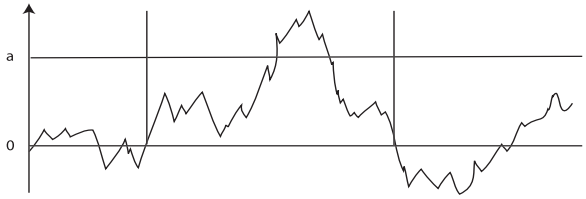


Figure 3.11: a-excursion sample paths

Then Neveu and Pitman (1989) show that the counting process $\{N_x^{(a)} : x \ge 0\}$ of the number of $a-$excursions (upcrossings from x to $x+a$) in the Brownian motion started at 0 is a continuous parameter critical Galton-Watson binary branching process with rate $\frac{1}{a}$. The corresponding embedded tree is depicted in Figure 3.12. In particular the leaves of the tree, denoted $\bullet$, correspond to local $a-$maxima, and branch points $\circ$ are local $a-$minima as defined below. From here one may deduce that the tree structure should be invariant under erasure of the tips of the tree at some unit rate. That this is in fact true is given in Neveu (1986) for continuous parameter Galton-Watson and in Burd, Waymire, Winn (1999) for discrete trees.

Remark. Lest one think all things are possible for this model via excursion theory, the following result does not seem amenable to this approach. Namely, using analytic/combinatoric methods Theorem 3.1 was extended to allow for certain weight distributions which depend on location through size of the drainage network in Waymire (1992). In particular one has

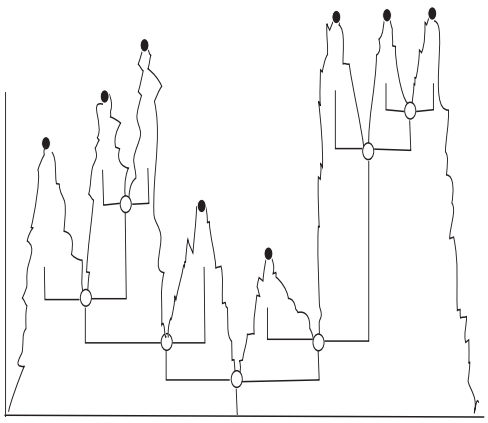


Figure 3.12: Tree embeddings in excursions

Theorem 3.8 *(Waymire, (1992)) Suppose that τ is a random treee distributed as a BGW with binary offspring distribution. Suppose that the weights are independently distributed such that the weight at a vertex of a subtree of size m has an exponential distribution with mean $e^{-(m-1)\theta}\mu, \theta > 0$. Then $F_n \Rightarrow F$ where F is uniquely determined by its moments given by*

$$\int_0^\infty h^k F(dh) = \frac{\mu^k e^{k\theta} \hat{r}(\frac{1}{4}e^{-(k+1)\theta})k!}{\hat{r}(\frac{1}{4}e^{-\theta}\sqrt{\prod_{j=1}^k (1-e^{-j\theta})})},$$

where

$$r(s) = \frac{1}{2}(1 - \sqrt{1-4s}).$$

The decay of the mean in Theorem 3.8 roughly corresponds to the *concave shape* of river basins in which the weights represent elevation drops and the larger the drainage area supported by a vertex the closer to the outlet and the smaller the elevation drop in an edge; i.e. a cheap way to incorporate "geometry" without spatial embedding. A special case of interest for which there are no precise results is that in which the exponentially decaying mean is replaced by a power law decay of the from $m^{-\theta}\mu$. In fact, data analysis in Gupta and Waymire (1989) suggests this power law form.

Let us now turn to other network models. The interest in main channel length is inspired by work of Hack (1957) on the behavior of the length of the main channel as a function of basin size. Hack's law is a set of empirical observations which reports the growth of the main channel to be $O(Area)^\alpha$ with $\alpha \approx .7$. A model which has been shown to naturally exhibit this behavior is the *coalescing random walk* Scheidegger (1967), Ngyuen (1990); see Troutman and Karlinger (1998) for a general perspective on these models. The basic idea is to consider the tree graph obtained by consideration of a system of coalescing simple random walks $S_k^{(0)}$ on a

triangular lattice initiated at a level L and conditioned to reach the orign; see Figure 3.13. The basin size A may be represented as the area between two independent (boundary)random walks, the difference of which is a random walk. That is A is the area under a random walk starting at 0 and conditioned to return to 0 in L steps. That is,

$$A = \sum_{n=0}^{L} S_n^{(0)} = L^{\frac{3}{2}} \sum_{n=0}^{L} \frac{S_k^{(0)}}{\sqrt{L}} \cdot \frac{1}{L} \sim L^{\frac{3}{2}} \int_0^1 B_t^{(0)} dt. \tag{3.12}$$

In particular, therefore, $L \sim A^{\frac{2}{3}}$ as $L \to \infty$. This and related models will be discussed in the research lecture by D. Winn.

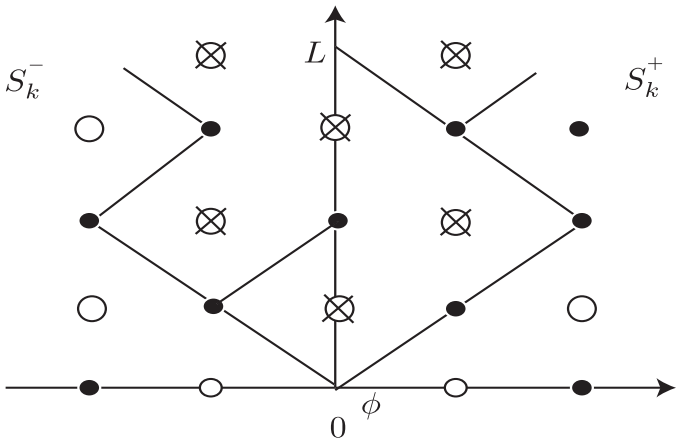


Figure 3.13: Coalescing random walks

Another special class of deterministic trees, referred to as *Recursive Replacement Trees,* may be constructed as *iterated function systems* in the plane as defined in Barnsley (1988), for example. For this let F be a closed subset of $\mathbf{R}^k$ and recall that a map $s : F \to F$ is called a *contraction transformation* if there is a number $0 < \alpha < 1$ such that $|s(x) - s(y)| \leq \alpha |x - y|, x, y \in F$. In the case of equality then the map is called a *similarity.* The set F is said to be *invariant* under a given set $s_1, \ldots, s_N$ of contractions on F if $F = \cup_{i=1}^N s_i(F)$ and the collection $s_1, \ldots, s_N$ is called an *iterated function system (IFS)* for F. The usual Cantor set is a familiar example of an invariant subset of the number line with similarities $s_1(x) = \frac{x}{3}, s_2(x) = \frac{x}{3} + \frac{2}{3}$, i.e. $\alpha_1 = \alpha_2 = \frac{1}{3}$ in this example. The following theorem is a standard result which provides the foundation for the trees considered here.

Theorem 3.9 *Let $s_1, \ldots, s_N$ be a family of contractions on a closed subset K of $\mathbf{R}^d$. Then there is a unique invariant compact subset $F \subseteq K$ for this iterated function system. Moreover,*

$$F = \cap_{j=1}^{\infty} s^j(K)$$

for any nonempty compact set K for which $s_i(K) \subseteq K, i = 1, \ldots, N$, where $s^1(K) = s(K) := \cup_{i=1}^N s_i(K)$ and $s^j(K) := s^{j-1}(s(K)), j = 2, 3, \ldots$.

A convenient method of defining similarities introduced by Mandelbrot (1982) is by specification of an *IFS-generator* G consisting of a number of straight-line segments and two specified points such that similarity maps $s_1, \ldots, s_N$ are associated with each line segment which map the two points onto the endpoints of the segment. Using the IFS-generator one obtains a sequence of sets approximating the limiting invariant set by iterating the process of replacing each line segment by a similar copy of the generator. While the planar similitudes are defined by the IFS-generator only to within reflection, the orientations may be prescribed by showing the first iteration $s(G)$. We are interested in trees, referred to as *recursive replacement trees*, for which the IFS-generator is itself a finite *b-ary tree graph* embedded in the plane, say with n *vertices* of *degree* $b+1$ of the form given in Figure 3.14. In particular, this generator defines an IFS in the plane consisting of $N = nb + 1$ similarity maps with common similarity ratio $\alpha = \frac{1}{n+1}$. We will refer to

$$F^{(m)} = \cap_{j=1}^m s^j(G) \tag{3.16}$$

as the mth scale resolution of G.

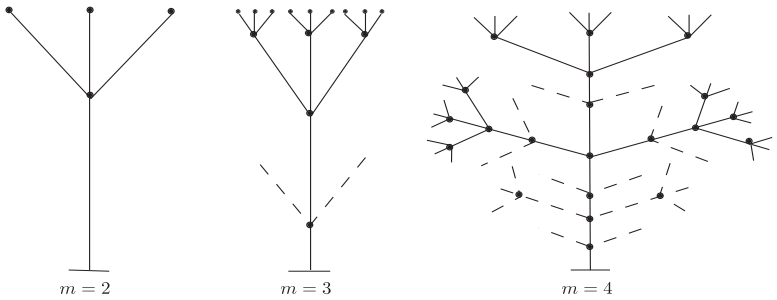


Figure 3.14: b-ary tree generator

The following theorem is a special case of much more general results known in geometric measure theory; Falconer (1990).

Theorem 3.10 *Suppose that F is a resursive replacement tree in the plane. Then F has finite positive Hausdorff measure. Moreover, the Hausdorff and box dimensions of F coincide and are given by the solution θ_c to*

$$\sum_{i=1}^N \alpha_i^{\theta_c} = 1.$$

As a Corollary to Theorem3.10 one readily obtains the following formula first argued along other lines by Peckham (1995).

Theorem 3.11 *The Hausdorff and box dimension of the recursive replacement tree is given by*

$$dim = \frac{\log(nb+1)}{\log(n+1)}.$$

Peckham (1995) has computed the generators of recursive replacement trees as follows: Write $T_k = T_{i,i-k}$.

$$T_1 = (b-1)(n-1), \quad T_k = (b-1)n^2(n+1)^{k-2}, \ k \geq 2. \tag{3.17}$$

The recursive structure of an SST gives rise to a recursion equation for the number N_j of streams of order j in the network from which Peckham (1995) also obtains the following result.

Theorem 3.12

$$\lim_{\omega \to \infty} \frac{N_\omega}{N_{\omega+1}} = R_b = nb + 1$$

where R_b is the bifurcation ratio defined above.

Let us consider the width function asympototics for two classes of special deterministic self-similar trees, the so called *Peano trees* and *uniform b-ary trees*. Note that two trees can have the same generators and still be different, owing to two freedoms in adding the edges.

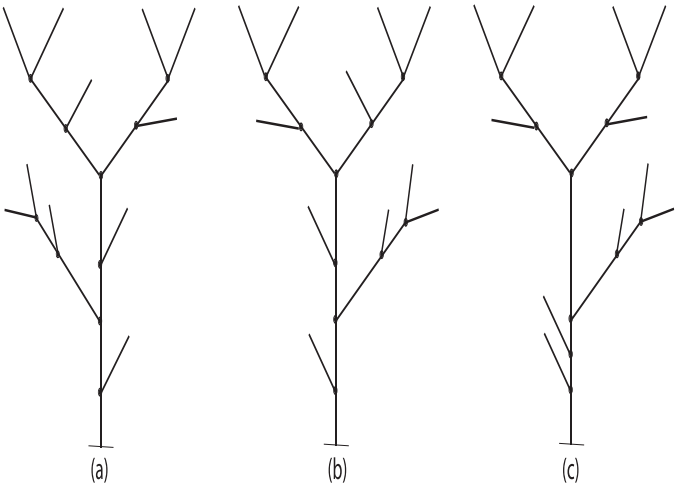


Figure 3.15: Self similar trees: $m = 3, \ b = 2, \ T_k = T_{i,i-k} = 2^{k-1}, k \geq 1$

The Peano tree is represented by a class of self-similar trees with branching number b = 3 and generators $\{T_1 = 0, T_k = 2^{k-1} : k = 2, 3, \ldots\}$. Figure 3.14 provides examples of the Peano trees with order 2, 3, and 4. The dashed lines are subtrees which are put in according to the generators.

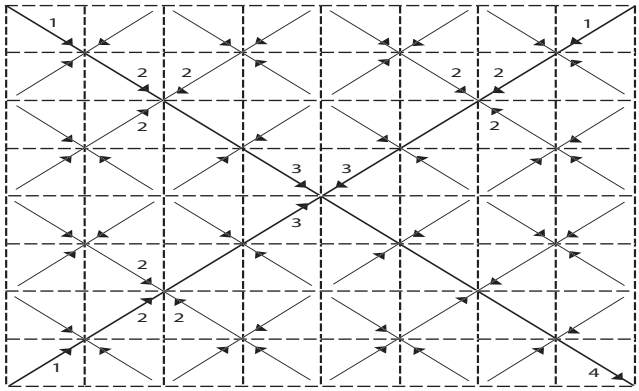


Figure 3.16: Order $m = 3$ basin partition by a Peano tree with $b = 3$

These trees are associated with a partition of a rectangular basin depicted in Figure 3.16.

Let $Z_{m,k}$ be the numbers of edges of the kth generation of a Peano tree of order m. The width functions of a self-similar tree of order m and generator $\{T_1 = 0, T_k = 2^{k-1}; k = 1, 2, \dots\}$ is

$$Z_m(h) = \sum_{k=0}^{2^{m-1}-1} Z_{m,k+1} \cdot I(k \le h < k+1), \tag{3.20}$$

where $m = 2, 3, \dots$. A simple induction argument shows that a Peano tree of order m has height 2^{m-1} and total progeny 4^{m-1}. We define a normalized width density f_m as

$$f_m(h) = \frac{Z_m(h)}{N_m}, \tag{3.21}$$

where $h \in [0, 2^{m-1})$ and $N_m = 4^{m-1}$. Then f_m defines a measure μ_m with distribution function

$$\mu_m[0, h] = \int_0^{2^{m-1}h} f_m(y)\, dy = \int_0^h 2^{m-1} f_m(2^{m-1}y)\, dy \tag{3.22}$$

for $h \in [0, 1)$.

At the initial scale of resolution on a unit segment network the width function is the unit mass distributed uniformly over [0,1]. At the first stage this channel is subdivided into half, and two side tributaries are added at the node in the middle. The width function can be written as,

$$f_1(h) = \begin{cases} 1/4 & \text{if } 1 \le h \le 1/2, \\ 3/4 & \text{if } 1/2 < h \le 1. \end{cases} \tag{3.13}$$

At the second stage of construction, each link of length 1/2 is subdivided into half, and two side tributaries are added at each node in the middle. The width

function can be expressed as,

$$f_2(h) = \begin{cases} 1/16 & \text{if } 1 \le h \le 1/4, \\ 3/16 & \text{if } 1/4 < h \le 1/2 \\ 3/16 & \text{if } 1/2 < h \le 3/4 \\ 9/16 & \text{if } 3/4 < h \le 1. \end{cases}$$

The following theorem formalizes observations of Marani et al (1991); see Ossiander et al (1997) for precise details.

Theorem 3.13 *μ_m converges weakly to a continuous singular probability measure on $[0,1]$ as $m \to \infty$, namely the induced infinite product measure $(\frac{1}{4}\delta_0 + \frac{3}{4}\delta_1)^N$ under the map $\phi(x) = (x_1, x_2, \dots), x = \sum_{i=1}^{\infty} x_i 2^{-i}, x \in [0,1]$.*

More generally, the second class of problems which we consider is that of computing the width function for the class of deterministic self-similar trees defined by generators $\{T_k(b) = (b-1)2^{k-1}$, k=1, 2, $\dots\}$. We set the problem up in this section but provide the main result in the next section.
The *uniform b-ary self-similar trees* are defined as the trees having generators $\{T_k(b) = (b-1)2^{k-1}, k = 1, 2, \dots\}$. The recursive equations of $\{Z_{m,k}, k = 1, 2, \dots, 2^m - 1\}$ are:

$$Z_{m,1} = 1 \tag{3.25a}$$

$$Z_{m,j+1} = bZ_{m-1,j} \tag{3.25b}$$

$$Z_{m,j+2^{n-1}} = bZ_{m-1,j}, \tag{3.25c}$$

for $j = 1, 2, \dots, 2^{m-1} - 1$. The total progeny is

$$N_m = 2bN_{m-1} + 1 = (2b)^{m-1} + (2b)^{m-2} + \cdots + 1 = \frac{(2b)^m - 1}{2b - 1}, \tag{3.26}$$

where $b = 2, 3, \dots$, $n = 2, 3, \dots$, and $Z_{1,1} = N_1 = 1$.

The width functions of a b-ary tree of given order, as a normalized probabilty measure, converge weakly to a uniform distribution function over [0, 1]. Let us now consider the width function of the uniform b-ary trees of order ω with generator $\{T_k(b) = (b-1)2^{k-1} k = 1, 2, \dots\}$. In particular, let

$$Z_\omega(x) = \sum_{k=0}^{2^\omega - 2} Z_{\omega,k+1} \cdot I(k \le x < k+1), \tag{3.25}$$

where $\omega = 2, 3, \dots$ and let

$$f_\omega(x) = \frac{Z_\omega(x)}{N_\omega}. \tag{3.27}$$

The height of b-ary tree of order ω is given by $H_\omega = 2^\omega - 1$, so that the distribution function is

$$F_\omega(x) = \int_0^{H_\omega x} f_\omega(y)\, dy = \int_0^{H_\omega x} \frac{Z_\omega(y)}{N_\omega}\, dy = \int_0^x \frac{Z_\omega(H_\omega y)}{\frac{N_\omega}{H_\omega}} dy. \tag{3.28}$$

If we let F_ω denote the distribution function defined by the density function

$$\frac{Z_\omega(H_\omega x)}{\frac{N_\omega}{H_\omega}}, \tag{3.29}$$

then we have the following theorem.

Theorem 3.14 *F_ω converges weakly to the uniform distribution on $[0,1]$ as $\omega \to \infty$.*

The following theorem of Troutman and Karlinger (1984) is of interest for comparison of these results to expected behavior of critical Galton-Watson trees.

Theorem 3.15 *Let $\tau_{n,\omega}$ be distributed as a Galton-Watson binary branching process conditioned to have order ω and total progency n. Then, the expected width function converges weakly to a uniform distribution.*

A randomization of the recursive replacement trees was introduced by Veitzer (1999), Veitzer and Gupta (2000), referred to as *random self-similar networks* (RSN), which may be viewed as a random iteration of the similarity maps in an IFS as follows. The similarity maps are pairs $(s_i, t_i), i = 1, 2 \ldots N$ possibly acting differently on segments designated interior or exterior. For example, consider the RSN which is obtained by replacing interior edges by the generator in Figure 3.17a with probability p, or Figure 3.17b with probability $q = 1-p$. Similarly, exterior edges are replaced by the generators in Figure c with probability p or those in Figure 3.17c with probability $q = 1 - p$. Veitzer and Gupta (2000) observe that the number of edges $C_m, m = 1, 2, \ldots$ in a stream of order m evolves according to a supercritical Galton-Watson branching process. This, for example, leads to a Horton law of stream numbers by application of the martingale convergence theorem to C_ω / EC_ω as $\omega \to \infty$.

4 Multiplicative Cascades: Statistical Rainfall and Turbulence Models

During the 1950's and 1960's a central theme of rainfall research in hydrology was to fit parameters of various time series models to point rainfall measurements on time scales ranging from hourly, daily, monthly and yearly. Examples of time series models include mth order wet/dry Markov chain models, renewal sequences, moving average models; see Katz (1985), Waymire and Gupta (1981). As attention turned to more physically based approaches to take into account observed clustering in

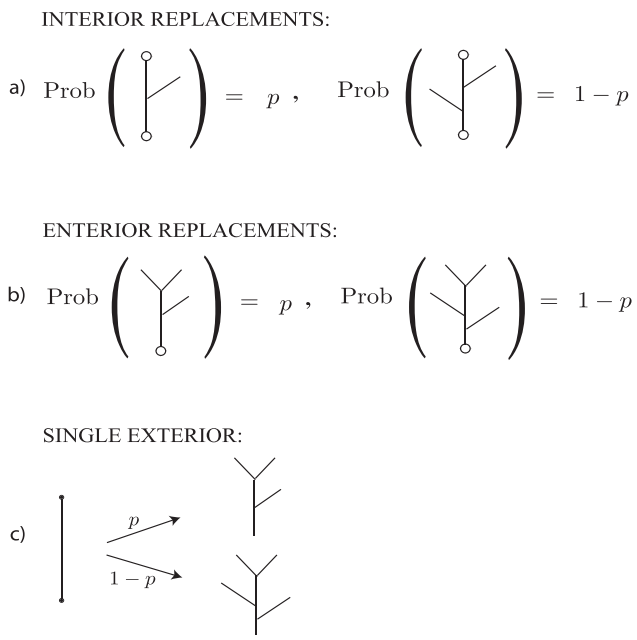


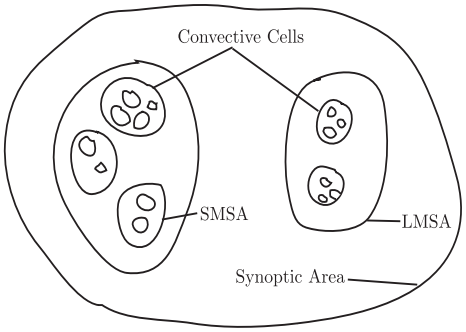
Figure 3.17: Random self-similar network generators

space and time, this theme was further explored into the following decades using compound Poisson and compound Neyman-Scott time series models; see Rodriguez-Iturbe (1986). The lesson learned was that up to second order, moment characteristics could be "reasonably well reproduced by a variety of models at a given scale". Better understanding of the temporal evolution would require the physical structure of storm events which is furnished by spatial observations.

The hierarchical structure of spatial rainfall fields takes the form of clusters of high intensity rain cells embedded in clusters of lower intensity regions, called small mesoscale areas (SMSA), which are in turn embedded in rainbands of identifiably lower intensity, called large mesoscale areas (LMSA), embedded in a still larger scale synoptic rain area of lower rainrate. This structure is supported by radar and raingage observations of the type analyzed by Austin and Houze (1972); see Figure 4.1 for a schematic.

While this structure is the supposed consequence of combined effects of vertical and horizontal motions, the precise dynamics of rainfall formation are not available. As a rule of thumb, the (possibly artificial) scales of these regions decrease by successive factors of $\frac{1}{10}$ from the synoptic scale through LMSA, SMSA, and down to a cell, while the corresponding rainrates nearly double at each level until the scale of a cell where this rule generally breaks down; supercells are possible where the rainrate may be larger than the SMSA by several orders of magnitude.

One of the earliest studies of the spatial and temporal variability of rainfall was that of LeCam (1961) based on spatial cluster point processes and random mea-



FEATURE	RAIN RATE
Synoptic ($< 10^4$ km^2)	
LMSA (10^3 to 10^4 km^2)	Doubling Rule
SMSA (50 to 10^3 km^2)	
CELL (< 50 km^2)	Factor of 10 possible

Figure 4.1: Schematic of spatial rainfall hierarchy

sures of the type also occuring in the study of the clustering of galaxies, earthquake aftershock sequences, population growth, etc. A substantial body of research has evolved over the past decade involving various problems associated with this approach to describing rainfall. Among these is a reasonably accurate computation of spatial/temporal correlation structure down to the scales of cells; see Zawadzki (1973), Waymire, Gupta, Rodriguez-Iturbe (1984), Rodriguez-Iturbe, Cox, Isham (1987), Phelan and Goodall (1990), Bell (1987), Smith and Karr (1985). However, the lower scale high intensity regions have not been adequately represented in this approach. Of course these high intensity localized rainfall events cannot be neglected from the point of view of floods.

It is interesting for us to note that the development of new mathematical/statistical methods to represent space-time rainfall phenomena in LeCam (1961) was also motivated by hydrologic flows. Namely, quoting from LeCam (1961): *"The problems encountered by this group included the evaluation of probabilities of excessive discharges, the evaluation of probabilities of excessive droughts, as well as the development of optimal management procedures for the big and small hydroelectric reservoirs. As the studies progressed, the need for a mathematically tractable description of the random structure of stream flow became more and more imperative. To obtain such a description it was found necessary to start with a description of the random structure of rainfall."* This is not a new problem but it continues to be a basic motivation for research in surface water hydrology.

Scaling notions for random fields are prominent in the probability and mathe-

matical physics literature largely motivated by symmetries in critical phenomena. A random field $\{R(x) : x \in \mathbf{R}^k\}$ is said to be *simple scaling* if

$$\{R(\lambda x)\} = \{\lambda^\theta R(x)\} \tag{4.1}$$

where equality is in the sense of joint distributions. The *scaling exponent* θ is a real number parameter.

A familiar example of a simple scaling stochastic process ($k = 1$) is standard Brownian motion starting at 0, and more generally Levy stable processes and fractional Levy noises. Of course these are non-stationary processes. That there are no nontrivial stationary simple scaling processes with minimal sample path regularity may be cast as follows.

Proposition 4.1 *If $\{R(x) : x \in \mathbf{R}^d\}$ is a translation-invariant simple scaling random field having stochastically continuous sample paths then $\{R(x) : x \in \mathbf{R}^d\}$ is constant with probability one and $\theta = 0$.*

As a result of Proposition 4.1 one is led to considerations of generalized random fields (random linear functionals) as models for statistically homogeneous rainfall distributions; e.g. Gaussian white noise is homogeneous simple scaling with $\theta = \frac{d}{2}$. An important class of random linear functionals to represent such phenomena continue to be classes of singular random measures. In fact, scaling observations have led to new classes of multiplicative structures which in this respect are not unlike the cluster models initiated with LeCam (1961).

One approach to determining scaling structure in spatial rainfall is to consider the behavior of rainfall moments over regions $\lambda\Delta$ of area λ^k where simple scaling would imply

$$\log ER^h(\lambda\Delta) = h\theta \log \lambda + c_h. \tag{4.2}$$

In particular simple scaling translates into two properties:
i. log-log linearity between a specified moment and length scale.
ii. a linear change in slope $s(h) = \theta h$ of the line as a function of moment order.

The analysis of spatial rainfall data leads to a very interesting phenomenon. Namely, property (i) is preserved but the slope function $s(h)$ in (ii) is nonlinear; see Gupta and Waymire (1993), Gupta and Over (1994); see Figures 4.2a, 4.2b.

A second important characteristic of rainfall is it observed *intermittancy* structure in space and time. It is both the observed moment structure and intermittency properties which have led to the study of *cascade measures* of the type originating in the statistical theory of turbulence as models for rainfall distributions. Some speculate that it is in fact the atmospheric turbulence patterns which are being displayed and modelled as rainfall!

Remark Early versions of multiplicative cascades were introduced by Kolmogorov (1941, 1962) in the statistical theory of turbulence for use in modeling the redistribution of energy under a rapid stirring motion as a repeated random splitting of

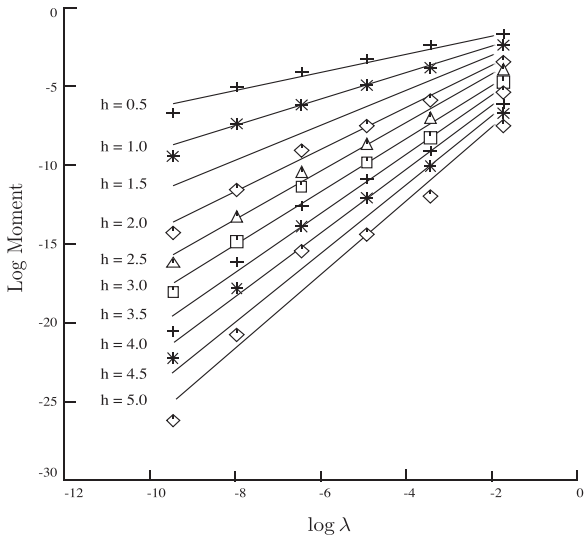


Figure 4.2a: Empirical multiscaling moment plots for rainfall

energy into finer scale eddies. These ideas followed earlier thinking of Richardson Refinements were developed further by Yaglom (1948) and by Mandelbrot (1974), and steps toward a rigorous mathematical foundation were initiated by Kahane and Peyrière (1976); also see Frisch (1998) in this context. Other highly variable and intermittent phenomena whose statistics appear to be reasonably represented by such models are internet traffic, financial markets, etc., eg. Gilbert et al (1999), Mandelbrot (1998), respectively. Moreover such models were also considered by Joffe, LeCam, and Neveu (1973) from a purely mathematical point of view. Our goal is to extend some of the existing statistical theory required for rigorous statistical tests and parameter estimations within this framework.

Before we consider random cascades in detail, let us consider a classical mathematical structure which illustrates the notion of a deterministic cascade and which already occured in our width function analysis of the preceeding section.

Example 4.1 (Deterministic Multinomial Cascades). Let $T = [0,1]$ and let $b \geq 2$ be an arbitrary but fixed integer. Let $p_0, p_1, \ldots, p_{b-1}$ be non-negative numbers such that $\sum_{i=0}^{b-1} p_i = 1$. For each $n = 1, 2, \ldots$, $t_1, \ldots, t_n \in \{0, 1, \ldots, b-1\}$ let $\{\Delta(t_1 \ldots t_n) \subseteq [0,1]\}$ denote the partitioning subinterval defined by

$$\Delta(t_1 \ldots t_n) := [\sum_{i=1}^{n} t_i b^{-i}, \sum_{i=1}^{n} t_i b^{-i} + b^{-n}]. \tag{4.1}$$

Then defining

$$\mu(\Delta(t_1 \ldots t_n)) = \prod_{j=0}^{b-1} p_j^{\gamma(j, t_1 \ldots t_n)}$$

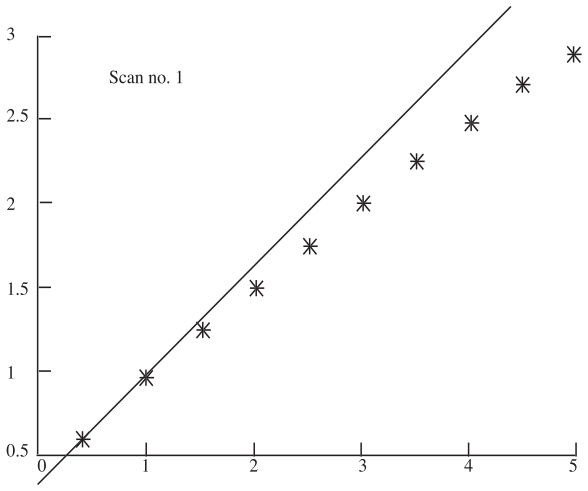


Figure 4.2b: Empirical multiscaling slope plots for rainfall

where

$$\gamma(j, t_1 \dots t_n) = \#\{i \le n : t_i = j\}, j = 0, 1, \dots, b-1,$$

uniquely determines a product measure μ on $[0,1]$ for the given parameters. This measure is referred to as a *deterministic multinomial cascade;* see Figure 4.3. Note that μ is the distribution of a point $t \in [0,1]$ with b-ary expansion $t = (t_1, t_2, \dots)$, i.e. $t = \sum_{i=1}^{\infty} t_i b^{-i}$, with coordinates $t_1, t_2, \dots$ selected independently according to the distribution $p_0, \dots, p_{b-1}$ on $\{0, 1, \dots, b-1\}$. In particular the Cantor distribution is obtained by taking $b = 3, p_0 = p_2 = \frac{1}{2}, p_1 = 0$.

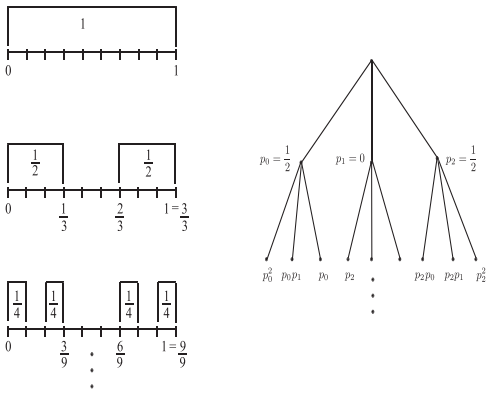


Figure 4.3: Multinomial cascade

From the point of view of dynamical systems μ may be viewed as an invariant measure on $[0,1]$ for the shift map $f(\sum_{i=1}^{\infty} t_i b^{-i}) = \sum_{i=1}^{\infty} t_{i+1} b^{-i}$. Also observe that the width function of the Peano and b-ary uniform measures computed in the previous section may also be viewed as deterministic multinomial cascades.

In the simplest models of random cascades one imagines repeated splittings of the unit interval into a number $b^n \geq 2$ subintervals of lengths $\frac{1}{b^n}, n = 1, 2, \ldots,$ such that the nth-stage mass of the interval $[\sum_{k=1}^n t_k b^{-k}, \sum_{k=1}^n t_k b^{-k} + b^{-(n+1)}), t_k \in \{0, 1, \ldots, b-1\}$ is given by $\mu_n([\sum_{k=1}^n t_k b^{-k}, \sum_{k=1}^n t_k b^{-k} + b^{-(n+1)})) = b^{-n} \prod_{k=1}^n W_{<t_1 \ldots t_k>}$, where $W_v, v \in \{0, 1, \ldots b-1\}^k, k \geq 1$ are i.i.d. non-negative mean one random variables; see Figure 4.4. The cascade measure μ_∞ is obtained as the a.s. vague limit of the sequence μ_n as $n \to \infty$.

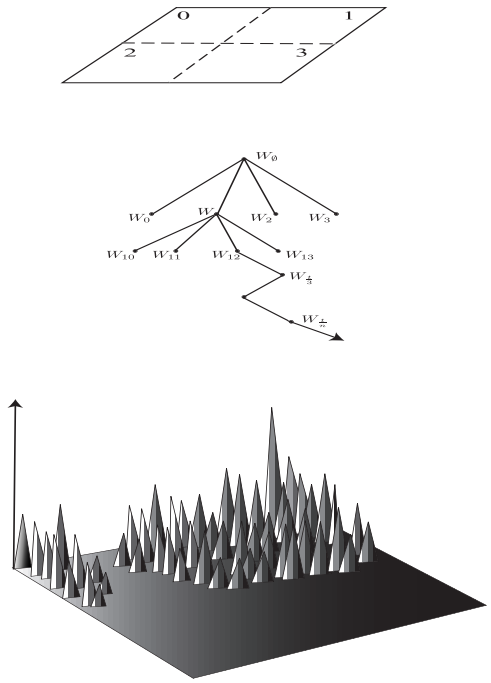


Figure 4.4: Multiplicative random cascade schematic

The main problem for applications is to infer the distribution of the random factors W_v from data on the random masses $\mu_\infty([\sum_{k=1}^n t_k b^{-k}, \sum_{k=1}^n t_k b^{-k} + b^{-(n+1)})), (t_1, \ldots, t_n) \in \{0, 1, \ldots, b-1\}^n$, at some prescribed *fine scale* b^{-n}. This problem will be addressed in the research talk by M. Ossiander.

Now to be more precise let $b \geq 2$ be a natural number and let $\mathbf{T}$ denote the product space

$$\mathbf{T} = \{0, 1, 2, \ldots, b-1\}^{\mathbf{N}} \tag{4.2}$$

equipped with the metric $\rho(s,t) = b^{-|s \wedge t|}, s, t \in \mathbf{T}$, where $|s \wedge t| = \inf\{n \geq 0 : s_{n+1} \neq t_{n+1}\}, s = (s_1, s_2, \ldots), t = (t_1, t_2, \ldots) \in \mathbf{T}$. $\mathcal{B}(\mathbf{T})$ will denote the corresponding Borel sigmafield on $\mathbf{T}$ for this metric.

For $t = (t_1, t_2, \ldots) \in \mathbf{T}$ let $t|n = (t_1, t_2, \ldots, t_n)$. If points $t \in \mathbf{T}$ are viewed as paths through a b-ary tree then $v = t|n$ denotes the nth generation vertex along t.

For $s \in \mathbf{T}, n \in \mathbf{N}$, let

$$\Delta_n(s) \equiv \Delta_n(s|n) = B_{b^{-n}}(s) = \{t \in \mathbf{T} : t_i = s_i, i \leq n\} \tag{4.3}$$

denote the closed ball of radius $r = b^{-n}$ centered at $s \in \mathbf{T}$. The normalized Haar measure μ on $\mathbf{T}$, viewed as a countable product of cyclic groups of order b, is specified by

$$\mu(\Delta_n(s)) = b^{-n}, s \in \mathbf{T}, n \geq 1. \tag{4.4}$$

Now let $\{W_v : v \in \{0, 1, \ldots, b-1\}^n, n \geq 1\}$ be a denumerable family of i.i.d. non-negative mean one random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Also let $\mathcal{F}_n, n \geq 1$, denote the filtration defined by

$$\mathcal{F}_n = \sigma\{W_v : |v| \leq n\}, n \geq 1, \tag{4.5}$$

where for $v = (v_1, v_2, \ldots, v_n), v_i \in \{0, 1, \ldots, b-1\}, n \geq 1, |v| = n$. The random variables W_v are referred to as the *cascade generators* and, as such, define a sequence of random measures μ_n on $(\mathbf{T}, \mathcal{B}(\mathbf{T})), n \geq 1$, via

$$\frac{d\mu_n}{d\mu}(t) = Q_n(t) = \prod_{i=0}^{n} W_{t|i}, t \in \mathbf{T} = W_\emptyset \prod_{i=1}^{n} W_{t|i}, t \in \mathbf{T}, \tag{4.6}$$

where $W_\emptyset$, referred to as the *cascade initiator,* is an a.s. positive random variable independent of $\mathcal{F}_n, n \geq 1$, and $W_{t|i}$ is the cascade generator at vertex $v = (t_1, \ldots, t_i)$.

It is well-known, e.g. see Kahane and Peyrière (1976), Joffe et al (1973), that there is a random measure μ_∞ on $(\mathbf{T}, \mathcal{B}(\mathbf{T}))$ such that

$$P(\mu_n \Rightarrow \mu_\infty \; as \; n \to \infty) = 1, \tag{4.7}$$

where, throughout, $\Rightarrow$ denotes vague convergence. In fact, for any countable family Φ of bounded Borel measurable functions, cf. Kahane (1989),

$$P[\lim_{n\to\infty} \int_{\mathbf{T}} f(t)\mu_n(dt) = \int_{\mathbf{T}} f(t)\mu_\infty(dt), \quad f \in \Phi] = 1. \tag{4.8}$$

The random measure μ_∞ defines the *multiplicative cascade.* The following basic structure theorem for μ_∞ is also well-known. First let

$$\chi_b(h) = \log_b \mathbf{E}[W^h \mathbf{1}[W > 0]] - (h-1), \tag{4.9}$$

where W is a generic cascade generator distributed as W_v for $v \neq \emptyset$. The function $\chi_b(h)$ is defined for all real numbers h but may be infinite, with the conventions that $0^0 = 0, 0 \cdot \infty = 0$.

Theorem 4.1 *(Kahane and Peyrière (1976)* $(i.)\mathbf{E}\mu_\infty(\mathbf{T}) > 0$ *iff* $\chi_b'(1-) < 0$. $(ii.)\mathbf{E}\mu_\infty^h(\mathbf{T}) < \infty$ *for* $0 \le h \le 1$, *and, if* $h_c := \sup\{h \ge 1 : \chi_b(h) \le 0\} > 1$, *then* $\mathbf{E}\mu_\infty^h(\mathbf{T}) < \infty$ *for* $1 < h < h_c$. $(iii.)dim(supp(\mu_\infty)) = (1 - \mathbf{E}W\log_b W) \wedge 0$.

An extension of this theorem allows for independent identically distributed random branching numbers at each generation and independent of the cascade generators. In particular, assuming the branching number distribution has moments of all orders $h \ge 1$, Peyrière (1977) notes without proof that the above theorem applies with b replaced by E[b]. This result was extended by Burd and Waymire (1999) assuming only a $EL\log L < \infty$ moment condition on the offspring distribution. Complete extensions of the Kahane-Peyrière theory have also been obtained in the case of certain types of dependent cascades (eg.Markov cascades) in Waymire and Williams (1996,1997). The basic idea for this is the *size-bias method* described below.

The theory of multiplicative random cascades is a special case of Kahane's positive T-martingales; eg. see Kahane (1991) and references therein, and from a mathematical point of view it is advantageous to view random cascades within this more general framework. For this, let T be a compact metric space with Borel sigmafield $\mathcal{B}$, and let $(\Omega, \mathcal{F}, P)$ be a probability space together with an increasing sequence $\mathcal{F}_n, n = 1, 2, \ldots$ of sub-sigmafields of $\mathcal{F}$.

Definition 4.1 A *positive T-martingale* is a sequence $\{Q_n\}$ of non-negative functions on $T \times \Omega$ such that (i.) For each $t \in T, \{Q_n(t,\cdot) : n = 0, 1, \ldots\}$ is a martingale adapted to $\mathcal{F}_n, n = 1, 2, \ldots$; (ii.) For $P - a.s.\omega \in \Omega, \{Q_n(\cdot,\omega) : n = 0, 1, \ldots\}$ is a sequence of Borel measurable non-negative real-valued functions on T.

Given $\{Q_n\}$ and a measure σ in the space $M^+(T)$ of finite measures on T define

$$Q_n\sigma(B) = \sigma_n(B) = \int_B Q_n(t)\sigma(dt), B \in \mathcal{B}. \tag{4.3}$$

Then essentially by the martingale convergence theorem one obtains $Q_\infty\sigma = \sigma_\infty = \lim Q_n\sigma$ a.s. in the sense of vague convergence. That is,

$$\lim_{n\to\infty} \int_T f(t)Q_n\sigma(dt) = \int_T f(t)Q_\infty\sigma(dt) f \in C(T), \tag{4.4}$$

where $C(T)$ is the space of (bounded) continuous real-valued functions on T. As suggested by the notation, it is the intention to view Q_∞ as a random operator acting on $M^+(T)$. Let $q(t) := EQ_n(t), t \in T$. While typically one has *mean conservation* in the form of $q(t) \equiv 1$, more generality is essential to the mathematical theory as indicated by the following basic proposition of Kahane (1987).

Theorem 4.2 *Let $\{Q_n\}$ be a positive T-martingale. Then corresponding to $\sigma \in M^+(T)$ such that $L_1(\sigma) \ni q$, there is a unique decomposition of $\{Q_n\}$ as a sum of two positive T-martingales $\{Q'_n\}$ $\{Q''_n\}$, such that for each $t \in T$*

$$Q_n(t) = Q'_n(t) + Q''_n(t)$$

$$[Q'_\infty lives\ on \sigma] : EQ'_\infty \sigma(B) = q'\sigma(B), B \in \mathcal{B}, q' := EQ'_n,$$

$$[Q''_\infty dies\ on \sigma] : Q''_\infty \sigma = 0\ a.s.$$

Remark. Theorem 4.2 makes a notion of cascade survival precise and is a major tool for calculations; see Waymire and Williams (1996). In particular we say that the *cascade survives* on (nonzero) σ iff it has a nontrivial living part Q'_∞ with positive probability.

Example 4.1(Independent Bernoulli Zero/Non-Zero Generators) The *β-model* is an important cascade defined by independent cascade generators distributed as

$$W = \begin{cases} \frac{1}{p} = b^\beta & \text{with probability } p = b^{-\beta} \\ 0 & \text{with probability } q = 1 - b^{-\beta}. \end{cases}$$

Observe that the total mass at the $n-$th generation $\mu_n(T) = (\frac{1}{p})^n Y_n b^{-n}$, where Y_n is the number of non-zero offspring in the n-th generation. In particular, $Y_n, n = 1, 2, ...$ is a Galton-Watson branching process with Binomial offspring distribution having mean bp. Thus the positive martingale $\frac{Y_n}{(bp)^n}$ is a.s. convergent to zero (i.e. degenerate) if and only if $bp \leq 1$.

Example 4.2 (Independent Uniform [0,2] Generators) For this example we consider generators W which are uniformly distributed on $[0, 2]$ with binary branching number $b = 2$. In this case one may check that the total mass $Z_\infty := \mu_\infty(\mathbf{T})$ satisfies the distributional recursion

$$Z_\infty = \frac{1}{2} W_0 Z_\infty^{(1)} + \frac{1}{2} W_1 Z_\infty^{(2)} \tag{4.10}$$

where $Z_\infty^{(1)}, Z_\infty^{(2)}$ are independent and identically distributed as Z_∞, and independent of W_0, W_1. From here one may check that the Gamma distribution is the unique non-negative solution.

$$P(Z_\infty \in dx) = 4xe^{-2x}dx, \quad x \geq 0. \tag{4.11}$$

More generally, in the case of mean one Beta distributed generators on b-ary trees the total mass has a Gamma distribution.

In a certain general sense it is clear that nondegenerate cascades owe their structure to large deviations from average behavior; e.g. with P-probability one a product of i.i.d. mean one variables along a given path tends to zero. Fortunately, the tree T contains uncountably many paths. We take advantage of this by transforming *(size-biasing)* the probability measure in such a way that the essential computations become law of large number computations. The trick is to find the

"right"probability measure! In the present framework a natural choice is to choose a path at random according to the cascade measure at level n and then average and pass to the limit as $n \to \infty$. More precisely, define a sequence $\mathcal{Q}_n$ of measures on $\Omega_n \times T, \Omega_n := [0,\infty)^{T^*(n)}$, by

$$\int_{\Omega_n \times T} f(\omega,t)\mathcal{Q}_n(d\omega \times dt) := E_P \int_T f(\omega,t)Q_n(t)\mu(dt), \tag{2.5}$$

for bounded measurable functions f; see Figure 4.5. Then one normalizes the masses of the $\mathcal{Q}_n$ by a factor $Z_\emptyset := EW_\emptyset$ and extends to a probability $\mathcal{Q}_\infty$ using the Kolmogorov extension theorem. Of course, the nontrivial initialization $Z_\emptyset > 0$ will be required wherever $\mathcal{Q}_\infty$ is applied.

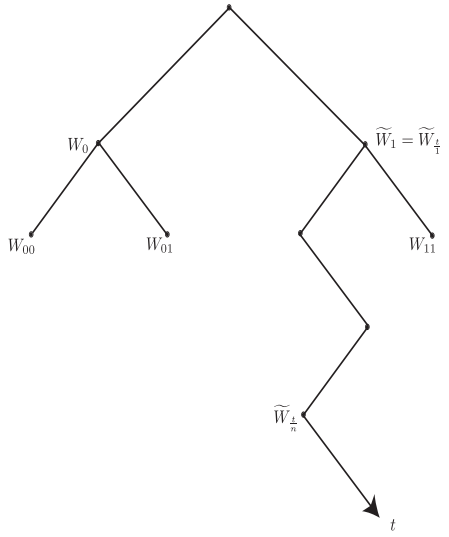


Figure 4.5: Size-biased cascade generators

Remark. The so-called *Peyrière probability* by Kahane was introduced in Peyrière (1976) under the apriori condition $P(\mu_\infty(T) > 0)$ of nondegeneracy of the cascade. The main difference here with Peyrière (1976) is that an apriori nondegeneracy is not required. As a result $\mathcal{Q}_\infty$ may be used to study the degeneracy vs. non-degeneracy problem.

The third ingredient to this theory is a corollary to a general *composition theorem* proved in Waymire and Williams (1996) which we refer to as the *percolation theorem* in the spirit of terminology introduced in Lyons (1990). The difference with Lyons (1990) is that here we consider a "measure valued"percolation in place of "set valued"percolation. The idea is still that multiplication by an independent cascade with i.i.d. zero/non-zero valued generators corresponds to an independent pruning of the tree under which one studies the critical parameters governing the survival of mass. This is an extremely powerful tool for analyzing dimension spectra.

Theorem 4.3 *(A Lebesgue Decomposition) Let π_Ω denote the coordinate projection map on $\Omega \times T$. Then,*

$$\mathcal{Q}_\infty \circ \pi_\Omega^{-1}(d\omega) = Z_\emptyset^{-1}\mu_\infty(T)(\omega)P(d\omega) + \mathbf{1}(\mu_\infty(T) = \infty)\mathcal{Q}_\infty \circ \pi_\Omega^{-1}(d\omega),$$

where $\mu_\infty = Q_\infty \mu$.

Corollary 4.1 *If $Z_\emptyset > 0$ then*

$$\mu_n(T) \to \mu_\infty(T) \text{in } L^1 \iff \mu_\infty(T) < \infty \ \mathcal{Q}_\infty - a.s. \tag{4.12}$$

$$\iff E_P\mu_\infty(T) = Z_\emptyset. \tag{4.13}$$

Theorem 4.4 *(A Size-Bias Transform Disintegration) For each $t \in T, n \geq 1$ define a probability $P_t << P$ on $\mathcal{F}_n$ by*

$$\frac{dP_t}{dP}|_n = Z_\emptyset^{-1} \prod_{i=0}^{n} W_{t|i}.$$

Also let P_t denote the Kolmogorov extension to $\mathcal{F}$. Then,

$$\mathcal{Q}_\infty(d\omega \times dt) = P_t(d\omega)\mu(dt),$$

Moreover, defining $\mathcal{F}_{t,n} := \sigma\{W_{t|n}, n \geq 0, W_\gamma, |\gamma| \leq n\}, t \in T, n \geq 0$, one has

$$\mu_n(T) = \sum_{j=0}^{n} b^{-j} \prod_{i=0}^{j} W_{t|i} M_{n,j}(t),$$

where each $M_{n,j}(t) := \sum_{|\gamma|=n-\frac{j}{\gamma}|1\neq t|j+1} b^{-(n-j)} \prod_{i=1}^{n-j} W_{t|j*\gamma|i}, 1 \leq j \leq n$, is the total mass in the (slightly irregular) subtree started at $t|j$, and given $\mathcal{F}_{t,0}$, the $M_{n,j}(t), j = 0, 1, \ldots n$ are independent. Moreover,

$$E_P[\mu_n(T)|\mathcal{F}_{t,0}] = \sum_{j=0}^{n} b^{-j} \prod_{i=0}^{j} W_{t|i}(\frac{b-1}{b} + \frac{1}{b}\delta_{j,n}),$$

where $\delta_{j,n}$ is the Kronecker delta.

Note. In the decomposition represented by the above Theorem4.4, at each node, say $t|j$, of the t-path one has a subtree generated at that site. However, there is a slight irregularity in the sub-tree since, at the first generation, the root $t|j$ subsequently generates only b-1 offspring, while each of these will generate b offspring. This is of no major consequence, however it tends to have subtle (arithmetic) effect on some calculations. Also, $\mu_n(T)$ need not be in $L^1(P_t)$, in which case the "conditional expectation" formula is an abuse of notation which is to be interpreted as a disintegration formula.

Corollary 4.2 *If $Z_\emptyset > 0$ then*

$$\mathcal{Q}_\infty(d\omega \times dt) \tag{4.14}$$
$$= Z_\emptyset^{-1}\mu_\infty(dt)P(d\omega) + \mathbf{1}(\mu_\infty(T) = \infty)\delta_{\tau(\omega)}(dt)\,\mathcal{Q}_\infty \circ \pi_\Omega^{-1}(d\omega), \tag{4.15}$$

where τ is a T-valued random path.

Remark 4.1 The proof of Corollary 4.2 makes the following heuristic precise. By definition the construction of $\mathcal{Q}_\infty$ involves renormalization of the mass by the total mass at each stage of the cascade and then passing to the limit. If the total is finite in the limit then one simply gets the total mass as the renormalization of P. On the other hand, if the limit is infinite then the renormalization results in a delta mass along a single surviving path.

Theorem 4.5 *(A 1st Departure Submartingale Bound) Let $\mathcal{F}_{t,n} := \sigma\{W_{t|i}, W_\gamma : i \geq 0, |\gamma| \leq n\}$. Fix $c_k \geq 0$, such that c_k is $\mathcal{F}_{t,0}-$measurable, $E_{\mathcal{Q}_\infty}\sum_k c_k < \infty$. Then, for arbitrary $t \in T$ (with $\frac{0}{0} := 1$),*

$$b^{-n}\prod_{i=0}^{n} W_{t|i} \leq \mu_n(T) \leq b^{-n}\prod_{i=0}^{n} W_{t|i} + \sup_j(b^{-j}\frac{\prod_{i=1}^{j}(W_{t|i})}{c_j})M_n,$$

where

$$M_n = \sum_{j=0}^{n-1} c_j b^{-(n-j)} \sum_{|\gamma|=n-\frac{j}{\gamma}|1\neq t|j+1} \prod_{i=1}^{n-j} W_{t|j*[\gamma|i}$$

is a non-negative submartingale with respect to $\mathcal{F}_{t,n}$, whose Doob decomposition has the predictable part $A_n = \frac{b-1}{b}\sum_{j\leq n} c_j$, and $\lim M_n$ exists $\mathcal{Q}_\infty-a.s.$

Let us refer to $c_j's$ satisfying the conditions of the previous theorem as *admissible.*

Corollary 4.3 *Fix $t \in T$. If*

$$\mu\{t : P_t(\sum_j b^{-j}\prod_{k=1}^{j} W_{t|k} < \infty) > 0\} > 0$$

then

$$P(\mu_\infty(T) > 0) > 0.$$

If for $\mu - a.e.t$,

$$P_t(\sum_j b^{-j}\prod_{k=1}^{j} W_{t|k} < \infty) = 1$$

then

$$E_P\mu_\infty(T) = Z_\emptyset$$

and

$$\mu_n(T) \to \mu_\infty(T) \text{ in } L^1(P).$$

If for $\mu - a.e.t$,

$$P_t(\limsup_{n\to\infty} b^{-n} \prod_{k=1}^{n} W_{t|k} = \infty) = 1$$

then P-a.s.

$$\mu_n(T) \to 0.$$

Remark 4.2 To obtain the convergence in Corollary 2.3 it is often useful to do a "root test"as the computation of $\sqrt[j]{b^{-j} \prod_{k\le j} W_{t|k}}$ is often calculable using an ergodic theorem. Alternatively, it is sometimes convenient to do a "comparison test" based on the simple fact that for admissible $c_j's$,

$$\sup_j \frac{b^{-j} \prod_{k\le j} W_{t|k}}{c_j} < \infty \Rightarrow \sum_{j=1}^{\infty} b^{-j} \prod_{k\le j} W_{t|k} < \infty,$$

and conversely.

The following the composition theorem and its corollary (the percolation theorem) complete this set of tools.

Theorem 4.6 *(Composition Theorem) Let T be an arbitrary compact metric space and let* $\{R_n\}$ *and* $\{S_n\}$ *be independent arbitrary positive T-martingales defined on a probability space* $(\Omega, \mathcal{F}, P)$, *with respect to independent increasing sigmafield sequences* $\{\mathcal{R}_n\}$ *and* $\{\mathcal{S}_n\}$, *respectively. Let* $\mathcal{F}_{n,m} := \sigma\{\mathcal{R}_n \cup \mathcal{S}_m\}$. *Define* $Q_n(t) := R_n(t)S_n(t), t \in T, n \ge 1$. *Then* $\{Q_n\}$ *is a positive T-martingale with respect to* $\mathcal{F}_{n,n}$ *and for* $\sigma \in M^+(T)$ *one has* $R_\infty(S_\infty\sigma) = Q_\infty\sigma \quad a.s.$

As an illustration of an application of Theorem 4.6 to get information about the support of an arbitrary cascade $Q_\infty\sigma$, one applies the following "percolation method"proposed by Kahane (1991); see Waymire and Williams (1995) for the proof. First apply the $\beta-$ *model,* denoted $Q_\infty^{(\beta)}$ to $Q_\infty\sigma$. The general criterion of composition theorem can then be applied to determine the non-degeneracy (i.e. survival) of the product. This can be equated with the critical parameter for the survival of the $\beta-$ model acting on a given measure σ which can be determined with the aid of the following proposition Kahane (1991).

Proposition 4.2 *For* $0 < \alpha < 1, 0 \ne \sigma \in M^+(T)$,

(a') $$Q_\alpha\sigma = 0 \Rightarrow \dim(\sigma) \le \alpha$$

(b') $$EQ_\alpha\sigma = \sigma \Rightarrow \dim(\sigma) \ge \alpha$$

Another application of the composition theorem shows that the log-infinitely divisible generators occupy a somewhat privileged role with respect to independent composition. Compositions, possibly dependent, arise naturally in extensions of the spatial cascade distribution to temporal evolutions where one replaces the generator random variables W_γ by generator stochastic processes $\{W_\gamma(s) : s \geq 0\}$. This induces a time evolution of the cascade operators $Q_\infty(s)$, $s \geq 0$. For the case of log-infinitely divisible generators taking the generator proceses $\{W_\gamma(s) : s \geq 0\}$ as exponentials of processes with stationary independent increments, one may use the composition theorem to compute the extinction time in terms of the spatial survival criteria; Waymire and Williams (1996). A simple Markovian space-time cascade was studied by Over (1995) as a model of GATE rainfall scans in which independent generator processes each evolved as a stationary two state (zero/nonzero) $\beta-$model. It is interesting to note that a $\beta-$ generator is log infinitely divisible as an extended real-valued random variable (in the sense of division into arbitrary n term independent sums). An attempt was made to interpret the rainfall probability parameter for this model in terms of large scale climatic forcing; see also Over (1995). Other applications of cascade compositions arise in consideration of temporal evolutions of spatial cascades and in the interaction of rainfall and landforms in the flood problem to be described in section 5.

Given the high degree of variability and intermittancy of rainfall it is the fine scale structure of cascades which is of particular interest in applications. As an illustration consider the multinomial cascade of Example 4.1. While purely mathematical, this example is presented as a way to simply illustrate the various mathematical concepts involved in the analysis of fine scale structure.

Example 4.1 In view of Borel's classic normal number theorem (Billingsley, 1986) or law of large numbers, one knows that the subset

$$F_p = \{t = (t_1, t_2, \dots) \in [0,1] : \frac{\#\{i \leq n : t_i = j\}}{n} \to p_j, j = 0, 1, \dots, b-1\}, \quad (4.10)$$

for prescribed non-negative frequencies $\mathbf{p} = (p_0, \dots, p_{b-1}), \sum_{j=0}^{b-1} p_j = 1$, has full Lebesgue measure one $\iff$ $p_j = \frac{1}{b}, j = 0, 1, \dots, b-1$. More generally, the multinomial cascade μ is supported by the set F_p. While the calculation of the Hausdorff dimension of F_p dates back to at least Eggleston (1949), the proof was dramatically improved by the following results of Billingsley (1960). Let μ, ρ be two probability measures on the Borel sigmafield of a metric space S and define a generalized Hausdorff measure by

$$H_{\rho,\theta}(F) = \lim_{\delta \to 0} \inf\{\sum_i \rho^\theta(J_i) : \cup_i J_i \supseteq F, \rho(J_i) < \delta\} \quad (4.11)$$

where J_i is a ball in S. Observe that in the case S is Euclidean space and ρ is Lebesgue measure then $\rho(J_i) \propto diam(J_i)$ and $H_{\rho,\theta}$ is usual Hausdorff measure.

Billingsley's generalized Hausdorff dimension is then the critical parameter defined by

$$dim_\rho(F) = \theta_c = \sup\{\theta : H_{\rho,\theta}(F) = \infty\} = \inf\{\theta : H_{\rho,\theta}(F) < \infty\}. \tag{4.12}$$

The following (heuristically simple) theorem demonstrates the power of these ideas in computation of fine scale structure.

Theorem 4.7 *If* $F \subseteq \{s \in S : \lim_n \frac{\log \mu(J_n(s))}{\log \rho(J_n(s))} = \alpha\}$, *where* $J_n(s)$ *are balls shrinking to* s, *then*

$$dim_\rho(F) = \alpha dim_\mu(F).$$

Note that since the multinomial cascade is supported on F_p, i.e. $\mu(F_p) = 1$, one has $dim_\mu(F_p) = 1$. In particular, from Billingley's Theorem 4.7 applied to the multinomial cascade one obtains the classic (1949) Eggleston formula for the Hausdorff dimension of F_p, that is the support of μ,

$$dim(F_p) = \alpha dim_\mu(F) = \alpha \tag{4.16}$$

where for $J_n(s) = \{t = (t_1, t_2, \dots) \in [0,1] : t_i = s_i, i \le n\}$ by the law of large numbers under the probability measure μ,

$$\alpha = \lim \frac{\sum_i \log p_{t_i}}{-n \log b} = -\sum_{j=0}^{b-1} p_j \log_b p_j. \tag{4.17}$$

More generally, the *singularity spectrum* defined by

$$f(\alpha) = dim\{t \in [0,1] : \lim_n \frac{\log \mu(J_n(t))}{-n \log b} = \alpha\} \tag{4.18}$$

may be computed using Billingsley's theorem as follows. For a real parameter q define

$$p_j(q) = P^{-1}(q) p_j^q, j = 0, 1, \dots, b-1, \tag{4.19}$$

where

$$P(q) = \sum_{j=0}^{b-1} p_j^q. \tag{4.20}$$

Then let μ_q denote the multinomial cascade for this choice of parameters. Now observe that

$$\begin{aligned} f(\alpha) &= dim\{t \in [0,1] : \lim_n \frac{\log \mu(J_n(t))}{-n \log b} = \alpha\} && (4.21)\\ &= dim\{t \in [0,1] : \lim_n \frac{\log \mu_q(J_n(t))}{-n \log b} = \log_b P(q) + \alpha q\} && (4.22)\\ &= \log_b P(q) + \alpha q, && (4.23) \end{aligned}$$

where q is selected such that $\mu_q(\{t \in [0,1] : \lim_n \frac{\log \mu(J_n(t))}{-n \log b} = \alpha\}) = 1$. That is

$$\log_b P(q) + \alpha q = -\sum_{j=0}^{b-1} \frac{p_j^q}{P(q)} \log_b(\frac{p^q}{P(q)}) \tag{4.24}$$

In particular

$$\alpha P(q) = -\sum_{j=0}^{b-1} p_j^q \log_b p_j = -P'(q), \tag{4.25}$$

or equivalently, q is selected to maximize $\log_b P(q) + \alpha q$ as a function of α and therefore the singularity spectrum is given by the *Legendre transform*

$$f(\alpha) = \sup_q \{\log_b P(q) + \alpha q\} \tag{4.26}$$

of the *cascade structure function*

$$\tau(q) = -\log_b P(q) = -\log_b \sum_{j=0}^{b-1} p^q. \tag{4.27}$$

The computation of these exponents are also required for the case of random cascades. Expected value computations in the case of random cascades often appear in the physical science literature and are easily obtained as follows. Let $\Delta_\lambda(i), i = 1, 2, \ldots$ denote a partition of a region X of d-dimensional space into cells at the length scale λ. Then

$$\log_b E[\sum_i \mu_\infty^h(\Delta_\lambda(i))] = -d\chi_b(h) \log(\lambda) + \log E\mu_\infty^h(X), \tag{4.28}$$

where

$$\chi_b(h) = \log_b \left[EW^h\right] - (h-1) \tag{4.29}$$

$$\frac{\log_b Prob[\mu_\infty(\Delta_\lambda) > \lambda^{d\alpha}]}{\log \mu} \to -d\chi_b^*(1-\alpha), \quad \lambda \to 0 \tag{4.30}$$

where

$$\chi_b^*(a) = \sup_h [ah - \chi_b(h)] \tag{4.31}$$

is the Legendre transform of $\chi_b(h)$. Note from (4.23) that the structure function exponent for random cascades is given by $-d\chi_b(h)$.

In order to apply these formulae to sample realizations one needs to be able to "drop the expectations". However because of long-range spatial correlations the cascade fields are nonergodic and one cannot simply replace expected values by spatial averages. Since data typically consists of a single spatial sample realization, this problem cannot be ignored. Holley and Waymire (1992) applied the idea described

for the multinomial cascade above to compute the singularity spectrum for a large class of random cascades; namely those with suitably bounded cascade generators. Define *nth-scale sample moments* by

$$M_n(h) = \sum_{|v|=n} \mu_\infty^h(\Delta_n(v)), h \in \mathbf{R}. \tag{4.32}$$

Then introduce *h-cascades,* denoted by the random measures $\mu_\infty(h;dt), h \in \mathbf{R}$, defined via the *h-cascade generators*

$$W_v(h) = \frac{W_v^h}{\mathbf{E}W_v^h}, h \in \mathbf{R}. \tag{4.33}$$

With this one may easily check that

$$\frac{\mu_n^h(\Delta_n(v))}{b^{n\chi_b(h)}} = \mu_n(h;\Delta_n(v)), \tag{4.34}$$

where

$$\frac{d\mu_n(h;\cdot)}{d\mu}(t) \equiv Q_n(h;t) = \prod_{i=0}^{n} W_{t|i}(h), t \in \mathbf{T}, \tag{4.35}$$

is the sequence of *nth level h-cascades,* $n = 1, 2, \ldots$.

Proposition 4.3 *For $h \in \mathbf{R}, n \geq 1$, one has*

$$\frac{M_n(h)}{b^{n\chi_b(h)}} = \sum_{|v|=n} Z_\infty^h(v)\mu_n(h;\Delta_n(v)) = \int_{\mathbf{T}} Z_\infty^h(t|n)\mu_n(h;dt)$$

where a.s.

$$Z_\infty(v) = \lim_{N\to\infty} \sum_{|u|=N-n} \prod_{i=1}^{N-n} W_{v*<u_1\ldots u_i>} b^{-(N-n)},$$

*and * denotes the concatenation*

$$< v_1, \ldots, v_n > * < u_1, \ldots, u_N > = < v_1, \ldots, v_n, u_1, \ldots, u_N > .$$

The crude idea is to then consider

$$\frac{\log_b M_n(h)}{n} = \frac{1}{n}\log_b(b^{n\chi_b(h)}\frac{M_n(h)}{b^{n\chi_b(h)}}) = \chi_b(h) + \frac{1}{n}\frac{\log_b(M_n(h)}{b^{n\chi_b(h)}} \tag{4.36}$$

and show that $\frac{\log_b(M_n(h)}{b^{n\chi_b(h)}}$ has a positive limit as $n \to \infty$. Related ideas have also been applied by Molchan (1996), Troutman and Vecchia (1999) and Ossiander and Waymire (2000) to obtain confidence intervals for statistical estimation of generator distributions from sample data. Although typically not present in current data

analysis, one may expect that scientific standards for testability of claims made in data analysis to be an achievable part of this theory. This will be discussed in the research lecture by M. Ossiander.

As noted in Section 3, in the simplest hydrologic context one imagines a rain of particles uniformly distributed over the network τ_n and traveling at constant rate v. Then the (hydrograph) proportion of particles which reach the outlet (θ) in time t is represented by the width function $\frac{Z(vt)}{n}$ suitably normalized. The (conditional) expected value is the best approximation in the mean square sense given the size of the network. This simple idealization illustrates the use of some basic results from Section 3 for prediction problems based on maps of river basins (in place of stream gauge networks). For example, using Shreve's random model one obtains a Rayleigh density for the hydrograph for large networks with constant velocity. The following simple calculations illustrate the basic elements of a theory which computes flow as a composition of rainfall and landform cascade structure. First consider the case of uniform rainfall over a square partitioned by the Peano network described in Section 3. As discussed in (3.24) and in view of Example 4.1 the width function for the Peano network can be viewed as a Binomial cascade with parameters $p_0 = 1/4, p_1 = 3/4$; see Figure 4.3. Taking unit velocity for the flow one sees immediately the scaling exponent for the flow from the definition of dimension as follows. Since $\lambda^{-dimD} \propto$ the number of cells which cover D at the scale λ, which in turn is $q(\lambda)$ normalized by λ^2, i.e. $q(\lambda)\lambda^{-2}$, one has for uniform rain and constant velocity that $q(\lambda)\lambda^{-2} \sim \lambda^{-dimD}$, where as computed in Theorem 3.13, $dimD = \frac{\log 3}{\log 2}$. Alternatively, the peak flow at the network outlet $q(\mu_n) = \max_t Q^{(}n)(t)$ at the nth scale of resolution (and length $\mu_n = 2^{-n}$) is $(\frac{3}{4})^n$. Thus one has

$$\log q(\mu_n) = n \log(\frac{3}{4}) = (2 - \frac{\log 3}{\log 2}) \log \mu_n. \tag{4.37}$$

In Gupta et al (1996) and in Troutman and Over (2001), the authors provide calculations via certain approximations and simulations to suggest a relationship with the peak flow exponent in the case of a $\beta-$model rainfall composed with the Peano network landform. These preliminary calculations illustrate the nature of the broad theoretical objective in (flow) predictions in ungauged basins (PUB). In fact, a significant part of the PUB problem is reduced to this and the corresponding mathematical problem for general cascades on the Peano network. The problem which will be addressed in lectures by B. Troutman and V. Gupta concern the flows from the network.

To accomodate spatial variability in the analysis of rain data over a network as a b-ary cascade one is first confronted with determination of the paramter b. However, within the framework of the present theory considerations of the parameter b arise also in connection with self-similar partitions of the drainage basin area by the network. In particular one hypothesises that the landform partitioned by the network provides a self-similar b-ary tiling of the plane over which rainfall is

distributed as a b-ary cascade. One approach to incorporate this hypothesis is to assume that the distribution of rainfall generators does not depend on the choice of b dictated by the landform structure beyond a scale factor. Over (1995) observes that this will be the case under the assumption that the generators have a log-infinitely divisible distribution. More precisely one has

Proposition 4.4 *If the generators of an independent b-ary cascade are log-infinitely divisible say,* $W = e^{X-a}, a = \log Ee^X$, *where* X *has an infinitely divisible distribution then*

$$\chi_b(h) = -\phi_{\frac{1}{\log b}X}(1) + \log \phi_{\frac{1}{\log b}X}(h) - (h-1),$$

where $\phi_{cX}(h) = Ee^{hcX}, c > 0$.

In view of Proposition 4.4 the choice of the parameter b simply sets a scale for the exponent of a log-infinitely divisible cascade generator. It is interesting to note that the familiar cascade generators Beta, lognormal, logPoisson which arise in the statistical analysis of turbulence data and theory are all log-infinitely divisible. Lovejoy and Schertzer (1990) argue that such forms of *universality* must hold.

We close this section with a discussion of the statistical estimation problem implied by the observations plotted in Figure 4.6 from Frisch (1995).

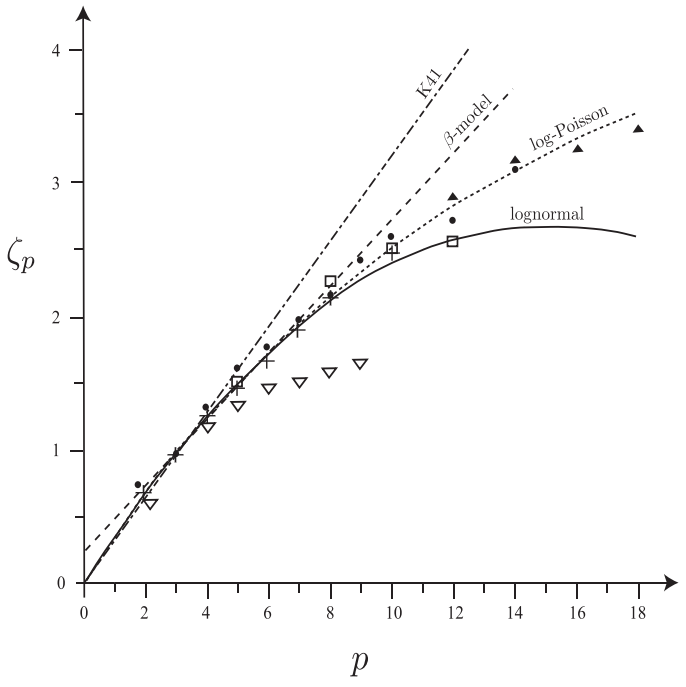


Figure 4.6: Empirical turbulence exponents

The *energy dissipation rate* ϵ defined by

$$\epsilon(x) = \frac{\nu}{2} \sum_{i,j=1}^{3} (\frac{\partial u_i}{\partial x_j})^2, \quad x \in R^3, \tag{4.38}$$

is computed in terms of the fluid velocity $u = (u_1, u_2, u_3)$ as the local rate of decay of kinetic energy $\frac{d}{dt}\frac{1}{2}\int_V |u(x)|^2 dx$ from the incompressible Navier-Stokes equation in a region V with viscosity parameter $\nu > 0$. However in Kolmogorov's statistical theory of turbulence the *energy dissipation rate,* denoted $\epsilon(x)$ per unit volume at the point x is a generalized random field represented by a multiplicative cascade in the sense that the amount of heat dissipated in a ball $B_r(x)$ located at x per unit time is $\epsilon_r(x) \equiv \int_{B_r(x)} \epsilon(y)dy = \mu_\infty(B_r(x))$. Data analysis depicted in Figure 4.6 from Frisch (1995) suggests that the competition between generators is resolved to that of logPoisson; see the discussion following (4.40) for relationship between $\zeta_p = \zeta(p)$ and $\tau(p)$. However, the question is much more delicate than indicated by *expected value analysis.*

The multiplicative cascade with *iid* non-negative mean one generators is assumed to be a valid statistical model for the turbulent redistribution of energy in the statistical model of the random dissipation field $\epsilon(dx)$ over an appropriate range of length scales, referred to as the *Kolmogorov inertial range.* The Kolmogorov inertial range is an interval of length scales from the largest length scale at which energy is input down to the smallest length scale where energy is dissipated by fluid viscosity. Actual observations of ϵ are one-dimensional cross sections wherein (4.38) is replaced by the surrogate measurement $15\nu(\frac{\partial u_1}{\partial x_1})^2$. As a result Jouault, Greiner, and Lipa (2000) have effectively argued that *iid* mean one generators provide the appropriate model for measurements of energy dissipation rates from the point of view of conservation laws. In particular, taking one-dimensional cuts through the three dimensional energy dissipation field makes the measurements non-conservative in an almost sure sense. The statistical model with *iid* mean one generators provides conservation on average.

The Kolmogorov lognormal hypothesis leads to quadratic structure function $\chi_b(h)$. Early data analysis revealed a departure from quadratic multiscaling exponents which is now understood to be remarkably adjusted by a linear correction. This effect had already been anticipated by preliminary calculations in the physics literature by Lovejoy and Schertzer (1991), and Molchan (1997). It is the subject of the lecture by Ossiander in these lectures.

Largely prompted by observations in the form of Figure 4.6, various adhoc alternatives to the lognormal hypothesis have been considered in the physics literature. However, the logPoisson distribution surfaced as an alternate hypothesis as a somewhat indirect consequence of an analysis by She and Lèvêsque (1994), Dubrulle (1994), and She and Waymire (1994,1995). Specifically, She and Lèvêsque (1994) obtain the following second order, linear, nonhomogeneous difference equation for

the scaling exponents.

$$\tau(h+2) - (1+\beta)\tau(h+1) + \beta\tau(h) + \frac{2}{3}(1-\beta) = 0, \tag{4.39}$$

where $\beta = \frac{2}{3}$, and $\tau(0) = \tau(1) = 0$ as a consequence of the following log-convexity hypothesis on the structure of the size-biased moments $\epsilon_l^{(h)} := E\epsilon_l^{h+1}/E\epsilon_l^h$ of energy dissipation:

$$\epsilon_l^{(h+1)} = A_h(\epsilon_l^{(h)})^\beta(\epsilon_l^{(\infty)})^{1-\beta}, \quad 0 < \beta < 1. \tag{4.40}$$

According to She (personal communication) this hypothesis was formulated in response to observations made in numerical simulations of Navier-Stokes equations. A plot of the scaling exponents with and without the linear correction for logPoisson generators against the Anselmet data is given in Figure 5.2a,b, respectively.

Obviously if one were permitted to use Figure 4.6 to test hypothesis of logNormal vs. logPoisson then the need for extensive statistical theory would hardly be justified. However, the problem to distinguish models based on the appropriate linear corrections requires the suitable error bars which will be described in the accompanying lectures by Ossiander; also see Ossiander and Waymire (2001,2002). The multiscaling estimations require data at the level of the measured dissipation rates. In the literature the reported data is often that of velocity exponents $\zeta(h)$ defined by $E(u_1(x+\lambda) - u_1(x))^h \sim \lambda^{\psi(h)}$ from which $\tau(h)$, defined by $E\epsilon^h(dx) \sim (dx)^{\tau(h)}$, is obtained by the assumed relationship based on dimensional arguments that $\psi(h) = \frac{h}{3} + \tau(\frac{h}{3})$.

The lecture by Greiner provides new and convincing alternative arguments in favor of lognormal hypothesis over logPoisson; see Jouault, Greiner, Lipa (2000). Another approach based on statistical analysis of the velocity data may be found in Barndorff-Nielsen, Jensen, and Sorensen (1990).

Finally, we wish to record that the statistical theory is complimented on the purely mathematical side by foundational efforts which seek to explain observed qualitative structure in turbulence measurements directly from analysis of incompressible Navier-Stokes equations. An extensive collection of references in this regard may be found in the recent monograph by Foias, Manley, Rosa, and Temam (2001). Recent alternative probabilistic analysis of Navier-Stokes is the subject of the lectures by Thomann and by Orum in this workshop. Some background material is provided in the next section.

5 Navier-Stokes Equations and Multiplicative Cascades

The Navier-Stokes equations provide a non-linear model of the flow of fluid as observed from a fixed location x, i.e. the so-called *Eulerian description.* These equations were first formally written down by the French hydrologist Claude Louis Navier and, some fifty years later, rediscovered in the mathematically more rigorous work

of George Gabriel Stokes. The equations are essentially a manifestation of Newton's law of motion obtained by resolving external body forces such as gravity and surface stress forces. Consider the velocity vector of a fluid parcel (material point) given by

$$\text{v(t,x)} = \text{velocity of fluid element at } x \text{ at time } t$$

If the flow carries a fluid parcel from location $X = \varphi(0, X)$ to $x = x(t) = \varphi(t, X)$ at time t, where $\varphi(t, X)$ is a smooth one-to-one map (Fig 5.1) with smooth inverse, then one may write

$$x = \varphi(t, X), \quad v(t, x) = \frac{\partial \varphi}{\partial t}(t, X) = \varphi_t(t, \varphi^{-1}(t, x)) \tag{5.1}$$

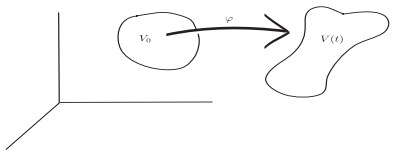


Figure 5.1: Schematic of fluid flow map

Thus, *acceleration* of the parcel is given by

$$\frac{d^2x}{dt^2} = (\varphi_t(t, \varphi^{-1}(t, x)))_t = \frac{\partial v}{\partial t} + \sum_{j=1}^{3} \frac{\partial v}{\partial x_j}\frac{dx_j}{dt}, \tag{5.2}$$

or, equivalently,

$$\frac{d^2x}{dt^2} = \frac{\partial v}{\partial t} + (v \cdot \nabla)v. \tag{5.3}$$

Most significant is the fact that the nonlinearity $(v \cdot \nabla)v$ is intrinsic to the Eulerian acceleration and is NOT due to modelling considerations.

Lemma 5.1 *(Euler) Let*

$$J(t, x) = det((\frac{\partial \varphi_i(t, X)}{\partial X_j})).$$

Then,

$$J_t(t, X) = J \cdot div(v),$$

where

$$div(v) = \sum_{j=1}^{3} \frac{\partial v_j}{\partial x_j}.$$

In particular, observe that if $div(v) = 0$ then $J_t = 0$. In this case the fluid is said to be *incompressible*. The *equation of mass balance* for this model is obtained as an application of Gauss's Divergence theorem to the mass in an arbitrary fluid element as follows. Let

$$\rho = \rho(t, x) = \text{density of fluid at location } x \text{ and time } t.$$

For an initial volume V_0, let

$$V(t) = \varphi(t, V_0).$$

Then the principle of mass balance may be expressed as

$$\int_{V_0} \rho dX = \int_{V(t)} \rho dx. \tag{5.4}$$

In particular

$$\frac{d}{dt} \int_V \rho dx = 0. \tag{5.5}$$

Thus, using Euler's lemma,

$$\begin{aligned} 0 &= \frac{d}{dt} \int_{V(t)} \rho dx \\ &= \frac{d}{dt} \int_{V_0} \rho(\varphi(t,x),t) J(t,x) dX \\ &= \int_{V_0} \{\rho_t(\varphi(t,x),t) + \sum_j \frac{\partial \rho}{\partial x_j} \frac{\partial \varphi_j}{\partial t}\} J dX + \int_{V_0} \rho J_t dX \\ &= \int_{V(t)} \{\rho_t + \nabla \rho \cdot v + \rho div(v)\} dx \\ &= \int_{V(t)} \{\rho_t + div(\rho v)\} dx. \end{aligned} \tag{5.6}$$

Since V_0 is arbitrary one has the following localized form of mass conservation, referred to as the *continuity equation.*

$$\rho_t + div(\rho v) = 0. \tag{5.7}$$

Note that if ρ is constant then the fluid must be incompressible. We will assume ρ is constant from here out.

According to Newton's Law of Motion, the time rate of change of momentum of a fluid parcel is the sum of the forces acting on a fluid parcel. Thus using (5.3), we write

$$\int_{V_0} \rho\{v_t + (v \cdot \nabla)v\} dx = \frac{d}{dt} \int_{V_0} \rho v = \int_{\partial V(t)} \mathcal{S} d\sigma + \int_{V(t)} g(t,x) dx, \tag{5.8}$$

where $g(t,x)$ represents a body force, e.g. $g(t,x) = \rho(0,0,-g)$ in the case of weight of parcel due to gravity, and $\mathcal{S}$ is a stress vector acting on the fluid parcel surface (as a force per unit area). One makes the modelling hypothesis that the stress forces act linearly as

$$\mathcal{S}(n) = \sigma n$$

where $\sigma = ((\sigma_{ij}))_{1\le i,j\le 3}$ is a *symmetric* matrix and $n = (n_1, n_2, n_3)$ is the unit normal to the surface. The symmetry of σ may be shown to be required for conservation of angular momentum. With this hypothesis one may apply the Gauss Divergence Theorem to (5.8) to obtain

$$\int_{V_0} \rho\{v_t + (v\cdot\nabla)v\}dx = \int_{V(t)} div(T)dx + \int_{V(t)} g(t,x)dx, \tag{5.9}$$

where

$$div(T) = (\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j})_{i=1,2,3}. \tag{5.10}$$

Write

$$\frac{\partial v_i}{\partial x_j} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) + \frac{1}{2}(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}) := D + \Omega.$$

Then D is referred to as the *deformation tensor*. Rigid motion may be defined by $D = 0$.

A *Stokes Fluid* is defined by the additional hypothesis that T is an affine linear function of the deformation rate D of the form

$$T = T(D) = -pI + 2\mu D,$$

where $-pI = T(0)$ defines the *pressure*. The parameter μ is referred to as the *shear viscosity*. Now, using incompressibility, one may compute that

$$div(D) = \frac{1}{2}\sum_j \frac{\partial}{\partial x_j}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) = \frac{1}{2}\nabla^2 v = \frac{1}{2}\Delta v.$$

In particular it is the divergence of the deformation tensor which is responsible for the Laplacian term in the equation. Now,

$$div(T) = -\nabla p + \mu\nabla^2 v = -\nabla p + \mu\Delta v.$$

The above modelling hypothesis thus provides the following incompressible 3d-Navier-Stokes equations as a model for fluid motion:

$$\rho(v_t + (v\cdot\nabla)v) = \mu\Delta - \nabla p + g, \quad \nabla\cdot v = 0.$$

In the case $\mu > 0$ this is referred to as the *viscuous* equation. The inviscid equation is also called the *Euler equation*. As noted earlier, the nonlinear *convection term* $(v\cdot$

$\nabla v)v$ is intrinsic to the definition of acceleration, while the linear diffusion term Δv occurs as a result of modelling hypotheses. For scale considerations it is convenient to write the equations in dimensionless form by introducing $u = U^{-1}v, y = L^{-1}x, s = T^{-1}t$, where U, L, and T are (somewhat arbitrarly) chosen characteristic velocity, length and time scale parameters. With this substitution the incompressible Navier-Stokes equations transform to

$$\frac{\partial u}{\partial s} + u \cdot \nabla u = -\nabla q + R^{-1}\Delta u + f, \quad \nabla u = 0, \tag{5.11}$$

where $q = \frac{p}{\rho U^2}$, $f = \frac{Lg}{\rho U^2}$ and the parameter $R = \frac{LU}{\mu}$ is the so-called *Reynolds number.* We will refer to these equations with the original notation of t in place of s, p in place of q, $\nu = R^{-1}$, f in place of $f = \frac{Lg}{U^2}$, but retain u for velocity. Thus our standard equation notation will be:

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + \nu\Delta u + g, \quad \nabla \cdot u = 0. \tag{5.12}$$

Remark Once again, the model is derived in $n = 3$ dimensions but admits a mathematical formulation in any $n \geq 2$ dimensions. The incompressibility condition $\nabla \cdot u = 0$ requires at least two space dimensions. The $n = 1-$dimensional equation

$$\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} + g, \tag{5.13}$$

is referred to as *Burgers equation.*

The 3d-Navier-Stokes equations is a system of 3 equations in the 3 unknown velocity components $v_i, i = 1, 2, 3$ and unknown pressure p. The incompressibility condition provides a 4th equation. There are a couple ways in which to proceed with these equations by first removing the pressure term, solve for v and then compute pressure from the velocity.

Method 1. (Vorticity method). If one takes the curl $\omega = \nabla \wedge u$, where $\wedge$ is used to denote the usual cartesian cross product then, since the curl of a gradient is zero, the pressure term is removed and one obtains an equation in the vorticity ω. (In 2d, one may write $\omega = (\omega, 0, 0)$ and obtain a scalar vorticity equation.) In particular, using the vector identities $(u \cdot \nabla)u = \frac{1}{2}\nabla|u|^2 + \omega \wedge u$, and $\nabla \wedge (\omega \wedge u) = u \cdot \nabla\omega - \omega \cdot \nabla u$, one obtains

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = \nu\Delta\omega + \nabla \wedge g, \quad \nabla \cdot \omega = 0. \tag{5.14}$$

Equivalently,

$$\frac{\partial \omega}{\partial t} + \nabla \wedge (\omega \wedge u) = \nu\Delta\omega + \nabla \wedge g, \quad \nabla \cdot \omega = 0. \tag{5.15}$$

If the vorticity is solved then one may compute the velocity field from vorticity, and then the pressure as will be explained below.

Method 2. (Orthogonal Projection). According to the Helmholtz-Hodge Decomposition, any smooth vector field $\mathbf{u}$ in $\mathbf{R}^3$ which falls off sufficiently fast at large distances can be uniquely represented as the superposition of a gradient and curl

$$u = u_1 + u_2 \tag{5.16}$$

where

$$u_1 = -\nabla\phi, \qquad u_2 = \nabla \wedge \Psi \tag{5.17}$$

for a scalar potential ϕ and vector potential Ψ obtained by solving

$$-\Delta\phi = \nabla \cdot u \quad (= \omega) \tag{5.18}$$

and

$$-\Delta\Psi = \Delta \wedge u, \qquad \nabla \cdot \Psi = 0 \tag{5.19}$$

In particular, therefore, u_2 is divergence free. Thus one may define a linear orthogonal projection operator, ie. bounded, linear, idempotent operator, onto the divergence free component by $\mathbf{P}u = u_2$. Now, for incompressible u satisfying Navier-Stokes, one has $\mathbf{P}u = u$ and $\mathbf{P}(-\nabla p) = 0$. Therefore,

$$\frac{\partial u}{\partial t} = \mathbf{P}(-(u \cdot \nabla)u + \nu\Delta u + g), \quad \nabla \cdot u = 0. \tag{5.20}$$

The point is that (5.20) eliminates the pressure term and expresses $\frac{\partial u}{\partial t}$ in terms of u alone. The pressure can be recovered from the solution u of (5.20) as the gradient part of $-u \cdot \nabla u \nu\Delta u + g$. Finally note also that in view of (5.18) one may recover the velocity from the vorticity formulation in the form, known as the Biot-Savart Law,

$$u(t,x) = K * \omega(t,x), \tag{5.21}$$

where K is a linear operator given by solving (5.18). Specifically, one may check by a change of variables that inverting the Laplacian via

$$\Psi(t,x) = -\frac{1}{4\pi}\int_{R^3} \frac{1}{|x-y|}\omega(t,y)dy \tag{5.22}$$

also solves $\nabla \cdot \Psi = 0$ since $\nabla \cdot \omega = 0$. Now one has

$$\begin{aligned} u(t,x) &= \nabla \wedge (-\frac{1}{4\pi}\int_{R^3} \frac{1}{|x-y|}\omega(t,y)dy) \\ &= \frac{1}{4\pi}\int_{R^3} \nabla_y \frac{1}{|x-y|} \wedge \omega(t,y)dy \\ &= K * \omega(t,x). \end{aligned} \tag{5.23}$$

Alternatively, in the Fourier domain one readily observes that incompressibility $\nabla \cdot u = 0$ is the orthogonality $-i\xi \cdot \hat{u} = 0$. Thus, one may take Fourier transforms

of the Navier-Stokes equation and project onto the plane orthogonal to the Fourier wave number ξ. Since $\hat{u}$ is in this plane it is invariant under the projection. However the Fourier transform of the pressure gradient ∇p, namely $-i\xi\hat{p}$, is in the plane of the wave number itself and will, hence, be projected out of the equation. Again one obtains an equation in the Fourier transform of the velocity field, and if the velocity is determined then the unkown pressure may be computed from the velocity field.

LeJan and Sznitman (1997) observed that the Fourier transformed version of the Navier-Stokes equations in 3-dimensions are amenable to a representation of the solution as an expected value of a certain product of Fourier transformed initial data and forcing over a branching random walk. We will see that the approach taken by LeJan-Sznitman (1997) may be extended to other dimensions, lower as well as higher, and to a wide variety of partial differential equations pertaining to flows, both linear and non-linear; cf Bhattacharya et al (2001). This will be discussed in more detail in related research lectures by E.Thomann and C. Orum. The basic idea for this approach is easily illustrated by the following simpler examples.

Example 5.1 (Simple Linear Diffusion). Consider one-dimensional heat equation on $(-\infty, \infty)$ with constant coefficients $a > 0, b \in \mathbf{R}$, given by

$$u_t = au_{xx} + bu_x \tag{5.24}$$

with initial data $u(0, x) = u_0(x)$. The (spatial) Fourier transform of equation (5.24) may be expressed as

$$\hat{u}_t(t, \xi) = -a\xi^2\hat{u}(t, \xi) + ib\xi\hat{u}(t, \xi), \tag{5.25}$$

where the Fourier transform $\hat{f}$ of an integrable function f on the real number line is defined by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)dx. \tag{5.26}$$

Regarding the first term on the right hand side as an integrating factor, the equation (5.25) readily integrates to the equivalent equation for $\xi \neq 0$,

$$\hat{u}(t, \xi) = \hat{u}_0(\xi)e^{-\lambda(\xi)t} + \int_0^t \lambda(\xi)e^{-\lambda(\xi)s} \frac{ib\xi}{\lambda(\xi)} \hat{u}(t - s, \xi)ds, \tag{5.27}$$

where

$$\lambda(\xi) = a\xi^2. \tag{5.28}$$

Now, consider the following lineal stochastic cascade. A root particle θ of type $\xi \neq 0$ holds for an exponentially distributed length of time S_θ with parameter $\lambda(\xi)$. When the clock rings the particle is replaced by 1 offspring $< 1 >$ of the same type ξ, i.e. one particle of type $\eta = \xi$ selected according to the probability kernel $K(\xi, d\eta) = \delta_\xi(d\eta)$. The clock is reset for the statistically independent new particle's waiting time $S_{<1>}$, and so on. This results in a linear graph $\tau = \{< \theta >, < 1 >$

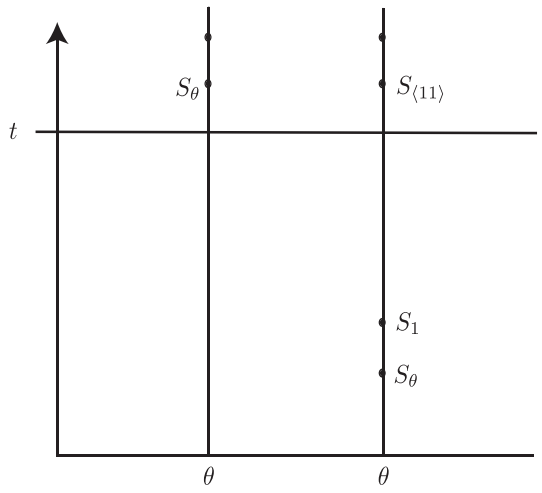


Figure 5.2: Linear cascade

$, < 11 >, < 111 >, \dots\}$ rooted at a vertex θ of type ξ, and i.i.d. exponentially distributed times $S_\theta, S_{<1>}, S_{<11>}, \dots$ as depicted in Figure 5.2.

Note The sequence of exponential holding times associated with the vertices of τ are implicit to future references to the "tree τ". For example, reference to the sub-tree τ re-rooted at $< 1 >$, denoted $\tau_{<1>}$, is defined by the shifted sub-sequences $\{\theta =< 1 >, < 11 >, \dots\}$ and $S_{<1>}, S_{<11>}, \dots$.

In the case of lineal trees it is convenient to identify a vertex $< 11 \dots 1 >$ of k 1's with its length $k = | < 11 \dots 1 > |, 0 = | < \theta > |$. Now consider the *product* of values over the vertices of the tree furnished by the recursively defined "times functional":

$$\chi(\hat{u}_0; \tau, t) = \begin{cases} \hat{u}_0(\xi), & \text{if } S_\theta > t, \\ \frac{ib\xi}{\lambda(\xi)} \cdot \chi(\hat{u}_0, \tau_{<1>}, t - S_\theta) & \text{if } S_\theta < t, \end{cases} \tag{5.29}$$

where $\tau_{<1>}$ is the lineal subtree of τ re-rooted at $< 1 >$ and S_θ is the replacement time for θ. In particular, performing the stochastic iteration one obtains

$$\chi(\hat{u}_0; \tau, t) = (\frac{ib\xi}{\lambda(\xi)})^{N_t} \hat{u}_0(\xi), \tag{5.30}$$

where N_t denotes the number of clock rings by time t as given by

$$N_t = \sup\{n : S_0 + S_1 + \cdots + S_{n-1} \le t < S_0 + S_1 + \cdots S_n\}, \tag{5.31}$$

$\sup \emptyset = 0$. In particular $\{N_t : t \ge 0\}$ is a Poisson process with parameter $\lambda(\xi)$. Now

observe that

$$\begin{aligned}\hat{u}(t,\xi) &= \exp\{-a\xi^2 t + itb\xi\}\hat{u}_0(\xi)\\ &= \exp\{(\frac{ib\xi}{\lambda(\xi)} - 1)\lambda(\xi)t\}\\ &= \mathbf{E}(\frac{ib\xi}{\lambda(\xi)})^{N_t}\hat{u}_0(\xi)\\ &= \mathbf{E}_\xi \chi(\hat{u}_0;\tau,t). \end{aligned} \tag{5.32}$$

Thus the expected product is the Fourier transform of the solution to the heat equation (5.24) with initial data $u_0(x)$. Observe that a similar graphical representation is valid for higher dimensions with only nominal change.

Example 5.2 *(Forced Heat Equation).* Next let us consider the one-dimensional heat equation (5.24) on $(-\infty,\infty)$ with a space-time forcing term $g(t,x)$ given by

$$u_t = au_{xx} + bu_x + g(t,x) \tag{5.33}$$

and initial data $u(0,x) = u_0(x)$. One may write for $\xi \neq 0$,

$$\hat{u}(t,\xi) = \hat{u}_0(\xi)e^{-\lambda(\xi)t} + \int_0^t \lambda(\xi)e^{-\lambda(\xi)s}\{p\frac{ib\xi}{p\lambda(\xi)}\hat{u}(t-s,\xi) + q\frac{1}{q\lambda(\xi)}\hat{g}(t-s,\xi)\}ds, \tag{5.34}$$

where

$$\lambda(\xi) = a\xi^2, \quad pq \neq 0, p+q = 1. \tag{5.35}$$

Now, consider the following lineal stochastic cascade. A root particle θ of type $\xi \neq 0$ holds for an exponentially distributed length of time S_θ with parameter $\lambda(\xi)$. When the clock rings the particle is removed and replaced by 1 offspring of the same type ξ with probability p, or it is replaced by 0 offspring with probability q. The clock is reset for the statistically independent new particle's waiting time $S_{<1>}$, and so on. This results in a linear graph as depicted in Figure 5.3. Let $\kappa_\theta, \kappa_{<1>}, \ldots$ be an i.i.d. sequence of Bernoulli 0-1 valued random variables and independent of $S_\theta, S_{<1>}, \ldots$, with $p = P(\kappa = 1) = 1 - P(\kappa = 0)$. The number N_t of clock rings and particle reproductions by time t is distributed as the smallest of a Poisson random variable and an independent Geometrically distributed random variable with parameters $\lambda(\xi)t$ and p, respectively.

Next form a *product* of values over the vertices of the tree by the recursively defined times functional:

$$\chi(\hat{g},\hat{u}_0;\tau,t) = \begin{cases} \hat{u}_0(\xi), & \text{if } S_\theta > t,\\ \frac{\hat{g}(t-S_\theta,\xi)}{q\lambda(\xi)}, & \text{if } S_\theta < t, \kappa_\theta = 0\\ \frac{ib\xi}{p\lambda(\xi)} \cdot \chi(\hat{g},\hat{u}_0;\tau_{<1>},t-S_\theta) & \text{if } S_\theta < t, \kappa_\theta = 1, \end{cases} \tag{5.36}$$

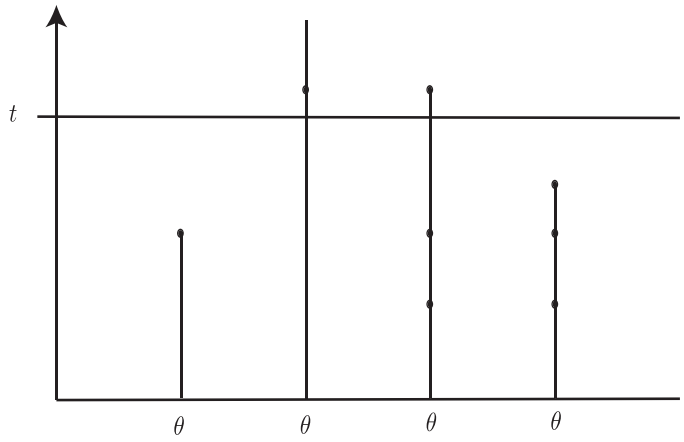


Figure 5.3: Linear cascade with forcing

where $\tau_{<1>}$ is again the lineal subtree of τ re-rooted at $< 1 >$ and S_θ is the replacement time for θ. Performing this iterated multiplication yields

$$\begin{aligned}
&\chi(\hat{g}, \hat{u}_0; \tau, t) = \hat{u}_0(\xi)\mathbf{1}[S_0 > t] + \frac{g(t - S_0, \xi)}{q\lambda(\xi)}\mathbf{1}[S_0 < t](1 - \kappa_0) \\
&+(\frac{ib\xi}{p\lambda(\xi)})^{N_t}\hat{u}_0(\xi)\mathbf{1}[S_0 + \cdots + S_{N_t-1} < t < S_0 + \cdots + S_{N_t}] \prod_{j=0}^{N_t-1} \kappa_j \mathbf{1}[N_t \geq 1] \\
&+(\frac{ib\xi}{p\lambda(\xi)})^{N_t-1}\frac{\hat{g}(t - S_0 - \cdots - S_{N_t-1}, \xi)}{q\lambda(\xi)}\mathbf{1}[S_0 + \cdots + S_{N_t-1} < t] \\
&\times \prod_{j=0}^{N_t-2} \kappa_j(1 - \kappa_{N_t-1})\mathbf{1}[N_t \geq 2]. \qquad (5.37)
\end{aligned}$$

Let $\{Y_t : t \geq 0\}$ denote a Poisson process with intensity $\lambda(\xi)$ and let W_q denote an independent geometrically distributed random variable with parameter p, ie. $P(W_q = j) = p^{j-1}q, j = 1, 2, \ldots$. Then

$$N_t =^{dist} Y_t \wedge W_q \qquad (5.38)$$

and

$$\prod_{j=0}^{N_t-1} \kappa_j =^{dist} \mathbf{1}[W_q > Y_t]. \qquad (5.39)$$

Note that the conditional distribution of N_t given the event $[\prod_{j=0}^{N_t-1} \kappa_j = 1]$ is Poisson with parameter $\lambda(\xi)$. Moreover, it is well-known that conditionally given Y_t, the jump times are distributed as the order statistics of Y_t iid values from the uniform distribution on $[0, t]$. In particular the conditional distribution of the jth jump time

has a density given by $j\binom{Y_t}{j}(s/t)^{j-1}(1-s/t)^{Y_t-j}(1/t)$. Thus we obtain a variant on the classical "Duhamel principle" viewed on the Fourier side. Namely,

$$\begin{aligned}\hat{u}(t,\xi) &= \mathbf{E}\chi(\hat{g},\hat{u}_0;\tau,t)\\ &= \mathbf{E}(\frac{ib\xi}{p\lambda(\xi)})^{Y_t}\hat{u}_0(\xi)\mathbf{1}[W_q > Y_t]\\ &+\mathbf{E}(\frac{ib\xi}{p\lambda(\xi)})^{W_q-1}\frac{\hat{g}(t-\sum_{j=1}^{W_q}S_j,\xi)}{q\lambda(\xi)}\mathbf{1}[W_q \le Y_t] \end{aligned} \tag{5.40}$$

so that

$$\begin{aligned}\hat{u}(t,\xi) &= \mathbf{E}\chi(\hat{g},\hat{u}_0;\tau,t)\\ &= \sum_{k=0}^{\infty}\sum_{j=k+1}^{\infty}p^{j-1}q(\frac{ib\xi}{p\lambda(\xi)})^k\frac{(\lambda(\xi)t)^k}{k!}e^{-\lambda(\xi)t}\hat{u}_0(\xi)\\ &+\sum_{k=1}^{\infty}\frac{(\lambda(\xi)t)^k}{k!}e^{-\lambda(\xi)t}\sum_{j=1}^{k}\int_0^t\frac{\hat{g}(t-s,\xi)}{q\lambda(\xi)}\frac{(ib\xi)^{j-1}}{(p\lambda(\xi))^{j-1}}p^{j-1}q\\ &\cdot\frac{k!}{(j-1)!(k-j)!}t^{-k}s^{j-1}(t-s)^{k-j}ds\\ &= \exp\{ib\xi t - a\xi^2 t\}\hat{u}_0(\xi) + \int_0^t\hat{g}(t-s,\xi)e^{\{ib\xi s - a\xi^2 s\}}ds\end{aligned} \tag{5.41}$$

Note that this imposes no further integrability constraints on the Fourier transform of the forcing than those required for the existence of the Fourier transform. Also, it is noteworthy that in the case of drift $b \neq 0$, the forcing adds an additional randomization to the offspring distribution ($p_0 = q, p_1 = p$) that was not required in the previous unforced case where $p_1 = 1$. In a sense which will become more transparent, this is the result of a competition of the lower order terms in the equation (i.e. lower order than the second order term).

Remark. Observe that in the above examples it is not necessary that the expectations be evaluated in closed form in order to prove that they solve the given equation. In particular, once it is known that the expectation is finite (well-defined) then it follows from the stochastic recursion that the equation must be satisfied by simply conditioning on the time S_θ relative to t. The closed form calculations were provided merely for familiarity as an aid to the exposition. Proceeding along these lines one may obtain a Feynman-Kac formula on the Fourier side by considerations of a linear production rate. In particular, as will be discussed in the lecture by Thomann, this provides a very simple probabilistic interpretation of the complex measure condition for Feynman-Kac obtained by Ito (1967).

Example 5.3 (Burgers Equation). Burgers equation is the one-dimensional nonlinear partial differential equation

$$u_t(t,x) + uu_x(t,x) = \nu u_{xx}(t,x) + f(t,x), \tag{5.42}$$

$\nu > 0$, equipped with various boundary and/or initial data. Let us consider Burgers equation (5.42) on $[0,1]$ with periodic L_1 initial data

$$u(0,x) = u_0(x) \tag{5.43}$$

and periodic boundary condition

$$u(t,0) = u(t,1), t \geq 0. \tag{5.44}$$

The term $f(t,x)$, referred to as a *forcing* term, is also minimally required to be an L_1-function. In view of a Galilei invariance lemma, cf. Foias (1994), without loss of generality we also assume

$$\int_{[0,1]} u_0(x)dx = \int_{[0,1]} f(t,x)dx = 0, \quad t \geq 0; \tag{5.45}$$

ie. in general there is a number $c = \int_{[0,1]} u_0(x)dx$ such that

$$w(t,x) := u(t, x + ct + \int_0^t \int_0^s \int_{[0,1]} f(\tau,x)dxd\tau ds) - \int_0^t \int_{[0,1]} f(s,x)dxds - c$$

solves the same problem with u_0 replaced by $u_0(x) = w(0,x) + c$ and f replaced by $\tilde{f}(t,x) = f(t,x) - \int_{[0,1]} f(t,x)dx$. Moreover

$$\int_{[0,1]} w(t,x)dx = \int_{[0,1]} \tilde{f}(t,x)dx = 0, \quad t \geq 0.$$

Let

$$u(t,x) = \sum_k \hat{u}(t,k)e^{ikt} \tag{5.46}$$

and

$$f(t,x) = \sum_k \hat{f}(t,k)e^{ikt}. \tag{5.47}$$

Then from (5.42) one obtains upon writing $uu_x = \frac{1}{2}(u^2)_x$ and denoting discrete convolution sums by $*$,

$$\frac{d\hat{u}(t,k)}{dt} + i\frac{1}{2}\hat{u} * \hat{u}(t,k) = -\nu k^2 \hat{u}(t,k) + \hat{f}(t,k). \tag{5.48}$$

Multiplying by the integrating factor $e^{\nu t^2}$ and making the indicated integration one obtains the multiplicative recursion for $k \neq 0$,

$$\frac{\hat{u}(t,k)}{\nu} = \frac{\hat{u}_0(k)}{\nu} e^{-\nu k^2 t}$$
$$+ \int_0^t \nu k^2 e^{-\nu k^2 (t-s)} [\frac{1}{2}\frac{1}{k}\sum_j (-i)\frac{\hat{u}(s,j)}{\nu}\frac{\hat{u}(s,k-j)}{\nu} + \frac{1}{2}\frac{2\hat{f}(s,k)}{\nu^2 k^2}] ds. \quad (5.49)$$

Thus it is convenient to introduce

$$\varphi(t,k) := \frac{2\hat{f}(t,k)}{\nu^2 k^2}, \qquad \chi(t,k) := \frac{\hat{u}(t,k)}{\nu}, \quad t \geq 0, k \in \mathbf{Z}. \quad (5.50)$$

Fix frequency $k \geq 1$ and time $t > 0$. Consider a branching random walk which holds in state k for an exponential time with parameter $\lambda(k) = \nu k^2$. When the clock rings, with probability $\frac{1}{2}$ the particle dies and replaced by no offspring, or with probability $\frac{1}{2}$ it is replaced by two frequencies j and $k-j$, with j picked uniformly over $1, \ldots k-1$. This process is repeated independently for each of the offspring frequencies; see Figure 5.4. Let $\partial_+(\omega_t)$ denote the vertices which are born before time t but not replaced by time t, and let $\partial_0(\omega_t)$ denote those vertices which are born before time t, but die and are replaced by no offspring before time t. At the input nodes we will attach the values $\chi_0(k_v)$ for $v \in \partial_0(\omega_t)$, and the values $\varphi(t-\tau_v, k_v)$ for $v \in \partial_+(\omega_t)$. For a given initial data, forcing, and viscosity, we build a multiplicative functional $\curlyvee(t,\omega)$ by recursive application of the following product: For each operational node, take $-i$ times the product of the pair of values attached to the corresponding pair of input nodes. Then $\curlyvee(t,\omega)$ is the value obtained by recursive application of this rule to ω_t.

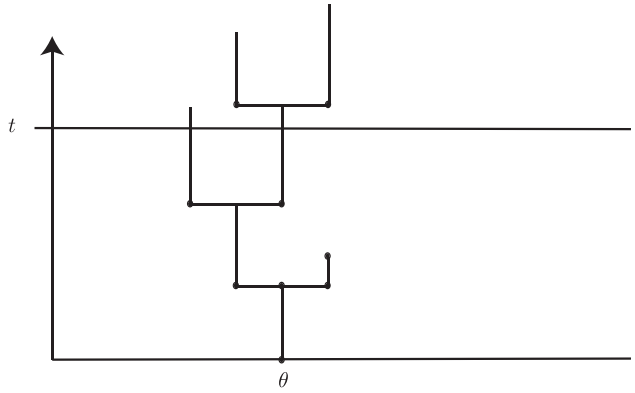


Figure 5.4: Branching random walk for Burgers' equation

Lemma 5.2 *Let $\chi_0(k)$ and $\varphi(t,k)$ be given measurable functions defined for each $k = 1, 2, \ldots, t \geq 0$. Define, for $k \geq 1, t > 0$*

$$\chi(k,t) = \mathbf{E}_k \curlyvee(t).$$

Then $\chi(t,k)$ satisfies

$$\begin{aligned}\chi(t,k) &= e^{-\nu k^2 t}\chi_0(k) \\ &+ \int_0^t \nu k^2 e^{-\nu k^2(t-s)}\{\frac{1}{2}(-i)\sum_{k_1,k_2\in\mathbf{Z}}\chi(s,k_1)\chi(s,k_2)p_k(k_1,k_2)+\frac{1}{2}\varphi(s,k)\}ds\end{aligned}$$

To prove this simply decompose the expected value according to the four categories of events which may occur and apply the Strong Markov property.

Lemma 5.3 *Suppose the progenitor of the branching process is of type $k_0 \neq 0$. Then,*

$$\sum_{v\in\partial_0(\omega_t)\cup\partial_+(\omega_t)} k_v = k_0.$$

To identify the class of initial data for which this recursion is valid we recall the theorem of Riesz characterizing functions whose Fourier transform vanishes for negative frequencies. Let us denote by $(H^p, ||\cdot||_p), 1 \leq p \leq \infty$, the usual Banach space of complex valued holomorphic functions h on the unit disc for which each $h_r(\theta) = h(re^{i\theta}), 0 < r < 1$ is bounded in L_p, with (eg. see Rudin (1966))

$$||h||_p = \begin{cases} \lim_{r\to 1}\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^+|f(re^{i\theta})|d\theta\right], & \text{if } p=1, \\ \lim_{r\to 1}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\theta})|^p d\theta\right]^{\frac{1}{p}}, & \text{if } 1<p<\infty \\ \lim_{r\to 1}\sup_\theta|f(re^{i\theta})|, & \text{if } p=\infty. \end{cases} \tag{5.51}$$

Equivalently by the Riesz Conjugacy Theorem, this space may be identified with the L_p-functions h on the unit circle whose Fourier coefficients $\hat{h}(k), k \in \mathbf{Z}$ vanish at negative frequencies, i.e.

$$\hat{h}(k) = \frac{1}{2\pi}\int_{[0,1]} h(x)e^{-ikx}dx = 0, \quad k = -1,-2,\ldots, \tag{5.52}$$

where extrapolation to the unit disc is via the Poisson Integral Formula. It is this latter version of H^p as a space of periodic functions which will be useful for us here.

Proposition 5.1 *Assume that $\hat{f}(t,k) = \hat{u}_0(k) = 0$, for all $k = 0,-1,-2,\ldots$. Also suppose that for some $\alpha > 0$ one has*

$$|\hat{u}_0(k)| \leq \nu e^{-\alpha k}, \qquad |\hat{f}(t,k)| \leq \frac{\nu^2 k^2}{2}e^{-\alpha k}, k = 1,2,\ldots, t \geq 0.$$

Then the complex Burgers equation has a unique classical solution $u(t,x)$. Moreover,

$$\hat{u}(t,k) = 0, k \leq 0 \quad \& \quad |\hat{u}(t,k)| \leq \nu e^{-\alpha k}, k > 0,$$

and $\hat{u}(t,k)$ is explicitly given by the formula

$$\hat{u}(t,k) = \mathbf{E}_k \chi(t).$$

To see this define

$$\chi_0(k) = \frac{\hat{u}_0}{\nu}, \quad \varphi(t,k) = \frac{2\hat{f}(t,k)}{\nu^2 k^2}, t \geq 0.$$

Then the key calculation is the following sample pointwise consequence of the conservation of frequency types given in Lemma 5.3:

$$|\chi(t)| = \prod_{v \in \partial_0(t)} |\chi_0(k_v)| \prod_{v \in \partial_+(t)} |\varphi(t-\tau_v, k_v)| \leq e^{-\alpha \sum_{k \in \partial_0(t) \cup \partial_+(t)} k_v} = e^{-\alpha k_0}.$$

Thus it follows that

$$|\chi(t,k)| \leq e^{-\alpha k}, \quad k = 1, 2, \ldots, t \geq 0.$$

Clearly

$$\chi(t,x) = \sum_{k=1}^{\infty} \chi(t,k) e^{ikx}$$

belongs to H^p for any $p \geq 1$. In particular, $\chi(t,x) \in L^2[0,1]$. From Lemma 5.2 it follows that

$$\frac{d\chi(t,k)}{dt} = -\nu k^2 \chi(t,k) + \frac{1}{2}(-i\nu k) \sum_{j=0}^{k} \chi(t,j)\chi(t,k-j) + \frac{1}{2}\nu k^2 \varphi(t,k).$$

Define

$$\hat{u}(t,k) = \nu \chi(t,k).$$

Then $\hat{u}(t,k) \in L^1[0,1]$, and

$$\frac{d\hat{u}(t,k)}{dt} = -\nu k^2 \hat{u}(t,k) + \frac{1}{2}(-ik) \sum_{j=0}^{k} \hat{u}(t,j)\hat{u}(t,k-j) + \hat{f}(t,k).$$

Applying the Fourier Inverson formula one now arrives at Burgers equation.

Example 5.4 (Fisher/KPP Equation). Consider

$$\frac{\partial u}{\partial t} = \nu u_{xx} + u^2 - u, \quad u(0,x) = u_0(x). \tag{5.53}$$

Taking Fourier transform and introducing the integrating factor $\exp\{-\lambda(\xi)t\}$, with

$$\lambda(\xi) = 1 + \nu|\xi|^2, \tag{5.54}$$

one has

$$\begin{aligned} \hat{u}(t,\xi) &= e^{\{-\lambda(\xi)t\}} \hat{u}_0(\xi) \\ &+ \int_0^t \int_R e^{-\lambda(\xi)s} \hat{u}(t-s, \xi-\eta)\hat{u}(t-s,\eta) d\eta. \end{aligned} \tag{5.55}$$

In general one may relate the representation of the Fourier transform of the solution of a general semilinear heat equation with quadratic nonlinearities in the form of an expected value, i.e.

$$\hat{u}(t,\xi) = h(t,\xi)\mathbf{E}_{(t,\xi)}(\chi(\hat{u}_0, \hat{g}, \tau_\xi, t)), \tag{5.56}$$

since the Fourier transform of a PDE with a quadratic nonlinearity can be written as

$$\frac{\partial \hat{u}}{\partial t}(t,\xi) = -\lambda(\xi)\hat{u}(t,\xi) + \gamma(\xi)\int \hat{u}(t,\xi-\eta) \otimes_\xi \hat{u}(t,\eta)d\eta + \hat{g}(t,\xi). \tag{5.57}$$

For example the meaning of λ, γ and the operation at the node for the Navier Stokes equation are given by

$$\lambda(\xi) = \nu|\xi|^2, \qquad \mathbf{a} \otimes_\xi \mathbf{b} = \left(\mathbf{a} \cdot \frac{\xi}{|\xi|}\right)\pi_\xi(\mathbf{b}) \tag{5.58}$$

where π_ξ denotes the projection perpendicular to the direction of ξ and $\gamma(\xi) = |\xi|$.

6 Branching Brownian Motion, Fisher/KPP Equation, Burgers Equation: Real Space Cascades

The role of branching Brownian motions in the representation of solutions to Fisher's equation were uncovered in a classic paper by McKean (1975). This is a real space solution, but has a dual expression on the Fourier side along the lines indicated for Navier-Stokes in the previous section; see Bhattacharya et al (2001). Thus it is natural to inquire about real space branching representations to Navier-Stokes. While this question is largely open for Navier-Stokes equation, the one-dimensional case of Burgers' equation admits such a real space representation.

Let us first consider McKean's (1975) representation of solutions to the Fisher/KPP equation. Recall that the solution of

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u^2 - u, \quad u(x,0) = u_0(x). \tag{6.1}$$

is represented as (see Figure 6.1)

$$u(t,x) = \mathbf{E}\prod_{j=1}^{N_t} u_0(x + B^j(t)) \tag{6.2}$$

This representation can be obtained by noting that from Duhamel's Principle discussed in Section 2, one may write

$$u(t,x) = e^{-t}\int_R g(t,x-y)u_0(y)dy + \int_0^t \int_R e^{-s} g(t-s,x-y)u^2(y,s)dyds$$

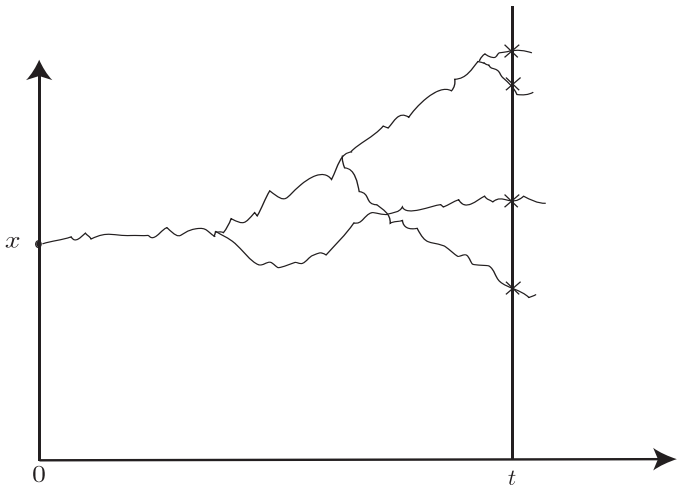


Figure 6.1: Branching Brownian motions

where $g(t, x-y) = p(t, y, x)$ is the Gaussian kernel (2.7) with $v = 0$ and $D = 1$. Since $\hat{g}(t, \xi) = e^{-\frac{1}{2}|\xi|^2 t}$ it follows that with $\lambda(\xi) = 1 + \frac{1}{2}|\xi|^2$

$$\hat{u}(t, \xi) = e^{-\lambda(\xi)t}\hat{u}_0(\xi) + \int_0^t e^{-\lambda(\xi)s} \int \hat{u}(t-s, \xi-\eta)\hat{u}(t-s, \eta)d\eta ds$$

One may check that the Fourier transform of the formula of McKean (1975) corresponds to the representation of the Fourier transform of solution as an expected value of the branching process on the Fourier side.

It is perhaps a bit more interesting that one may obtain similar dual branching random walk products in the case of Burgers' equation as well; cf Bhattacharya et al (2001). Suppose that u solves Burgers equation (5.42) and define

$$v(t, x) = e^{-\delta t}u(t, x). \tag{6.3}$$

Then one has (taking $\mu = \frac{\nu}{2}$ for convenience)

$$\frac{\partial v}{\partial t} = \frac{\nu}{2}v_{xx} + e^{\delta t}vv_x \tag{6.4}$$

We refer to this transformation as the *δ-method.* This will be revisited in the research talk by C. Orum for the 2-d Navier-Stokes equation.

Regarding the non-linearity $vv_x = \frac{1}{2}(v^2)_x$ as a forcing term one has again by a Duhamel principle that

$$\begin{aligned} v(t, x) &= \int_R \tilde{g}(t, x-y)v_0(y)dy \\ &+ \int_0^t \int_R \tilde{g}(t-s, x-y)\frac{1}{2}(v^2)_x(s, y)dyds, \end{aligned} \tag{6.5}$$

where $\tilde{g}(s,z) = e^{-\delta s}g(s,z) = e^{-\delta s}\frac{1}{\sqrt{2\pi\nu s}}e^{\frac{-1}{2\nu s}z^2}$ is the Green's function for the heat equation with sink term $-v$ (i.e. a Feynman-Kac term). Now, performing an integration by parts in the second integral and noting that $\frac{\partial g(s,z)}{\partial z} = -\frac{z}{\nu s}g(s,z)$, yields

$$\begin{aligned} v(t,x) &= \int_R e^{-\delta t}g(t,x-y)v_0(y)dy \\ &+ \int_0^t \int_R e^{-\delta s}(\frac{x-y}{\nu(t-s)})g(t-s,x-y)\frac{1}{2}(v^2)(s,y)dyds. \end{aligned} \tag{6.6}$$

This recursion now admits a representation as a Branching Brownian motion with multiplicative factors as follows: First, let us introduce a canonical model for binary Branching Brownian motions. Let

$$\mathcal{V} = \cup_{n=0}^{\infty}\{1,2\}^n, \quad \{1,2\}^0 = \{\theta\}.$$

Each particle performs a Brownian motion until its death, upon which it is replaced by two Brownian particles started at the same location as the parent upon its death. Brownian particles have independent exponentially distributed life-lengths S_v. Individual motions are then paths $\{B_v(t) : t_v \le t \le t_v + S_v\}$, where $t_v = \sum_{j=0}^{|v-1|} S_{<v_1,\dots,v_j>}$ is the time of birth of the "$v-$th Brownian motion", such that $B_\theta(0) = 0, B_{vj}(t_{vj}) = B_v(t_v + S_v), v \in \mathcal{V}, j = 1,2$.

$$\curlywedge(t,x) = \begin{cases} u_0(x+B_t), & \text{if } S_\theta > t, \\ \frac{e^{t-S_\theta}}{2S_\theta}B_{S_\theta}\cdot\curlywedge^{(1)}(t-S_\theta,B_{S_\theta})\curlywedge^{(2)}(t-S_\theta,B_{S_\theta}) & \text{if } S_\theta < t, \end{cases} \tag{6.7}$$

Thus, iterating this stochastic recursion one obtains

$$\begin{aligned} v(t,x) &= E\curlywedge(t,x) \\ &= E\big\{(\frac{e^t}{2})^{N_t^{(0)}}\prod_{v\in A(t)}u_0(x+B_{<v>}(t)) \\ &\qquad \prod_{v\in A^{(0)}(t)} e^{(t-\sum_{j=0}^{|v|}S_{<v_1,\dots,v_j>})}\frac{\Delta B_{<v>}(t_v)}{S_v}\big\}, \end{aligned} \tag{6.8}$$

where

$$A(t) = \{v : t_v \le t \le t_v + S_v\} \tag{6.9}$$

is the set of particles alive at time t,

$$A^{(0)}(t) = \{v : t_v + S_v \le t\} \tag{6.10}$$

the set of those particles born and replaced before time t, and

$$\Delta B_{<v>}(t_v) = B_{<v>}(t_v + S_v) - B_{<v>}(t_v). \tag{6.11}$$

In particular, in view of the nonlinearity uu_x it is interesting to note the presence of the factors $u_0(x + B_{<v>}(t))\frac{\Delta B_{<v>}(t_v)}{S_v}$.

A more complete treatment of the methods given in this note are being prepared for a monograph by Bhattacharya et al (2001). These will include applications to linear and nonlinear equations of the type discussed in these lecture notes.

Acknowledgements The author is grateful to the workshop participants for many helpful comments and remarks during the course of these lectures. Special thanks go to Ole Barndorff-Nielsen for organizing and hosting this workshop as part of the MaPhySto program at Aarhus University. Also the author is very grateful to Matthias Winkel for providing a detailed errata for the original draft of these lecture notes.

References

[1] David Aldous (1991): The continuous random tree II: an overview. Durham Symp. Stochastic Analysis, eds M.T. Barlow and N.H. Bingham. *Cambridge University Press.*

[2] Aldous, D. (1993): The continuous random tree III. *Ann. of Probab.* **21** , 248-289.

[3] Austin, P. M. and R. A.Houze (1972): Analysis of the structure of precipitation patterns in New England. *J. Appl. Met.* **11**, 926-935.

[3b] Barndorff-Nielsen, O.E., J.L. Jensen and M. Sørensen (1990) Parametric modeling of turbulence. *Phil. Trans. R. Soc. Lond. A* **332**, 439-455.

[4] M. Barnsley (1988): Fractals Everywhere. *Academic Press* NY.

[5] Bell, T. L. (1987): A space-time stochastic model of rainfall for satellite remote sensing studies. *J. Geophys. Res.* **92**(D8), 9631-9643.

[6] Bhattacharya, R.N. and Waymire, E.C. (1990): *Stochastic Processes with Applications.* Wiley, New York.

[7] Bhattacharya R., L.Chen, S. Dobson, R. Guenther, C. Orum, M. Ossiander, E. Thomann, E. Waymire (2001): Multiplicative Cascades and Majorization of Linear, Semilinear and Quasilinear Partial Differential Equations (in preparation).

[8] Burd, G. and E. Waymire (2000): Independent random cascades on Galton-Watson trees. *Proc. AMS* **128**), 2753-2761.

[9] Burd, G., E. Waymire, R. Winn (2000): Self-similar invariance of critical binary Galton-Watson trees. *Bernoulli* **6**, 1-21.

[10] Chabra, A. and R.V. Jensen (1989): Direct determination of the $f(\alpha)$ singularity spectrum. *Phys. Rev. Let.* **62**, 1327-1330.

[11] Chayes, J., Chayes, and L. Durrett, R. (1988): Connectivity properties of Mandelbrot's percolation. *Probab. Th. Rel. Fields* **77**, 307-324.

[12] K.L. Chung (1976): Excursions in Brownian motion. *Ar. Mat.* **14**, 155-177.

[13] Collet, P. and F. Koukiou (1992): Large deviations for multiplicative cascades. *Commun. Math. Phys.* **147**, 329-342.

[14] Constantin, P. (1994): Geometric statistics in turbulence. *SIAM Rev.* **36**, 73-98.

[15] Constantin, P. and C. Fefferman (1994): Scaling exponents in fluid turbulence: Some analytic results. *Nonlinearity* **7**, 41-57.

[16] Courant, R. and D. Hilbert (1953): Methods of mathematical physics I. *Interscience Publ.,* New York.

[17] Drmota, M. and B. Gittenberger (1997): On the profile of random trees. *Rand. Structures Alg.* **10**, 421-451.

[18] Dubrulle, B. (1994): Intermittancy in fully developed turbulence: logPoisson statistics and generalized scale invariance. *Phys. Rev. Lett.* **73**, 959-962.

[19] R. T. Durrett and D. L. Iglehart (1977): Functionals of Brownian meander and Brownian excursion. *Ann. Probab.* **5**, 130-135.

[20] R.T. Durrett, H. Kesten, and E. Waymire (1991): On weighted heights of random trees. *Journal Theor. Probability* **4**, 223-237.

[21] Eggleston, H.G. (1949): The fractional dimension of a set defined by decimal properties. *Quart. J. Math. Oxford Ser.* **20**, 31-36.

[23] Ershov, A.P. (1958): On programming of arithmetic operations. *Comm. ACM* **1**, 3-6.

[24] Falconer, K. (1990): *Fractal Geometry.* Wiley, N.Y.

[25] Falconer,K. (1993): The Multifractal Spectrum of Statistically Self - Similar Measures (preprint).

[26] Fife, P. (1979): Mathematical Aspects of Reacting Diffusing Systems. Lecture Notes in Biomathematics **28**. Springer-Verlag, NY.

[27] Fisher, R.A. (1937): The wave of advance of advantageous genes. *Ann. of Eugenics* **7**, 355-369.

[28] Foias, C. (1995): What do the Navier-Stokes equations tell us about turbulence. *The Inst. for Scientific Comput. and Applied Math.* **9620** (preprint), Indiana University.

[29] Flajolet, P. and H. Prodinger (1986): Register allocation for unary-binary trees. *SIAM J. Comput.* **15**, 629-640.

[29b] Foias, C., O. Manley, R. Rosa, R. Temam (2001): *Navier-Stokes Equations and Turbulence.* Encyclopedia of Math. and its Appl. Cambridge Univ. Press.

[30] Franchi, J. (1995): Chaos multiplicatif: un traitment simple et complet de la fonction de partition. *Lecture Notes in Math: Sem. de Prob.* **1613**. Springer-Verlag, Berlin.

[31] Frisch, U. (1999): *Turbulence: The legacy of Kolmogorov.* Cambridge Univ. Press, Cambridge.

[32] Gilbert, A.C., W. Willinger, A. Feldman (1999): Scaling analysis of conservative cascades with applications to network traffic. *IEEE Trans. Inform. Theory.* **45**, 971-991.

[33] Gupta, V.K. and E. Waymire (1993): A statistical analysis of mesoscale rainfall as a random cascade. *Jour. Appl. Meteor.* **32**, 251-267.

[34] Gupta, V. K., S. Castro and T. M. Over (1996): On scaling exponents of spatial peak flows from rainfall and river network geometry. *J. Hydrol.* **187**, 81-104.

[35] Gupta, V. K. and D. R. Dawdy (1994): Regional analysis of flood peaks: multiscaling theory and its physical basis. *Advances in Distributed Hydrology,* R. Renzo (editor), 149-168. Water Resources Publications, Highlands Ranch, Colo.

[36] Gupta, V.K., O. Mesa, E. Waymire (1900): Tree-dependent extreme values: The exponential case. *J. Appl. Prob.* **27**, 124-133.

[37] Gupta V.K. and E. Waymire (1998): Scale invariance and regionalization of floods. in: *Scale Dependence and Scale Invariance,* ed. G. Sposito, Cambridge University Press, Cambridge, pp. 88-135

[38] Hack, J.T. (1957): Studies of longitudinal stream profiles in Virginia and Maryland. *Geol. Survey Professional Paper* **294-B**.

[39] Holley, R. and E. Waymire (1992): Multifractal dimensions and scaling exponents for strongly bounded random cascades. *Ann. Applied Probab.* **2**, 819-845.

[40] Harris, T.E. (1985): *The Theory of Branching Processes.* Dover Publ. Inc., New York.

[41] Helgason, H. (1983): *Harmonic Analysis,* 2nd ed. The Crescent, Berkeley.

[42] Horton, R. (1945): Erosional development of streams and their drainage basins. *Geol. Soc. Amer. Bull.* **59**, 275-370.

[43] Ito, K. (1970): Poisson Point Processes Attached to Markov Processes. *Proc. Sixth Berkeley Symp. Math. Stat. Probab.* 225-23. UC Berkeley Press, Berkeley.

[43b] Ito, K. (1967): Generalized uniform complex measures in the Hilbertian metric space with application to the Feynam path integral, *Proc. 5th Berkeley Symp. on Math. Stat. and Probab.* Vol II, Part I, 145-161. Univ. Cal. Berkeley.

[44] Jarvis, R. S., and M. J. Woldenberg, (Eds.) (1984): River Networks *Benchmark Papers in Geology* **80**. Hutchinson Ross. Stroudsburg, PA.

[45] Joffe, A., L. LeCam, and J. Neveu (1973): Sur la loi des grands nombres pour des variables aléatoires de Bernoulli attachées à un arbre dyadique. *C.R. Acad. Sci. Paris Sér. A-B.* **277**, A963-A964.

[45b] Jouault, B. M. Greiner, P.Lipa (2000): Fix-point multiplier distributions in discrete turbulent cascade models. *Physica D* **136**, 125-144.

[46] Kahane, J.P. (1989): Random multiplications, random coverings, and multiplicative chaos, *Proc. Special Year in Modern Analysis. Cambridge Univ. Press.* (Eds. E. Berkson, N. Tenny Peck, J. Jerry Uhl.) **137**, 196-255.

[47] Kahane, J.P. and J. Peyrière. (1976): Sur certaines martingales de B. Mandelbrot. *Adv. in Math.* **22**, 131-145.

[48] Kaigh, W.D. (1976): An invariance principle for random walk conditioned by a late return to zero. *Ann. Probability* **4**, 115-121.

[49] Katz, R.W. (1985): Probabilistic models (eds A.H. Murphy, R.W. Katz). *Probability, Statistics, and Decision Making in the Atmospheric Sciences.* NCAR, Boulder, CO.

[50] Kesten, H. (1994): A limit theorem for weighted branching process trees. *The Dynkin Festschrift: Prog. in Prob.* **34**, 153-166. Birkhauser Boston, Boston,

[51] Kolmogorov, A.N. (1941): The local structure of turbulence in incompressible viscous fluid for very large Reynolds number. *Dokl. Akad. Nauk SSSR.* **30**, 9-13.

[52] Kolmogorov, A.N. (1962): A refinement of previous hypothesis concerning the local structure of turbulence in a viscous incompressible fluid at high Reynold number. *J. Fluid Mech.* **13**, 82-85.

[53] Kolmogorov, A.N., I.G. Petrovskii, N.S. Piskunov (1937): A study of the equation of diffusion with increase in the quantity of matter, and its application to a biological problem. *Bjul. Moskovskovo Gos. Univ* **17**, 1-72.

[54] Kreiss, H.O. and J. Lorenz (1989): Initial-Boundary Value Problems and the Navier-Stokes Equations. Academic Press, New York.

[55] LeCam, L. (1961): A stochastic description of precipitation. *4th Berkeley Symposium on Mathematical Statistics,and Probability* **3**, 165-186. Univ. of California,Berkeley, CA.

[56] Leheny, R.L., and S.R. Nagel (1993): Model for the evolution of river networks. *Phys. Rev. Letters* **71**, 1470-1473.

[57] LeJan, Y. and A.S. Sznitman (1997): Stochastic cascades and 3-dimensional Navier-Stokes equations. *Prob. Theory and Rel. Fields* **109**, 343-366.

[58] Leopold, L. B., M. G. Wolman, and J. P. Miller (1964): *Fluvial Processes in Geomorphology.* W. H. Freeman, San Francisco.

[59] Lovejoy, S. and D. Schertzer (1990): Multifractals, universality classes, and satellite and radar measurements of cloud and rain fields. *J. Geophys. Res.* **95**,(D3) 2021-2031.

[60] Lovejoy, S. and D. Schertzer (1991): Universal hard multifractal turbulence: theory and observations. *Nonlinear Dynamics of Structures* (Eds R.Z. Sagdeev, U. Frisch, F. Hussain, S.S. Moisdeev, N.S. Erokhin.) World Scientific Press.

[61] Lyons, R. (1990): Random Walks and Percololation on Trees. *Ann.Prob.* **18**, 931-958.

[62] Mandelbrot, B. B. (1982): *The Fractal Geometry of Nature* Freeman, San Francisco, CA.

[63] Mandelbrot, B. (1998): *Fractals and scaling in finance: Discontinuity, concentration, risk.* Springer-Verlag, New York.

[64] Mandelbrot, B. (1974): Intermittent turbulence in self-similar cascades: Divergence of high moments and dimension of the carrier. *J. Fluid Mech.* **62**, 331-333.

[65] Marani A., R. Rigon, A. Rinaldo (1991): A note on fractal channel networks. *Water Resour. Res.* **27**, 3041-3049.

[66] Marani, M, A. Rinaldo, R. Rigon and I. Rodriguez-Iturbe (1994): Geomorphological width functions and random cascade. *Geop. Res. Letters* **21**, 2123-2126.

[67] Meir, A., J.W. Moon, J.R. Pounder (1980): On the order of random channel networks. *SIAM J. Alg. Disc. Meth.* 25-33.

[68] Molchan, G. M. (1996): Scaling exponents and multifractal dimensions for independent random cascades. *Comm. Math. Phys.* **179**, 681-702.

[69] Molchan, G. M. (1997): Turbulent cascades: limitations and a statistical test of the lognormal hypothesis. *Phys. Fluids* **9**, 2387-2396.

[70] Neveu, J. (1986): Erasing a branching tree. *J. Appl. Probab.* 101-108.

[71] Neveu, J. and J. Pitman (1989): The branching process in a Brownian excursion. *Lecture Notes in Mathematics*, 248-257. Springer-Verlag, NY.

[72] Ngyuen, B. (1990): Percolation of coalescing random walks. *J. Appl. Probab.* 269-277.

[73] Ossiander, M., E. Waymire, Q. Zhang (1997): Some width function asymptotics for weighted trees. *Ann. Appl. Probab.* **3**, 972-995.

[74] Ossiander, M.E. (2000): Support fragmentation for multiplicative cascade measures. *Progress in Probability, Proceedings of High Dimensional Probability II, Seattle 1999.* (Eds. E. Gine, D. M. Mason, J. A. Wellner.) Birkhauser, Boston. (In press).

[75] Ossiander, M.E. and E.C. Waymire (2000): Statistical estimation theory for multiplicative cascades. *Ann. Stat.* **28**, 1-21.

[75b] Ossiander, M.E. and E.C. Waymire (2002): On estimation theory for multiplicative cascades. *Sankhya* (to appear)

[76] Over, T. (1995) Modeling space-time rainfall at the mesoscale using random cascades. Ph.D. dissertation, Hydrologic Sciences Program, *University of Colorado,* Boulder.

[77] Over, T. and V.K. Gupta (1994): Statistical analysis of meso-scale rainfall: Dependence of a random cascade generator on large-scale forcing. *J. Appl. Meteorology* **33**, 1526-1542.

[78] Paley, R. and N.Wiener (1934): Fourier transforms in the complex domain. NY. 140-162.

[79] Peckham, S. (1995a): New results for self-similar trees with applications to river networks. *Water Resour. Res.* **31**, 1023-1029.

[80] Peyrière, J. (1977): Calculs de Dimensions de Hausdorff. *Duke Mathematical Journal* **44**, 591-601.

[81] Phelan, M. and C. Goodall (1990): An assessment of generalized WGR model for GARP Atlantic Tropical Experiment rainfall. *J. Geophys. Res.* **95**, 7603-7615.

[82] Resnick, S., A. Gilbert, W. Willinger (2000): Wavelet analysis of conservative cascades. (Pre-print).

[83] Rodriguez-Iturbe, I., D. Cox, V. Isham (1987): Some models for rainfall based on stochastic point processes. *Proc. R. Soc. London, Ser A* **(410)**, 269-288.

[84] Rodriguez-Iturbe, I. (1996): Scale of fluctuation of rainfall models. *WRR* 15S-37S.

[85] Rudin, W. (1966): *Real and Complex Analysis.* McGraw-Hill, New York.

[86] Scheidegger, A.E. (1967): A thermodynamic analogy for meander systems. *Water Res. Res.* **3**, 1041-1046.

[87] She, Z.S. and E. Waymire (1994): LogPoisson statistics in fully developed turbulence. *MSRI Series* **No. 046-94** Berkeley, CA.

[88] She, Z.S. and E. Waymire (1995): Quantized energy cascade and logPoisson statistics in fully developed turbulence. *Phys. Rev. Lett.* **74**, 262-265.

[89] R.L. Shreve (1966): Statistical law of stream numbers. *Journal of Geology* **74**, 17-37.

[90] Shreve, R. L. (1967): Infinite topologically random channel networks. *J. Geol.* **75**, 178-186.

[91] Smith, J and A. Karr (1985): Parameter estimation for a model of space-time rainfall. *Water Resour. Res.* **21**, 1251-1257.

[92] Strahler, A.N. (1957): Quantitative analysis of watershed geomorphology. *Eos Trans. AGU* **38**, 913-920.

[93] Tokunaga, E. (1966): The composition of drainage networks in Toyohira river basin and valuation of Horton's first law (In Japanese with english summary). *Geophys. Bull. Hokkaido Univ.* **15**, 1-19.

[94] Tokunaga, E. (1978): Consideration on the composition of drainage networks and their evolution. *Geogr. Rep.* **13**, Tokyo Metrop. Univ.

[95] Troutman, B. (1996): Inference for a channel network model and implications for flood scaling. *Reduction and Predictability of Natural Disasters, Edited by J. Rundle, W. Klein and D. Turcotte, SantaFe Institute, XXV* Addison-Wesley.

[96] Troutman, B. and A. Vecchia (1999): Estimation of Renyi exponents in random cascades. *Bernoulli* **5**, 191-207

[97] Troutman, B. and M. Karlinger (1993): A note on subtrees rooted along the primary path of a binary tree. *J. Appl. Discrete Math.* **42**, 87-93.

[98] Troutman, B., and M. Karlinger (1984): On the expected width function of topologically random channel networks. *J. Appl. Prob.* **22**, 836-849.

[99] Troutman and Karlinger (1988): Asymptotic Rayleigh instantaneous unit hydrograph. *Stoch. Hydrol. Hydraul.* **2**, 73-78.

[100] Troutman, B. and M. Karlinger (1998): Spatial channel network models in hydrology. *Stoch. Meth. in Hydrology* eds O.E. Barndorff-Nielsen, V.K. Gupta, V. Perez-Abreu, E. Waymire, Adv. Series on Stat. Science & Appl. Probab. **7**, World Scientific Press, Singapore, 85-121.

[101] Veitzer, S. (1999) A theoretical framework for understanding river networks: Connecting topology, processes, and geometry across many scales. PhD Thesis, Dept. of Physics, U. of Colorado, Boulder.

[102] Veitzer, S. and V.K. Gupta (2000): Random self-similar river networks and derivations of generalized Horton laws in terms of statistical simple scaling. *Water Resour. Res.* **36**, 1033-1048.

[103] Wang, X. and E. Waymire (1991): A large deviation rate and central limit theorem for Horton ratios. *SIAM J. Disc. Math.* **4**, 575-588.

[104] Waymire, E. (1992): On network structure function computations. *New Direction in Time Series Analysis* **II.** Springer-Verlag, Berlin.

[105] Waymire, E. C., V. K. Gupta, and I. Rodriguez-Iturbe (1984): A spectral theory of rainfall intensity at the meso-β scale. *Water Resour. Res.* **20**, 1453-1465.

[106] Waymire, E. and S.C. Williams (1994): A general decomposition theory for random cascades. *Bull Amer. Math. Soc.* **31**, 216-222.

[107] Waymire, E. and S.C. Williams (1995): Multiplicative cascades: Dimension spectra and dependence. *Jour. Fourier Anal. and Appl.* **Special Edition** 589-609.

[108] Waymire, E. and S.C. Williams (1996): A cascade decomposition theory with applications to Markov and exchangeable cascades. *Trans. Amer. Math. Soc.* **348**, 585-632.

[109] Waymire, E. and S.C. Williams (1997): Markov cascades *In: Classical and Modern Branching Processes.* R.B. Athreya and P. Jagers, eds., **84**, 305-321. Springer Verlag, NY.

[110] Whitham, G.B. (1974): *Linear and Nonlinear Waves.* Wiley, New York.

[111] Yaglom, A.M. (1966): Effect of fluctuations in energy dissipation rate on the form of turbulence characteristics in the inertial subrange. *Dokl. Akad. Nauk SSSR.* **166**, 49-52.

[112] Yekutieli, I. and B. Mandelbrot (1994): Horton-Strahler ordering of random binary trees. *J. Phys. A. Math. Gen.* **27**, 285-293.

[113] Zawadski, I. (1973): Statistical properties of precipitation patterns. *J. Appl. Meteor.* **12**, 459-472.

MULTISCALE DIFFUSION EQUATIONS

RABI BHATTACHARYA
Department of Mathematics, Indiana University, Bloomington, IN 47405, U.S.A.

In geosciences multiscale phenomena occur in problems of transport in inhomogeneous media with evolving heterogeneities, such as solute transport in saturated aquifers. For an understanding of this complex phenomenon consider a Fokker-Planck equation governing solute concentration

$$c(t,y) : \partial c/\partial t = -\left(b(y) + \beta(y/a)\right) \cdot \bigtriangledown c + \frac{1}{2} \sum_{i,j} \partial^2/\partial y_i \partial y_j (Dij(y)c), \tag{1}$$

with initial condition $\delta_{x_0}(dy)$. Here 'a' is a large parameter, $b(\cdot)$ and $\beta(\cdot/a)$ represent 'local' and 'largescale' divergence free velocity fields, respectively, $D(y) \equiv ((Dij(y)))$ is positive definite. Since the solution to this equation is $c(t,y) = p(t; x_0, y)$, where $p(t; x, y)$ is the transition probability density of a Markov process (diffusion) $\{X_t\}$ with drift $b(\cdot) + \beta(\cdot/a)$ and diffusion matrix $D(y)$, one may derive asymptotic properties of $c(t,y)$ from those of $\{X_t\}$.

It turns out that the fluctuations of the slowly evolving large scale velocity field $\beta(\cdot/a)$ may be ignored, i.e., $\beta(\cdot/a)$ may be replaced by its constant initial value, for times $t << a^{2/3}$ during which time the dynamics is governed essentially by the fluctuations of $b(\cdot)$. For larger times the large scale fluctuations of $\beta(\cdot/a)$ come into play and are often dominant.

To gain further insight consider some classes of velocity fields $b(\cdot), \beta(\cdot)$ which give rise to different Gaussian phases and phase transitions as time progresses, as observed in many hydrologic experiments. For this let $b(\cdot)$ and $\beta(\cdot)$ be divergence free and periodic with the same period, 'a' positive integral (parameter) and $D(y) = D$ a constant positive definite matrix. For times $1 << t << a^{2/3}$ the process $\{X_t\}$ (and, therefore, $c(t,y)$) is asymptotically Gaussian whose dispersion depends on $b(\cdot)$, but not on $\beta(\cdot)$. For times $t >> a^2$ a second (and final) Gaussian phase occurs whose dispersion coefficients either grow quadratically with 'a', or to a constant (in 'a'), depending on the geometry of $\beta(\cdot)$. This provides an explanation of the so-called 'scale effect' in dispersion observed in many experiments. For a special class of stratified media, a range of non-Gaussian phase transitions are shown to occur in between the two Gaussian phases considered above.

Precise statements of the results outlined above and their detailed proofs may be found in [3], where a logarithmic factor $\log a$ or $(\log a)^2$ appears with the time scale $t >> a^2$ in the final phase. These results first appeared in [5], but with a gap in the proofs rectified in [3]. Apart from the CLT for diffusions with periodic coefficients (see [1],[2]), these proofs make use of the spectral theory of compact skew symmetric operators to estimate the growth in dispersion with 'a' ([3],[5],[7]), and estimates of the speed of convergence to equilibrium of a diffusion on a 'big' torus $\{x \in \mathbb{R}^k : x \bmod a\}$ ([3],[5],[11],[12],[19]). The removal of the logarithmic factor from, $a^2 \log a$ or $a^2 (\log a)^2$ (see [3]) is achieved by using a recent result on the speed of convergence to equilibrium given in [13].

For earlier literature on the 'scale effect' in dispersion we refer to [6],[15],[17],[18],[22]. For general surveys in the engineering literature on this and related topics, see [14],[10].

It is of much interest to consider more general models (1) where the coefficients comprise a realization from an ergodic random field. In the absence of a scale parameter 'a', CLT's under this ergodic random field assumption are proved in [20],[21], when the parabolic operation in (1) is in divergence form. Formal derivations for more general parabolic equations (with coefficients realized from an ergodic random field), appropriate for solute transport in a porous medium, are given in [15],[22], again without the introduction of multiple scales.

From a mathematical point of view it is of interest to consider nondivergence-free velocity fields $b(\cdot), \beta(\cdot)$. The techniques used for the divergence free case in [3],[5],[7] break down in this case. For the one-dimensional case, however, one may investigate this problem combining explicit computations and asymptotic analysis (see [3],[4]), thus providing an insight into the general nondivergence-free case.

References

[1] BENSOUSSAN, A., LIONS, J. L. and PAPANICOLAOU, G. C. (1978). *Asymptotic Analysis for Periodic Structures.* North-Holland, Amsterdam.

[2] BHATTACHARYA, R. N. (1985). A central limit theorem for diffusions with periodic coefficients, *Ann. Probab.* **13** 385-396.

[3] BHATTACHARYA, R. N. (1999). Multiscale diffusions with periodic coefficients and an application to solute transport in porous media. *Ann. Appl. Prob.* **9** 951-1020.

[4] BHATTACHARYA, R. N., DENKER, M. and GOSWAMI, A. (1999). Speed of convergence to equilibrium and to normality for diffusions with multiple periodic scales. *Stochastic Process. Appl.* **80** 55-86.

[5] BHATTACHARYA, R. N. and GÖTZE, F. (1995). Time-scales for Gaussian approximation and its break down under a hierarchy of periodic spatial heterogeneities. *Bernoulli* **1** 81-123. Correction, ibid.(1996), **2** 107-108.

[6] BHATTACHARYA, R. N. and GUPTA, V. K. (1983). A theoretical explanation of solute dispersion in saturated porous media at the Darcy scale. *Water Resour. Res.* **19** 938-944.

[7] BHATTACHARYA, R. N., GUPTA, V. K. and WALKER, H. F. (1989). Asymptotics of solute dispersion in periodic porous media. *SIAM J. Appl. Math.* **49** 86-98.

[8] BHATTACHARYA, R. N. and RAMASUBRAMANIAN, S. (1988). On the central limit theorem for diffusions with almost periodic cofficients. *Sankhyã Ser. A* **50** 9-25.

[9] BHATTACHARYA, R. N. and WAYMIRE, E. C. (1990). *Stochastic Processes with Applications*. Wiley, New York.

[10] CUSHMAN, J. (ed.) (1990). *Dynamics of Fluid in Hierarchical Porous Media.* Academic Press, New York.

[11] DIACONIS, P. and STROOCK, D. W. (1991). *Geometric bounds for eigenvalues of Markov chains. Ann. Appl. Probab.* **1** 36-61.

[12] FILL, J. A. (1991). Eigenvalue bounds on convergence to stationarity for nonreversible Markov chains, with an application to the exclusion process. *Ann. Appl. Probab.***1** 62-87.

[13] FRANKE, B. (2001). *Heat Content Inequalities for Diffusions on Manifolds.* Dissertation zur Erlangung des Doktorgrades. Georg-August-Universität zu Göttingen.

[14] FRIED, J. J. and COMBARNOUS, M. A. (1971). Dispersion in porous media. *Adv. Hydrosci.* **7** 169-282.

[15] GELHAR, L. W. and AXNESS, C. L. (1983). Three-dimensional stochastic analysis of macrodispersion in aquifers. *Water Resour. Res.* **19** 161-180.

[16] GLIM, J. and SHARP, D. H. (1997). Multiscale Science: A challenge for the 21st century. *Siam News* **30**, No. 8.

[17] GUPTA, V. K. and BHATTACHARYA, R. N. (1986). Solute dispersion in multidimensional periodic saturated porous media. *Water Resour. Res* **22** 156-164.

[18] GUVEN, O. and MOLZ. F. J. (1986). Deterministic and stochastic analysis of dispersion in an unbounded stratified porous medium. *Water Resour. Res.* **22** 1565-1574.

[19] HOLLEY, R. A., KUSUOKA, S. and STROOCK, D. W. (1989). Asymptotics of the spectral gap, with applications to simulated annealing. *J. Funct. Anal.* **83** 333-347.

[20] KOZLOV, S. M. (1979). Averaging operators with almost periodic, rapidly oscillating coefficients. *Math. USSR-Sb.* **35** 481-498.

[21] PAPANICOLAOU, G. C. and VARADHAN, S. R. S. (1979). Boundary problems with rapidly oscillating coefficients. *Colloq. Math. Soc. János Bolyai* **27** 835-875.

[22] WINTER, C. L., NEWMAN, C. M. and NEUMAN, S. P. (1984). A perturbation expansion for diffusion in a random velocity field. *SIAM J. Appl. Math.* **44** 425-442.

TURBULENT RANDOM MULTIPLICATIVE BRANCHING PROCESSES

MARTIN GREINER
Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Str. 38, D–01187 Dresden, Germany

All of us know about turbulence. It is a real-life phenomenon describing far-out-of-equilibrium situations: turbulent human relationships, turbulent ball games or turbulent stock markets, just to name a few. As those events come and go, no one is really able to predict, control and understand them. The latter is also exactly the case for the physics process coining the name: fully developed turbulence of fluid flows. Some even say that this process might represent the biggest challenge of classical statistical physics. In fact, this fortress has not been taken by cohorts of scientists besides many heroic attempts during the past century [1, 2]. How comes that turbulence is so tremendously difficult to understand?

To give a feeling for this, let us face the underlying equation of motion for incompressible fluid flows, the Navier-Stokes equation:

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} p + \nu \vec{\nabla}^2 \vec{v} + \vec{f} \; . \tag{1}$$

It represents Newton's second law of classical mechanics: the left-hand side is the acceleration of a fluid particle moving with velocity $\vec{v}$ and the right-hand side are the forces acting on it. $\vec{f}$ stands for external forces representing in some respect the boundary conditions; this term introduces a large length-scale L. The friction force with viscosity ν introduces another distinct length-scale, which is called the dissipation scale η and which is still macroscopic, but much smaller than L. In order to see what role the advection term $(\vec{v} \cdot \vec{\nabla})\vec{v}$ and the pressure term $-\vec{\nabla} p$ do play, we transform Eq. (1) into an energy balance equation by multiplication with $\vec{v}$, subsequent integration over a large volume and assuming stationarity:

$$\left\langle \vec{v} \cdot \vec{f} \right\rangle - \left\langle \frac{\nu}{2} \sum_{i,j=1}^{3} (\partial_i v_j + \partial_j v_i)^2 \right\rangle = 0 \; . \tag{2}$$

The first term with the external force feeds energy into the turbulent flow at the large scale L and dissipation takes it out at the small scale η as kinetic energy is then converted into heat. Somehow the energy has to be transported from L down to η. This is what the nonlinear local advection term and the nonlinear nonlocal pressure

term are in charge for. Hence, turbulence is a multiscale process. The number of degrees of freedom are about $(L/\eta)^3$; since $L/\eta = \mathcal{O}(10^3 - 10^4)$ for realistic jet, wind-tunnel or atmospheric boundary layer turbulence this number is of the order 10^{10}. Because of the nonlinearities and nonlocalities involved in the energy cascade transport the interaction between these too-many degrees of freedom is strong. This is exactly the point why all analytical and numerical approaches starting with the Navier-Stokes equation have more or less failed to provide a complete understanding of turbulence.

The very large number of degrees of freedom justifies a statistical approach. Here two different routes have to be distinguished: a top-down approach, which again starts right away with the Navier-Stokes equation, but is again haunted by notorious difficulties. The other approach is bottom-up, aims at understanding the data phenomenology with simple empirical models first and only then tries to connect all the way back to the Navier-Stokes equation. It is the latter approach we are conveying.

Typically data is taken with an anemometer, which records components (typically only one) of the velocity field as a time series, as the turbulent flow is sweeping over it. Assuming fluctuations around the mean velocity to be small, such time-series can be interpreted as one-dimensional spatial series by the frozen flow hypothesis. These signals look very irregular, i.e. turbulent. The standard observables are velocity structure functions, which are moments of velocity increments $\Delta v_l = v(x+l) - v(x)$. They show an approximate scaling $\langle \Delta v_l^n \rangle \sim l^{\zeta_n}$ for distances $\eta << l << L$; scaling exponents are found to be $\zeta_n \approx n/3 - \tau_{n/3}$, where $\tau_{n/3}$ represent the intermittency corrections to the $n/3$-behavior of Kolmogorov's K41 picture [3] of fully developed turbulence. This scale-invariance is to be expected since, as we have already argued before, no distinct length scales are present in the Navier-Stokes equation other than the large and small scales L and η, respectively. However, the observed scaling is not perfect. Hence, legitimate questions to ask are: are velocity structure functions the ideal observational tool for scaling? Maybe other observables are more suited? Perhaps other fields than the velocity field should be looked at?

Picking up the last question and remembering our intuitive picture on the energy transport process through the scales, a kind of "energy" field appears to be more appealing. A good representative is the so-called energy dissipation field $\varepsilon(\vec{r})$, which is the quantity given within the angle bracket of the second term of Eq. (2). The surrogate field $\varepsilon(x) = 15\nu(\partial_x v_\parallel)^2$ is easily constructed from the streamwise component of the measured velocity field and reveals how energy flux is taken out at the dissipation scale: spatial fluctuations $\varepsilon(x)/\langle\varepsilon\rangle$ around the average are anomalously strong and intermittent, values ranging between about 10^{-2} to 10^2 for large-Reynolds number turbulent flows. The spatial statistics of these fields now serve as guidance to learn about the details of the energy cascade process and to construct respective models.

A particular attractive and robust class of such data-driven models are random

multiplicative branching processes. They introduce a nested hierarchy of scales $l_j = L/2^j$ ranging from the large scale L down to the small scale $\eta = L/2^J$. The synthetic field construction uses a cascade generator, which redistributes the energy flux density $\Pi_{k_1\cdots k_j}$ associated to the parent interval with length l_j nonuniformly onto the left $(k_1\cdots k_j 0)$ and the right $(k_1\cdots k_j 1)$ offspring intervals with length l_{j+1}:

$$\begin{aligned} \Pi_{k_1\cdots k_j 0} &= q_{k_1\cdots k_j 0}\Pi_{k_1\cdots k_j} \ , \\ \Pi_{k_1\cdots k_j 1} &= q_{k_1\cdots k_j 1}\Pi_{k_1\cdots k_j} \ ; \end{aligned} \tag{3}$$

the random variables q are drawn from a scale-independent probabilistic splitting function $p_{\rm sf}(q_{k_1\cdots k_j 0}, q_{k_1\cdots k_j 1})$ and are independent from those belonging to other branchings. Starting at L the cascade generator is applied over and over again until η is reached. Then the energy flux amplitudes are set equal to the energy dissipation amplitudes:

$$\varepsilon_{k_1\cdots k_J} = \Pi_{k_1\cdots k_J} = q_{k_1} q_{k_1 k_2} \cdots q_{k_1\cdots k_J} \ . \tag{4}$$

They are a product of J iid random numbers with mean $\langle q\rangle = 1$, so that with some small probability the observed strong intermittent fluctuations around $\langle\varepsilon\rangle = 1$ are easily reproduced. A synthetic model field realization comes with support $[0, L]$ and can be expressed as

$$\varepsilon(x) = \sum_{k_1,\ldots,k_J=0}^{1} \varepsilon_{k_1\cdots k_J}\, \Theta\left(\frac{L}{\eta}\left(x - \sum_{j=1}^{J} k_j l_j\right)\right) \ , \tag{5}$$

with the index function $\Theta(x) = 1$ for $0 \leq x \leq L$ and 0 otherwise.

The qualitative reproduction of the observed intermittent fluctuations is only a first test of the model. A much more sophisticated and complete test would be to compare the spatial statistics. The latter is characterized by the multivariate probability distribution function

$$p_{\rm field}(\varepsilon(x)) = p_{\rm field}(\varepsilon_{0\cdots 0}, \ldots, \varepsilon_{1\cdots 1}) \ , \tag{6}$$

which gives the probability that a specific cascade field configuration with the 2^J amplitudes $\varepsilon_{k_1\cdots k_J}$ occurs. Another, but equivalent characterization of the field statistics is given by the Fourier transform of (6):

$$\begin{aligned} Z[\lambda_\kappa] &= \int d\varepsilon_{0\cdots 0}\cdots d\varepsilon_{1\cdots 1}\, p_{\rm field}(\varepsilon_{0\cdots 0}, \ldots, \varepsilon_{1,\ldots,1}) \exp\left(i\sum_\kappa \lambda_\kappa \varepsilon_\kappa\right) \\ &= \left\langle \exp\left(i\sum_\kappa \lambda_\kappa \varepsilon_\kappa\right)\right\rangle \\ &= 1 + i\sum_{\kappa_1} \lambda_{\kappa_1}\rho_{\kappa_1} + \frac{i^2}{2!}\sum_{\kappa_1}\sum_{\kappa_2} \lambda_{\kappa_1}\lambda_{\kappa_2}\rho_{\kappa_1\kappa_2} + \ldots \ ; \end{aligned} \tag{7}$$

here we have used the abbreviation $\kappa = (k_1 \cdots k_J)$ for the binary address of the field amplitude ε_κ. $Z[\lambda_\kappa]$ contains the same information as $p_{\text{field}}(\varepsilon(x))$. The second step of (7) is simply a shorthand notation of the first line, where the angle brackets symbolize an expectation value. The coefficients of the Taylor expansion of (7) are the n-point correlation densities

$$\rho_{\kappa_1 \cdots \kappa_n} = \langle \varepsilon_{\kappa_1} \cdots \varepsilon_{\kappa_n} \rangle = \frac{1}{i^n} \left. \frac{\partial^n Z[\lambda_\kappa]}{\partial \lambda_{\kappa_1} \cdots \partial \lambda_{\kappa_n}} \right|_{\lambda=0} . \tag{8}$$

This equation is the reason why $Z[\lambda_\kappa]$ is called a generating or characteristic functional. All this demonstrates that

$$p_{\text{field}}(\varepsilon(x)) \quad \longleftrightarrow \quad Z[\lambda_\kappa] \quad \longleftrightarrow \quad \rho_{\kappa_1 \cdots \kappa_n} \tag{9}$$

all access the spatial statistics in a complete way and are thus equivalent to each other. Experimentalists like to analyse the n-point correlation densities, whereas mathematicians and theoretical physicists like to deal with the generating functional.

For the random multiplicative branching process of (3)–(5) an analytic expression for the generating functional can be given [4, 5]. The secret here is to use the logarithm of (4), i.e.

$$\ln \varepsilon_{k_1 \cdots k_J} = \sum_{j=1}^{J} \ln q_{k_1 \cdots k_j} , \tag{10}$$

and the modified generating functional

$$K[\lambda_\kappa] = \ln \tilde{Z}[\lambda_\kappa] = \ln \left\langle \exp \left(\sum_\kappa \lambda_\kappa \ln \varepsilon_\kappa \right) \right\rangle . \tag{11}$$

For demonstration we pick a branching process of only J=2 cascade steps; then (11) becomes:

$$\begin{aligned} & K[\lambda_{00}, \lambda_{01}, \lambda_{10}, \lambda_{11}] = \\ & \quad = \ln \left\langle \varepsilon_{00}^{\lambda_{00}} \varepsilon_{01}^{\lambda_{01}} \varepsilon_{10}^{\lambda_{10}} \varepsilon_{11}^{\lambda_{11}} \right\rangle \\ & \quad = \ln \left\langle (q_0 q_{00})^{\lambda_{00}} (q_0 q_{01})^{\lambda_{01}} (q_1 q_{10})^{\lambda_{10}} (q_1 q_{11})^{\lambda_{11}} \right\rangle \\ & \quad = \ln \left\langle q_0^{\lambda_{00}+\lambda_{01}} q_1^{\lambda_{10}+\lambda_{11}} \right\rangle + \ln \left\langle q_{00}^{\lambda_{00}} q_{01}^{\lambda_{01}} \right\rangle + \ln \left\langle q_{10}^{\lambda_{10}} q_{11}^{\lambda_{11}} \right\rangle \\ & \quad = Q[\lambda_{00}+\lambda_{01}, \lambda_{10}+\lambda_{11}] + Q[\lambda_{00}, \lambda_{01}] + Q[\lambda_{10}, \lambda_{11}] . \end{aligned} \tag{12}$$

In the third step we have used that the bituples of random multiplicative weights (q_0, q_1), (q_{00}, q_{01}) and (q_{10}, q_{11}) belong to different branchings and are thus independent from each other. The branching generating function

$$Q[\lambda_L, \lambda_R] = \ln \left(\int dq_L dq_R \, p_{\text{sf}}(q_L, q_R) q_L^{\lambda_L} q_R^{\lambda_R} \right) \tag{13}$$

is directly linked to the splitting function $p_{\rm sf}(q_L, q_R)$ of the cascade generator and is the building block for the generating functional (12): one Q for the only branching of the first cascade step and two Qs for the two branchings of the second cascade step; see last line of (12). It is this structure, which directly generalizes to an arbitrary number J of cascade steps [4, 5]: the generating functional (11) is a sum of branching generating functions over all branchings.

From the analytic solution of the generating functional various types of n-point correlation densities can be calculated straightforwardly: n-point correlation densities between logarithmic field amplitudes,

$$\langle \ln \varepsilon_{\kappa_1} \cdots \ln \varepsilon_{\kappa_n} \rangle = \left. \frac{\partial^n \exp(K[\lambda_\kappa])}{\partial \lambda_{\kappa_1} \cdots \partial \lambda_{\kappa_n}} \right|_{\lambda=0} , \tag{14}$$

n-point cumulants between logarithmic field amplitudes,

$$\langle \ln \varepsilon_{\kappa_1} \cdots \ln \varepsilon_{\kappa_n} \rangle_c = \left. \frac{\partial^n K[\lambda_\kappa]}{\partial \lambda_{\kappa_1} \cdots \partial \lambda_{\kappa_n}} \right|_{\lambda=0} , \tag{15}$$

as well as n-point correlation densities between conventional field amplitudes,

$$\langle \varepsilon_{\kappa_1} \cdots \varepsilon_{\kappa_n} \rangle_c = \exp\left(K[\lambda_{\kappa=\kappa_i=1}, \lambda_{\kappa\neq\kappa_i=0}]\right) . \tag{16}$$

Their values depend on a set of mutual ultrametric distances $D(\kappa_1, \kappa_2)$ between two bins κ_1 and κ_2, which is a quantity inherent to the hierarchical binary tree-like structure of the random multiplicative branching process. An experimentalist is not able to observe it. Consider for example a two-point correlation density, which is sampled over point pairs with Euclidean two-point distance d. In order to realistically compare quantities like (14)–(16) to their experimental counterparts, we have to introduce an additional experimental-like nonrestrictive sampling using conditional probabilities like $p_{\rm samp}(D|d)$; see Refs. [6, 7] for more details. Then two-point correlators take on the form [7]

$$\frac{\langle \varepsilon^{n_1}(x+d)\varepsilon^{n_2}(x)\rangle}{\langle \varepsilon^{n_1}(x+d)\rangle \langle \varepsilon^{n_2}(x)\rangle} = a_{n_1 n_2} \left(\frac{L}{d}\right)^{\tau[n_1,n_2]} f_{n_1 n_2}(d/L) , \tag{17}$$

where $f_{n_1 n_2}(d \ll L) = 1$ represents a scaling function and $\tau[n_1, n_2] = \tau_{n_1+n_2} - \tau_{n_1} - \tau_{n_2}$ with $\tau_n = \log_2\langle q^n \rangle$. Analogously, two-point cumulants become [6]

$$\left\langle \ln^{n-1}\varepsilon(x+d)\, \ln \varepsilon(x) \right\rangle_c = G(d, J) \langle \ln^n q \rangle_c , \tag{18}$$

where the geometric function G does not depend on the order n and is related to the first moment of $p_{\rm samp}(D|d)$. Observables like (17) and (18) help to test the validity of random multiplicative branching processes for fully developed turbulence and to learn about the best approximate cascade generator $p_{\rm sf}(q_L, q_R)$ via the extractable

multifractal scaling exponents τ_n and the same-lineage cumulants $\langle \ln^n q \rangle_c$; consult Ref. [8] for more details.

There are also other observables which test the validity of random multiplicative branching processes and allow to bridge the latter to other data-driven models. One of these non-standard observables are breakup coefficients, often also called multipliers,

$$M(x,l) = \frac{\overline{\varepsilon}(x - l/4, l/2)}{2\overline{\varepsilon}(x,l)} \,, \tag{19}$$

where

$$\overline{\varepsilon}(x,l) = \frac{1}{l} \int_{x-l/2}^{x+l/2} \varepsilon(x')dx' \tag{20}$$

is the coarse-grained energy dissipation field. Experimental distributions $p_{\rm M}(M)$ of breakup coefficients have been found to be scale-independent in the range $\eta << l \leq L$ and, at first, have been interpreted in terms of a conservative cascade generator $p_{\rm sf}(q_L, q_R) = p(q_L)\delta(q_L + q_R - 2)$, invoking the association $M \leftrightarrow q_L$. Conditional distributions $p_{\rm M}(M(l)|M(2l)) \neq p_{\rm M}(M(l))$ have been observed with scale-correlations and have been interpreted to be in conflict with the random multiplicative branching processes, where branchings are independent from each other. However, these two conclusions are not correct. For a realistic comparison with data, these model observables have to be sampled unrestrictively over positions x as in experiments. Also, a non-conservative splitting function $p_{\rm sf}(q_L, q_R) = p(q_L)p(q_R)$ should be taken, since it accounts best for the fact, that three-dimensional turbulence is observed in only one dimension. Including these two points into the model simulations, the scale-independent unconditional distribution $p_{\rm M}(M)$ comes out right and turns out to be a scale-independent fixed-point resulting from the coarse-graining (20). The observed conditional distributions $p_{\rm M}(M(l)|M(2l))$ are also reproduced, but only if the splitting function $p(q)$ has a positive skewness. This rules out log-Poisson and log-stable distributions. For more details consult Refs. [9, 10]. – Another class of non-standard observables are Markovian Kramers-Moyal coefficients, which address the issue of observed scale correlations in a different way. As demonstrated in Ref. [11], random multiplicative branching processes are capable to quantitatively describe the experimental findings, too.

Fundamentally the presented random multiplicative branching processes have the drawback of having no time degree of freedom. However, generalizations in this direction are possible [12] and might help to learn about dynamical properties of the turbulent energy cascade. – For the ultimate goal to understand the random multiplicative branching processes in terms of the Navier-Stokes equation still something can be learned from the geometric cascade models: a wavelet representation of the respective two-point correlation densities leads to a more-or-less complete diagonalisation [13, 14, 15, 16], revealing that wavelets $\Psi_{jm}(x)$ appear to be the correct

degrees of freedom to describe turbulence. A very severe truncation

$$\vec{v}(\vec{r},t) \longrightarrow v(x,t) = \sum_{jm} v_{jm}(t)\Psi_{jm}(x) \ , \tag{21}$$

of the velocity field, once inserted into (1), motivates the dynamical toy-model amplitude equations

$$\partial_t v_1 = ik_1 \sum_{2,3} c(1;2,3)v_2 v_3 - \nu k_1^2 v_1 + f_1 \ , \tag{22}$$

known as the hierarchical shell model [17]. The last two terms only act at the small and large scales. Here it would be very interesting to find out which, if any, structure of the coupling $c(1{=}j_1m_1;2{=}j_2m_2,3{=}j_3m_3)$ between wavelet modes is required to reproduce the spatial statistics of the random multiplicative branching processes. This then would serve as a first link between the empirical random multiplicative cascade processes and the Navier-Stokes equation. Work and more dreams in this direction are presently in progress.

Acknowledgements
I thank J. Schmiegel, J. Cleve, T. Dziekan, H. Eggers, B. Jouault, J. Giesemann and P. Lipa for their cooperation over many years; without them these studies would not have come to such a mature stage.

References

[1] Monin, A.S. and Yaglom, A.M. (1971): *Statistical Fluid Mechanics*, Vols. 1 and 2 (MIT Press, Cambridge).

[2] Frisch, U. (1995): *Turbulence* (Cambridge University Press, Cambridge).

[3] Kolmogorov, A.N. (1962): *J. Fluid Mech.* **13**, 82.

[4] Greiner, M., H. Eggers, P. Lipa (1998): *Phys. Rev. Lett.* **80**, 5333.

[5] Greiner, M., J. Schmiegel, F. Eickemeyer, P. Lipa, H. Eggers (1998): *Phys. Rev. E* **58**, 554.

[6] Eggers, H.C., T. Dziekan, M. Greiner (2001): *Phys. Lett. A* **281**, 249.

[7] Schmiegel, J., T. Dziekan, B. Jouault, J. Cleve, M. Greiner (2001): *preprint.*

[8] Dziekan, T. (2001): *Zwei-Punkt Kumulanten in synthetischer Turbulenz* (diploma thesis, TU Dresden).

[9] Jouault, B., P. Lipa, M. Greiner (1999): *Phys. Rev. E* **59**, 2451.

[10] Jouault, B., M. Greiner, P. Lipa (2000): *Physica D* **136**, 125.

[11] Cleve, J., M. Greiner (2000): *Phys. Lett. A* **273**, 104.

[12] Schmiegel, J., H. Eggers, M. Greiner (2001): *preprint* cond-mat/0106347.

[13] Greiner, M., P. Lipa, P. Carruthers (1995): *Phys. Rev. E* **51**, 1948.

[14] Greiner, M., J. Giesemann, P. Lipa, P. Carruthers (1996): *Z. Phys. C* **69**, 305.

[15] Giesemann, J., M. Greiner, P. Lipa (1997): *Physica A* **247**, 41.

[16] Greiner, M., J. Giesemann (1998), In: *Wavelets in Physics*, ed. by L.Z. Fang, R.L. Thews (World Scientific, Singapore), pp. 89-131.

[17] Benzi, R., L. Biferale, R. Tripiccione, E. Trovatore (1997): *Phys. Fluids* **9**, 2355.

MULTISCALE FLOOD ANALYSIS ON SELF-SIMILAR RIVER NETWORKS

VIJAY K. GUPTA
Cooperative Institute for Research in Environmental Sciences, University of Colorado, Boulder, USA 80309

River flows shape landforms and create channel networks. Once formed, river networks determine the space-time dynamics of river flows. This reciprocal coupling between channel networks and river flows is depicted in empirical power-law relationships, which can be written in a generic form, $Y \sim Xb$, where b is an empirical parameter. These empirical power laws have been obtained using multiple regression. Well known examples in hydrology come from regional frequency analyses of peakflow and lowflow quantiles representing Y, and drainage area representing X; see Gupta and Dawdy (1995), Goodrich et al. (1997); Cathcart (2001) for peakflows, and Furey and Gupta (2000) for lowflows. Examples in geomorphology include the systematic down-stream variations of hydraulic-geometric variables on a channel network representing Y, e.g., velocity, depth, width, slope and friction, and the bankfull discharge representing X (Leopold, Wolman and Miller, 1964, Ibbitt et al. 1998). In ecology, myriad ecological phenomena from metabolic rate to biome formation are organized with respect to specific spatial and temporal scales. This hierarchy has been represented as a power law (Delcourt et al. 1982, Milne 1998a). Eventhough, the above empirical relationships have been known for a long time, only within the last decade it has been recognized that some of these relationships may be physically understood on the basis of mathematical ideas of *scale invariance and scale dependence* (Gupta and Waymire, 1998a, 1998b).

A long-term objective of our research is to develop a new mathematical 'scaling theory', which can unify diverse empirical observations from hydrology, geomorphology, ecology and meteorology based on conservation laws and other fundamental biophysical principles for medium to large drainage networks. Another long-term goal is to solve the long-standing classic applied problem of prediction of peakflows and lowflows from basins that have inadequate stream flow and other biophysical data in space and time. Referred to as, *poorly gauged and ungauged*, most basins worldwide fall into this category. We believe that a scientific unification of hydrology, geomorphology, ecology and meteorology is necessary to solve the problem of ungauged basins. In this extended abstract, I will illustrate some elements of this newly developing mathematical theory with examples.

The signature of network geometry on the dynamics of river flows can be illustrated through regional, or spatial, flood frequency analyses. The US Geological Survey has performed these analyses on a state by state basis throughout the US since the middle 1960s. They contain a wealth of regional information on flood response to precipitation input, and they served as the first major impetus to systematically identifying signatures of scale invariance in data (Gupta and Dawdy, 1995). Let $Q(A)$ be a random variable denoting the annual maximum flows from subbasins A with drainage areas $|A|$ within a basin or a region D. Then the regressions of the p-th quantile, $q_p(A)$, defined as,

$$P\{Q(A) > q_p(A)\} = p, \tag{1}$$

where $1/p$ is the *return period*, against $|A|$ for different p values generally show log-log linearity,

$$q_p(A) = q_p(1)|A|^{\theta(p)} \tag{2}$$

Here, both the coefficient $q_p(1)$ and the exponent $\theta(p)$ can be functions of p. When the scaling exponent $\theta(p) \equiv \theta$, is a constant, then (2) exhibits *simple scaling*. The floods generated by snowmelt and stratiform rainfall have been observed to obey (2). When $\theta(p)$ varies systematically as a function of p, then (2) exhibits *multiscaling*. Floods generated by convective rainfall have been observed to obey (3) (Gupta and Dawdy, 1995). These notions can be made mathematically precise in terms of scale invariance of probability distributions of the random field $\{Q(A)\}$ (Gupta and Waymire, 1998a, 1998b). Simple- and multi- scaling may be viewed as working hypotheses that have been guiding the development of a physical-statistical theory of floods. The main goal of this theory is to provide a rigorous physical understanding of the empirical scaling exponent $\theta(p)$, and the intercept $q_p(1)$, on the basis of conservation principles and other biophysical processes that govern the generation of floods from precipitation.

Before explaining some elements of this mathematical theory, let me mention another recent empirical observation that complements (2). The physical process of flood generation takes place in a river basin, which consists of a nested channel network and a collection of hills on both sides of a network. Hills produce runoff and feed it into a channel network. The US Department of Agriculture has operated a nested basin as an experimental facility in the semiarid south western US, known as Walnut Gulch. An empirical study on peak flow quantiles for Walnut Gulch showed that (Goodrich et al., 1997),

$$q_p(A) = q_p(1)|A|^{\theta(p)}, \quad \begin{cases} 0.85 \le \theta \le 0.9 & |A| < |A_C| = 1km^2 \\ 0.55 \le \theta \le 0.58 & |A| > |A_C|. \end{cases} \tag{3}$$

Equation (3) shows that there exists a critical drainage area, $|A_C| = 1km^2$, above and below which the scaling exponents are different. Moreover, the scaling exponent is lower in the upper range of drainage areas than in the lower range (Goodrich et.

al., 1997, Fig. 4). This empirical observation concerning a break in the scaling of peak flows on nested subbasins is the first one to my knowledge.

Another recent study has found a break in scaling of mean annual floods in the western coast of Oregon (Cathcart, 2001). However, this study was carried out on a regional basis, where each region contains several basins that are unnested. These results suggest that the existence of a critical area at which the scaling exponents for the mean and other quantiles change is a general feature of spatial flood statistics. Therefore, a mathematical theory should explain why this break happens and give a physical explanation of the two sets of scaling exponents. In the rest of this abstract, I will focus on explaining some recent mathematical results, which illustrate a physical basis of eqs. (2) and (3).

Consider a recent formulation of a mass balance equation for a network as our starting point. The reader can refer to Gupta and Waymire (1998a, 1998b) for details of various assumptions leading to this equation. Let $q(e,t)$, $e \in \tau, t \geq 1$, be a space-time field representing river discharge, or the volume of flow per unit time, across a link e in a channel network τ in the time interval $(t-1)$ of unit length $\triangle t$. A *link* is defined as a segment of a channel between two consecutive junctions, or a junction and a source. The flow out of a link from its bottom vertex $\underline{e}$ is defined as $q(\underline{e},t) = -q(e,t)$. Let $R(e,t)a(e)$ denote the volume of runoff into a link e from the two adjacent hills in time interval $(t-1,t)$, where $a(e)$ denotes the area of the hillsides, and $R(e,t)$ is the runoff intensity measured in the units of length per unit time. This term can also include the volume of runoff depletion per unit time due to channel infiltration losses or evaporation from channel surface in link e, but we will not introduce these physical features to keep our problem simple. Let $S(e,t)$ denote the total volume of runoff stored in link in time interval $(0,t)$, and the change in the total volume of runoff stored in time interval $(t-1,t)$ be denoted by $\triangle S(e,t)$. Then the equation of mass balance for a network-hills system can be written as,

$$\frac{\triangle S(e,t)}{\triangle t} = -q(e,t) + \sum_{f:\underline{f}=\overline{e}} q(f,t-1) + R(e,t)a(e), \quad e \in \tau,\ t \geq 1. \tag{4}$$

The sum on the right hand side of (4) consists of discharges from all the links that join with link e at its top node $\overline{e}$. The above formulation does not assume that the channel network is binary, although river networks are generally binary. However, it assumes that no loops are present in the network.

Solutions of (4) depend on the branching and geometric structure of a network through the summation term on its right-hand side, and on the physical-statistical structure of storage field, $S(e,t)$ and the runoff generation field, $R(e,t)$. A key physical issue is to explicitly understand the role of these three terms on the spatial scaling statistical of peak flows given by (2) and (3). Eq. (4) can be solved iteratively to obtain $q(e,t)$, $e \in \tau, t \geq 1$, which represents the runoff hydrograph at the bottom of every link, and then analyze the scaling of peak flows. Gupta et. al (1996),

Gupta and Waymire (1998a, 1998b), and Troutman and Over (2001) have obtained analytical results under certain simplifying assumptions. Manabde et al. (2001), and Manabde and Sivapalan (2001) have carried out iterative numerical simulations to solve (4) by relaxing some of these assumptions. I will give a brief summary of assumptions, and some results by these investigators, and point out connections to the empirical observations reported in eqs. (2) and (3). However, this body of research is in its infancy, and not much is known about general statistical solutions of (4) within a space-time context.

Analytical solutions of (4) can be obtained under idealized conditions, which provide a glimpse into a physical interpretation of the scaling exponent in (2). Assuming that $\triangle S(e,t) = 0$, the runoff field $a(e)R(e,t) = aR(t)$, $t \geq 0, e \in \tau$, is *spatially uniform*, where a denotes the combined area of hills on either side of a channel link. It follows from (4) that the river runoff in each link is given by a *convolution*,

$$q(e,t) = a\sum_{s=0}^{t-1} W_e(t-1-s)R(s), \quad t \geq 1, e \in \tau, \tag{5}$$

where $W_e(x)$, $x \geq 0$, is called the *local geomorphologic width function* of a network for each link $e \in \tau$. It is defined as the number of links at a distance x from the bottom most vertex known as an *outlet*, denoted by ϕ. In order to introduce the notion of distance, define the lengths of vertices as, $|\overline{e}| = |\underline{e}| + 1$, and $|\phi| = 0$. This definition can be used iteratively to assign a distance to every link from the outlet and thereby compute the width function.

To further understand the importance of the width function in the present context, assume that runoff is applied instantaneously on a network; that is, $R(s) = R$ if $s = 0$ and $R(s) = 0$ otherwise. This simplifies (5) to,

$$q(e,t) = W_e(t-1)Ra, \quad t \geq 1, e \in \tau. \tag{6}$$

This means that the width function, up to a scale transformation from space to time via $x = t/v$, where v is a constant velocity of flow through out a network, determines the response of the channel network to an instantaneous, spatially uniform, runoff . This key concept is called a *geomorphologic instantaneous unit hydrograph* (GIUH). Defining peak flows as the maximum of $q(e,t)$ over an arbitrary time interval, it follows from (6) that,

$$Q(e) = \max_t q(e,t) = \max_t W_e(t-1)Ra, \quad e \in \tau. \tag{7}$$

Eq. (7) shows that, to a first-order approximation, the scale invariance of peak flows can be understood from that of the width function, which is purely a geometric network function. This insight is physically very significant.

The branching and the geometric patterns of a channel network determine the width function. Three comprehensive channel network models have been introduced

within the last 35 years. The first and the most well known of these three is the random model (Shreve, 1967). It assumes that a channel network is a binary Galton-Watson critical branching process. A classic book on channel networks containing many of the key papers published between 1945 and 1975 is by Jarvis and Woldenberg (1984). The second model, known as the Tokunaga model (Tokunaga, 1994), is based on the assumption of mean self-similarity in network topology. For example, the random model has been shown to exhibit mean self-similarity. Tokunaga model has gained considerable recognition in the recent literature (Peckham, 1995; Tarboton, 1996; Turcotte, 1997; Dodds and Rothman, 1999) because it gives certain predictions of empirical observations, which the random model can not. However, it is severely limited, because it does not incorporate statistical variability that is so pervasive in real channel networks. A newly introduced third model, known as the random self-similar model (RSN), remedies this situation (Veitzer and Gupta, 2000). It gives many empirical predictions that the random model can not. Moreover, under certain conditions, the predictions of the random model and the Tokunaga model can be obtained as special cases of the RSN model. Preliminary solutions of (4) have been investigated under all these three classes of network models.

Gupta et al. (1996) considered a Peano network as an idealized example of a mean self-similar network (Mandelbrot, 1983; Marani et. al, 1991). They showed, as can be easily checked, that $\max_t W_e(t-1)$ scales with respect to the size of the subnetworks, $|\tau(e)| = |A(e)|$, the upstream drainage area, in a power law manner with an exponent $\log 3/\log 4$. Therefore, it follows from (7) that the flood scaling exponent θ in (2) is $\log 3/\log 4 = 0.792$. Manabde et al. (2001) relaxed the assumption that $\triangle S(e,t) = 0$, and considered dynamic changes in storage through iterative numerical solutions of (4). They observed that scaling in peak flows holds asymptotically as drainage area increases, but the scaling exponent θ decreased from 0.792 to 0.63 due to dynamic storage effects. An analytical demonstration of this finding remains open. Manabde and Sivapalan (2001) also analyzed the effect of spatially variable velocity on scaling exponent by incorporating the hydraulic-geometric relations described at the very beginning. They found that the scaling exponent is further reduced over its value under a constant velocity. The above results give a first set of evidence that the empirical scaling exponent θ has a fundamental physical basis. Other analytical results for the flood scaling exponent have been obtained by assuming that a random cascade model describes spatial rainfall input on a Peano network (Gupta et al., 1996). This result has been generalized to more realistic mean self-similar networks by Troutman and Over (2001).

Finally, we relax the assumption of instantaneous runoff input in (4) and consider a finite duration runoff forcing, i.e., $R(t)$, $0 \leq t \leq T$, and 0 otherwise. Physically runoff generation is attenuated in time due to finite rainfall duration, and in addition due to subsurface runoff contribution to stream flows in humid climates. By contrast, in semi-arid climate of the Walnut Gulch basin, there is no subsurface runoff contribution to stream flows, and the runoff attenuation is purely due to rainfall.

Castro (1998) illustrated the effect of duration T on the scaling exponent within the context of the Peano example. She showed that the scaling exponent for smaller basins is 1, and it switches to $\log 3/\log 4$ after a critical basin size is reached. In other words, she observed a break in scaling, which is qualitatively similar to the effect shown in (3). To understand this effect physically, assume that the runoff term is not instantaneous but persists for a longer time T. Then smaller-order subbasins begin to "saturate". This means that the entire drainage basin contributes to the peak flow, rather than only a fraction of it given by the peak of the width function in (7). Therefore, the peak flow becomes proportional to the drainage area with an exponent 1. This time to saturation is called the *concentration time*, at which the river discharge reaches its maximum value. The discharge remains in a steady state as long as $R(t) > 0$ after the concentration time is reached. If $j(T)$ denotes the scale index for the largest subbasin in the Peano example that has become saturated at a fixed duration T, then the peak flow scaling exponent for subbasins with scale index $k < j(T)$ is 1. In larger basins with $k > j(T)$, it is $\log 3/\log 4$. The saturation extends to still larger subbasins as duration T increases. This argument illustrates that runoff attenuation is responsible for scale break and may provide a physical explanation of the empirical observation shown in (3). This effect is analytically illustrated in Gupta and Waymire (1998a) using the width function calculations for the random model (Troutman and Karlinger, 1984), as shown in Figure-1.

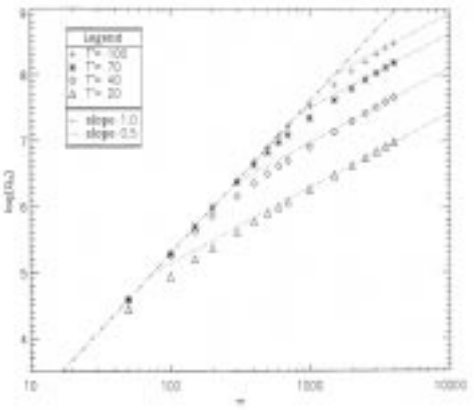


Figure 1: Schematic depiction of the effects of rainfall duration and basin size on the scaling exponents for floods (Gupta and Waymire, 1998a).

Manabde and Sivapalan (2001) also noted the presence of scaling break using numerical solutions of (4) under more realistic physical conditions than considered in Castro (1998), or in Gupta and Waymire (1998a). They also found that the effects of dynamic storage and nonlinear velocity decrease the scaling exponents on both sides of the critical area $|A_C|$.

The above physical explanation for a change in the flood scaling exponent also suggests that the critical basin area, $|A_C|$, would be smaller in semi-arid climate of

Arizona than in the humid climate of western Oregon. The reason is that rainfall events are typically of longer duration in Oregon than in Arizona, and the presence of baseflow in stream flow in Oregon would further increase the effect of duration on the size of $|A_C|$. It is noteworthy, that this prediction is borne out by data. Cathcart (2001) observed that $|A_C| \approx 50km^2$ for a humid region in western Oregon compared to $|A_C| \approx 1km^2$ in the Walnut Gulch basin located in a semi arid part of southern Arizona (Goodrich et al., 1997). Further tests of our theoretical prediction are needed on more basins from different climates.

Acknowledgements: This research was supported by grants from NSF and NASA.

References

[1] Castro, S., 1998: Scaling exponents of floods from scale-invariant rainfall and river networks, M. S. Thesis, University of Colorado, Boulder, CO., USA.

[2] Cathcart, J., 2001: The Effects of Scale and Storm Severity on the Linearity of Watershed Response revealed through the Regional L-Moment Analysis of Annual Peak Flows, Ph.D. dissertation, University of British Colombia, Vancouver, Canada.

[3] Delcourt, H.R., P.A. Delcourt, and T. Webb III: 1982. Dynamic plant ecology: the spectrum of vegetational change in space and time. Quat. Sci. Rev. 1:153-75.

[4] Furey, P., and V. K. Gupta, 2000: Space-time varaibility of low streamflows in river networks, Water Res. Res., 36(9), 2679-90.

[5] Goodrich, D., L. J. Lane, R. M. Shillito, S. N. Miler, K. H. Syed, and D. A. Woolhiser, 1997: Linearity of basin response as a function of scale in a semiarid watershed, Water Resour. Res., 33(12), 2951-65.

[6] Gupta, V. K., and D. R. Dawdy, 1995: Physical interpretations of regional variations in the scaling exponents of flood quantiles, Hydrological Processes, 9(3/4), 347-61.

[7] Gupta, V.K. and E. Waymire, 1998a: Spatial variability and scale invariance in hydrologic regionalization, In: Scale dependence and scale invariance in hydrology, Edited by G. Sposito, pp. 88-135, Cambridge. London.

[8] Gupta, V.K. and E. C. Waymire, 1998b: Some mathematical aspects of Rainfall, Landforms and Floods, In: Stochastic Methods in Hydrology: Rainfall, landforms and floods, Ed. O. E. Barndorff-Nielsen, V. K. Gupta, V. Perez-Abreu, and E. C. Waymire, Advanced Series on Statistical Sciences and Applied Probability, V. 7, World Scientific, pp. 129-71.

[9] Gupta, V. K., Castro, S. and Over, T. M., 1996: On scaling exponents of spatial peak flows from rainfall and river network geometry. J. Hydrol., 187: 81-104.

[10] Ibbitt, R. P., A. I. McKerchar, and M. J. Duncan, 1998: Taieri river data to test channel network and river basin heterogeneity concepts, Water Resour. Res., 34(8), 2085-8.

[11] Jarvis, R. S., and M. J. Woldenberg, (Eds.) 1984: River Networks, Benchmark Papers in Geology, V. 80, Hutchinson Ross, Stroudsburg, PA.

[12] Leopold, L. B., Wolman, M. G., and Miller, J. P. 1964. Fluvial Processes in Geomorphology. San Francisco: W. H. Freeman.

[13] Manabde M., S. Veitzer, V. K. Gupta, and M. Sivapalan, 2001: Tests of peak flow scaling in simulated self-similar river networks, Adv. Water Resour. (In press).

[14] Mandelbrot, B. B., 1983: The Fractal Geometry of Nature, Freeman, San Francisco, California.

[15] Marani A., R. Rigon, A. Rinaldo, 1991: A note on fractal channel networks, Water Resour. Res., 27(12), 3041-49.

[16] Menabde M, and Sivapalan M., 2001: Linking space-time variability of rainfall and runoff fields: A dynamic approach. Adv. Water Resour. (In press).

[17] Milne, B.T. 1998a. Motivation and benefits of complex systems approaches in ecology. Ecosystems 1:449-56.

[18] Peckham, S. 1995. New results for self-similar trees with applications to river networks, Water Resour. Res. 31: 1023-29.

[19] Dodds, P. S., and D. Rothman, 1999: A unified view of scaling laws for river networks, Physical Review E, 59, 4865-77.

[20] Shreve, R. L., 1967: Infinite topologically random channel networks, J. Geol., 75: 178-86.

[21] Tarboton, D. 1996: Fractal river networks, Horton laws and Tokunaga cyclicity, J. Hydrology, 187: 105-17.

[22] Tokunaga, E., 1994: Self-similar nature of drainage basins, In: Research in pattern formation, R. Takaki (Ed), KTK Scientific publishers, Tokoyo.

[23] Troutman, B., and Karlinger, M. 1984. On the expected width function of topologically random channel networks. J. of Appl. Prob. 22: 836-49.

[24] Troutman, B. and Karlinger, M. 1998. Spatial channel network models in hydrology, In: Advanced Series in Statistical Sciences and Applied Probability, Vol. 7 : Statistical Methods in Hydrology: Rainfall, Landforms and Floods, ed. O. E. Barndorff-Nielsen, V. K. Gupta, V. Perez-Abreu, and E. C. Waymire. Singapore: World Scientific.

[25] Troutman B, and Over T., 2001: River Flow Mass Exponents with fractal Channel Networks and Rainfall, Adv. Water Resour (In press).

[26] Turcotte, D., 1997: Fractals and chaos in geology and geophysics, 2nd Edition, Cambridge.

[27] Veitzer, S. and V. Gupta, 2000: Random self-similar river networks and derivations of generalized Horton laws in terms of statistical simple scaling, Water Resour. Res. 36(4), 1033-48.

STOCHASTIC CASCADES AND 2D FOURIER NAVIER-STOKES EQUATIONS

JOHN C. ORUM
Oregon State University

Certain solutions of the three dimensional Fourier transformed Navier-Stokes (FNS) equations may be represented, pointwise, as the expected value of a kind of branching process statistic. The statistic samples the Fourier transforms of both the initial data and the forcing at various random frequencies as directed by the leaves of the branching process, and then combines the results together, in a multiplicative way, according to the nodes of the branching process. This method began with the paper of LeJan and Sznitman [3] who remark that the method depends on the three dimensional situation. We review this representation method by considering, as far as possible, both the three- and two-dimensional FNS equations simultaneously. Then at a certain point a discriminating problem for the two-dimensional case appears. This involves a nonlinear convolution equation for the majorizing kernel, namely $h*h(\xi) = |\xi|h(\xi)$, or even the inequality $h*h(\xi) \leq |\xi|h(\xi)$. The concept of a majorizing kernel was introduced after the appearance of [3] and its role in the theory is now more explicit. The problem here is that while solutions to this convolution equation may be obtained in three dimensions, it has no known solution in two dimension. Nevertheless, we may get around this problem by introducing a time dependent transformation $\hat{v}(\xi,t) = \hat{u}(\xi,t)e^{-\delta t}$, where $\hat{u}(\xi,t)$ solves the two dimensional FNS. This is discussed here; the three dimensional case is considered by Thomann [this volume]. These results are new enough that their position and importance in the developing theory is not clear. Consequently the emphasis here is on an informal discussion of the route to the results.

The representation method rests on writing the FNS equations in a form that admits a probabilistic interpretation. This is equation (10) below. The most transparent way to get there is to begin with the Navier-Stokes equations for an incompressible fluid filling all of $\mathbb{R}^n$ ($n = 2$ or 3). These equations for an unknown velocity vector $u = (u_k(x,t)_{1\leq k\leq n}) \in \mathbb{R}^n$ and pressure $p = p(x,t) \in \mathbb{R}$ are given by

$$\frac{\partial}{\partial t}u_k + u \cdot \bigtriangledown u_k = \nu \bigtriangleup u_k - \frac{\partial p}{\partial x_k} + f_k \tag{1}$$

$$div\ u = \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_k} = 0 \tag{2}$$

with initial data $u(x,0) = u_0(x)$, $x \in \mathbb{R}^n$. The initial data and external forcing $f = (f_k(x,t))_{1\leq k\leq n}$ are assumed to be divergence free. The ensuing computations are facilitated by writing everything componentwise and incorporating the incompressibility condition into the nonlinear term:

$$\frac{\partial u_k}{\partial t} + \frac{\partial}{\partial x_j}(u_k u_j) = \nu \frac{\partial^2 u_k}{\partial x_j \partial x_j} - \frac{\partial p}{\partial x_k} + f_k; \qquad \frac{\partial u_k}{\partial x_k} = 0. \tag{3}$$

Repeated indices denote summation. The Fourier transform is applied, where $\hat{u} = \hat{u}(\xi,t) = (2\pi)^{-n/2} \int e^{-i\xi\cdot x} u(t,x)dx$, resulting in the set of FNS equations

$$\frac{\partial \hat{u}_k}{\partial t} + \frac{i}{(2\pi)^{n/2}} \xi_j (\hat{u}_k * \hat{u}_j) = -i\xi_k \hat{p} - \nu|\xi|^2 \hat{u}_k + \hat{f}_k; \qquad \xi_k \hat{u}_k = 0. \tag{4}$$

These are converted to a system of integral equations by combining $\partial_t \hat{u}_k$ and $\nu|\xi|^2\hat{u}_k$ via the integrating factor $\exp(\nu|\xi|^2 t)$, and then integrating between time $s = 0$ and time $s = t$:

$$\hat{u}_k(\xi,t) = e^{-\nu|\xi|^2 t}\hat{u}_k(\xi,0) + \int_0^t e^{-\nu|\xi|^2(t-s)} \left\{ \frac{-i}{(2\pi)^{n/2}} \xi_j (\hat{u}_k * \hat{u}_j) - i\xi_k \hat{p} + \hat{f}_k \right\} ds. \tag{5}$$

The convolution terms are expanded as integrals and everything is recombined into the vector integral equation

$$\hat{u}(\xi,t) = e^{-\nu|\xi|^2 t}\hat{u}_0(\xi) + \int_0^t e^{-\nu|\xi|^2(t-s)} \left\{ \frac{-i}{(2\pi)^{n/2}} \int_{R^n} \hat{u}(\eta,s)\cdot\xi \hat{u}(\xi-\eta,s)d\eta - i\xi\hat{p}(\xi,s) + \hat{f}(\xi,s) \right\} ds. \tag{6}$$

The standard technique of removing the pressure term is the application of the orthogonal projection $P_\xi : \mathbb{C}^n \to \langle\xi\rangle^\perp$. This projects onto the orthogonal complement of ξ, i.e. $P_\xi(\xi) = 0$. An equation for pressure may be recovered by taking the divergence of (1) using (2) [4]. Using the assumption that the initial data and forcing are divergence free, application of P_ξ gives

$$\hat{u}(\xi,t) = e^{-\nu|\xi|^2 t}\hat{u}_0(\xi) + \int_0^t e^{-\nu|\xi|^2(t-s)} \left\{ \frac{-i}{(\sqrt{2\pi})^n} \int_{R^n} \hat{u}(\eta,s)\cdot\xi P_\xi \hat{u}(\xi-\eta,s)d\eta + \hat{f}(\xi,s) \right\} ds. \tag{7}$$

The next adjustments are the multiplication and division by $|\xi|$, and the substitution $s \to t - s$ giving

$$\hat{u}(\xi,t) = e^{-\nu|\xi|^2 t}\hat{u}_0(\xi) + \int_0^t e^{-\nu|\xi|^2 s} \left\{ \frac{-i|\xi|}{(\sqrt{2\pi})^n} \int_{R^n} \hat{u}(\eta,t-s)\cdot\frac{\xi}{|\xi|} P_\xi \hat{u}(\xi-\eta,t-s)d\eta + \hat{f}(\xi,s) \right\} ds. \tag{8}$$

The following *times operation* is introduced because of its ability to simplify notation, and its persistence in the solution representation where it appears as the predominant algebraic operation at the nodes of the branching process: $a \otimes_\xi b = -i|\xi|^{-1}(a \cdot \xi)P_\xi(b)$, $a, b \in \mathbb{C}^n$. This results in

$$\hat{u}(\xi,t) = e^{-\nu|\xi|^2 t}\hat{u}_0(\xi) + \int_0^t e^{-\nu|\xi|^2 s}\left\{\frac{|\xi|}{(\sqrt{2\pi})^n}\int_{R^n}\hat{u}(\eta,t-s)\otimes_\xi \hat{u}(\xi-\eta,t-s)d\eta + \hat{f}(\xi,t)\right\}ds. \tag{9}$$

A key idea of the representation method is the introduction a class of *majorizing kernels*. These are positive functions $h(\xi)$ with the property that $h*h(\xi)$ can normalize $h(\eta)h(\xi-\eta)d\eta$ to a probability measure. Generally this measure is expressed symmetrically as the law $K_\xi(d\xi_1, d\xi_2)$ (supported on the set $\xi_1 + \xi_2 = \xi$) of a pair of correlated random vectors summing to ξ. Certain other conditions are put on $h(\xi)$ stemming from the particular equation under consideration. See Waymire [this volume]. For the FNS equations a majorizing kernel should satisfy

1. $0 < h(\xi) < \infty$ almost everywhere,
2. $h*h(\xi) < \infty$ almost everywhere,
3. $h*h(\xi) \le C|\xi|h(\xi)$ where C is a constant.

The reason for the third condition is explained below. The term majorizing kernel refers to any function used in the analysis of a PDE as it is here, and its name comes from its role in tracking the size of solutions from the size of the Fourier transformed initial data. (Hence its utility in tracking the regularity and support of solutions in physical space.)

The following adjustments involving a majorizing kernel $h(\xi)$ (that is yet to be made explicit) are now made to equation (9): Divide through by $h(\xi)$ and also multiply and divide by $h(\eta)$, $h(\xi-\eta)$, $h*h(\xi)$, $\nu|\xi|^2$, and the number 2. The result is

$$\chi(\xi,t) = e^{-\nu|\xi|^2 t}\chi_0(\xi) + \int_0^t \nu|\xi|^2 e^{-\nu|\xi|^2 s}\{\cdots\}\,ds \tag{10}$$

$$\{\cdots\} = \frac{1}{2}m(\xi)\int_{R^n}\chi(t-s,\eta)\otimes_\xi \chi(t-s,\xi-\eta)K_\xi(\eta,\xi-\eta)d\eta + \frac{1}{2}\phi(t-s,\xi)$$

where

$$\chi(\xi,t) = \frac{\hat{u}(\xi,t)}{h(\xi)}, \qquad \chi_0(\xi) = \frac{\hat{u}_0(\xi)}{h(\xi)}, \qquad m(\xi) = \frac{2|\xi|h*h(\xi)}{\nu|\xi|^2(\sqrt{2\pi})^n h(\xi)},$$

$$K_\xi(\eta,\xi-\eta) = \frac{h(\eta)h(\xi-\eta)}{h*h(\xi)}, \qquad \phi(\xi,t) = \frac{2\hat{f}(\xi,t)}{\nu|\xi|^2 h(\xi)}.$$

This is the form of FNS that admits the interpretation in terms of a branching process: Let S be an exponentially distributed random variable with parameter $\nu|\xi|^2$. Let κ be a Bernoulli random variable with parameter $\frac{1}{2}$. Let η_1 and η_2 be a pair of correlated random variables distributed as $K_\xi(d\xi_1, d\xi_2)$. Assume S, κ, and η_1 (or η_2) are independent. Then equation (10), interpreted probabilistically, may be written

$$\begin{aligned}\chi(\xi,t) &= \mathbb{E}\left\{\chi_0(\xi)\mathbf{1}_{[S\geq t]} + \phi(\xi,t-S)\mathbf{1}_{[S<t]}\mathbf{1}_{[\kappa=0]} + \right. \\ &\qquad \left. m(\xi)\chi(\eta_1,t-S)\otimes_\xi \chi(\eta_2,t-S)\mathbf{1}_{[S<t]}\mathbf{1}_{[\kappa=1]}\right\}. \end{aligned} \tag{11}$$

Consider the following heuristic: Let $M^{(1)}(\xi,t)$ denote the random variable under the expectation on the right, so that $\chi(\xi,t) = \mathbb{E}M^{(1)}(\xi,t)$. Let $M^{(2)}(\xi,t)$ denote the result of replacing $\chi(\eta_1,t-S)$ and $\chi(\eta_2,t-S)$ by similarly defined random variables $M^{(1)}(\eta_1,t-S)$ and $M^{(1)}(\eta_2,t-S)$, respectively. Iterate this replacement process, obtaining for $k=1,2,\ldots,$

$$\begin{aligned}M^{(k+1)}(\xi,t) &= \chi_0(\xi)\mathbf{1}_{[S\geq t]} + \phi(\xi,t-S)\mathbf{1}_{[S<t]}\mathbf{1}_{[\kappa=0]} + \\ &\qquad m(\xi)M^{(k)}(\eta_1,t-S)\otimes_\xi M^{(k)}(\eta_2,t-S)\mathbf{1}_{[S<t]}\mathbf{1}_{[\kappa=1]}. \end{aligned} \tag{12}$$

We would like to define a random variable $M^{(\infty)}(\xi,t)$ that is the result of iterating this replacement process as far as possible, and write

$$\begin{aligned}M^{(\infty)}(\xi,t) &= \chi_0(\xi)\mathbf{1}_{[S\geq t]} + \phi(\xi,t-S)\mathbf{1}_{[S<t]}\mathbf{1}_{[\kappa=0]} + \\ &\qquad m(\xi)M^{(\infty)}(\eta_1,t-S)\otimes_\xi M^{(\infty)}(\eta_2,t-S)\mathbf{1}_{[S<t]}\mathbf{1}_{[\kappa=1]}, \end{aligned} \tag{13}$$

$$\chi(\xi,t) = \mathbb{E}M^{(\infty)}(\xi,t). \tag{14}$$

The formal construction suggested by this heuristic is built on the following:

1. An underlying random process.

2. A multiplicative functional defined on this process, whose expected value is $\chi(\xi',t')$ when it is started, or restarted, at the possibly random ξ', t'.

3. In view of the random restart times the underlying random process must have the strong Markov property.

The underlying random process is a multi-type branching process whose particle types are the non-zero frequencies $\xi \in \mathbb{R}^n/\{0\}$. It is the result of reducing the procedure of iterated replacements given by equations (11)-(14) to a skeleton of random frequencies and their exponential lifetimes. Next, the definition of the multiplicative functional becomes transparent when the remaining information in (11)-(14) is included. Both the branching process and the multiplicative functional are described here informally. A formal treatment is given by Waymire [this volume] and in [1].

The branching process may be viewed as evolving backward or forward in time. Here we depict the backward view: Starting at (ξ, t), a particle ξ lives for an exponentially distributed length of time S_θ with holding parameter $\nu|\xi|^2$, and then it dies. The branching process evolves backwards in time, since the temporal argument of $\chi(\xi, t - S_\theta)$ is previous to t. At the death of ξ a coin κ_θ is tossed. If $\kappa_\theta = 0$ no new particles are born. If $\kappa_\theta = 1$, then two new particles, η_1, η_2 are born, distributed as $K_\xi(d\xi_1, d\xi_2)$. The process is repeated independently for each of the new particle types η_1, η_2 whose exponential lifetimes S_1, S_2 have holding parameters $\nu|\eta_1|^2$, $\nu|\eta_2|^2$, respectively. This process is iterated and branching continues long as there are living particles above the threshold $t = 0$. This results in a random tree, $\tau(\xi, t)$, as illustrated in Figure 1.

Such a tree $\tau(\xi, t)$ may have two types of nodes: operational nodes, and input nodes. Operational nodes ($\bullet$) occur at branch points, when a particle dies and is replaced by two new particles. Input nodes ($\circ$) occur at the leaves, when a particle either dies and is not replaced, or else when it dies below the threshold $t = 0$. If an input node, (ξ^*, t^*) say, occurs above $t = 0$, the tree samples the forcing evaluated at this point: $\phi(\xi^*, t^*)$. If this input node occurs below $t = 0$, then the tree samples the initial data, $\chi_0(\xi^*)$. The multiplicative functional combines these sampled values through the non-associative binary operation $(a, b) \mapsto m(\xi)a \otimes_\xi b$ where the nesting of the operations corresponds to the branching structure of the tree: Starting at the leaves the input vectors are sent up the tree until an operational node is encountered. At an operational node, for particle type ξ say, whose two branches have sent in the vectors a and b, combine a and b according to the times operation $m(\xi)a \otimes_\xi b$ and send this up the tree. Multiply like this, working upward until the value of the multiplicative functional is attained at the root (ξ, t). The multiplicative functional is called the *times functional* and is denoted $\curlywedge(\tau(\xi, t))$. As an example, the times functional on the tree $\tau^*(\xi, t))$ shown in Figure 1, would assume the value

$$\curlywedge(\tau^*(\xi, t)) = m(\xi)[m(\eta_1)\chi_0(\eta_{11}) \otimes_{\eta_1} \chi_0(\eta_{12})] \otimes_\xi \phi(\eta_2, t - S_\theta - S_2). \qquad (15)$$

The times functional $\curlywedge(\tau(\xi, t))$ replaces the heuristic random variable $M^{(\infty)}(\xi, t)$ that is anticipated by equations (13) and (14).

The representation $\chi(\xi, t) = \mathbb{E}\curlywedge(\tau(\xi, t))$ depends on the integrability of the times functional. The following results address this [1], [3]:

1. With probability one, the number of values $m(\cdot)$, $\chi_0(\cdot)$, $\phi(\cdot, \cdot)$ combined together by the times functional $\curlywedge(\tau(\xi, t))$ is finite.

2. If the bounds $|\chi_0(\xi)| \leq 1$, $|\phi(\xi, t)| \leq 1$, and $m(\xi) \leq 1$ hold for all $\xi \in \mathbb{R}^n$ and for all $t \geq 0$, then $\chi(\xi, t) = \mathbb{E}\curlywedge(\tau(\xi, t))$ solves (10), and this solution is unique among solutions that remain pointwise bounded by 1 on any finite time interval $0 \leq t \leq T$.

This first result is ancillary to the second. By standardizing all the particle lifetimes to a single epoch in discrete time, the resulting critical binary Galton-Watson process

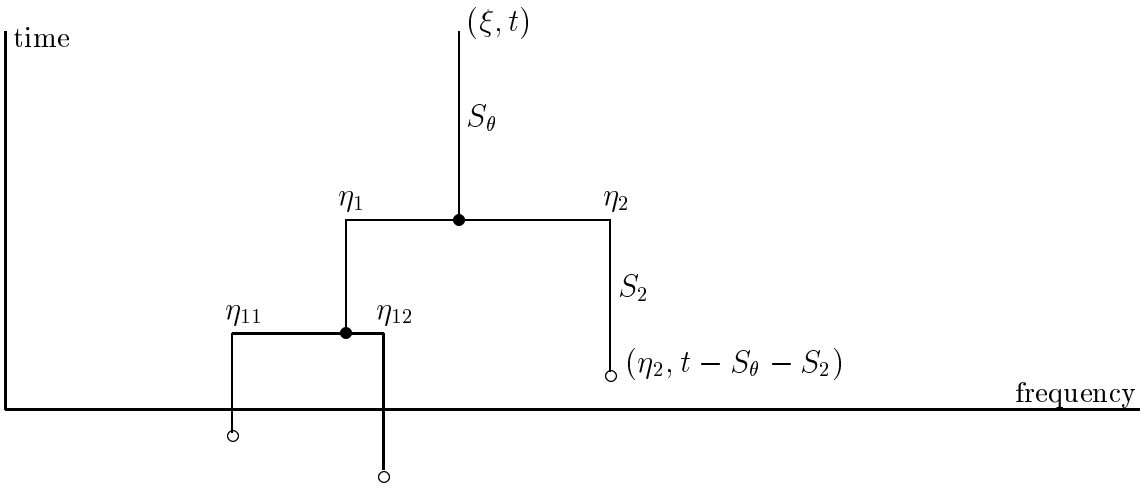


Figure 1: The branching process and multiplicative functional: A particle of type $\xi \in \mathbb{R}^n$ lives for an exponentially distributed lifetime with parameter $\nu|\xi|^2$, then is replaced by either zero particles or two particles distributed as $K_\xi(d\xi_1, d\xi_2)$, and the process begins anew. The resulting random tree samples the initial data and forcing at the input nodes ($\circ$) and then combines the results together at the internal nodes ($\bullet$) according to the operation $(a, b) \mapsto m(\xi)a \otimes_\xi b$.

terminates with probability one. Consequently the same holds for the tree $\tau(\xi, t)$ in continuous time, and with probability one, the number of input nodes is finite. Using this, and the fact that $|a \otimes_\xi b| \leq |a||b|$, it follows that if the stated bounds hold, then times functional is bounded by 1 in absolute value almost surely. The first two of these bounds may be met by assuming $|\hat{u}(\xi)| \leq h(\xi)$ and $2|\hat{f}(\xi, t)| \leq \nu|\xi|^2 h(\xi)$. The bound on $m(\xi) \leq 1$ is the source of the third condition on the FNS majorizing kernel. In three dimensions this bound is obtained by solving the nonlinear convolution equation $h * h(\xi) = C|\xi|h(\xi)$, where $C = \nu(\sqrt{2\pi})^3/2$. The known solutions to $h*h(\xi) = |\xi|h(\xi)$ in three dimensions are

$$h(\xi) = \frac{1}{\pi^3|\xi|^2} \qquad \text{and} \qquad h(\xi) = \frac{\beta}{2\pi}\frac{e^{-\beta|\xi|}}{|\xi|} \quad \beta \geq 0. \tag{16}$$

Rescaling either the dependent or independent variable attains the constant C. Even more solutions to the inequality $h*h(\xi) \leq |\xi|h(\xi)$ in three dimensions are known [1].

The discriminating problem for the two-dimensional FNS equations is that there are no known solutions to the inequality $h*h(\xi) \leq |\xi|h(\xi)$ (with $h(\xi)$ non-negative and fully supported). That is, there are no known 2d-FNS majorizing kernels. Moreover, it is not even known if such functions exists.

It is not clear if this dimensional dependent problem of the majorizing kernels is intrinsically related to the difference between FNS equations in different dimensions.

The above bounds were invoked simply to force the absolute value of the times functional to be bounded by 1 almost surely, and hence integrable. More subtle analysis on the branching process may allow the bounds on the multiplier $m(\xi)$ to be relaxed, and hence conditions on the majorizing kernel $h(\xi)$ to be relaxed as well.

The present method requires the bound $m(\xi) \leq 1$, but we may get around this problem in two dimensions by making a time dependent transformation $\hat{v}(\xi,t) = \hat{u}(\xi,t)e^{-\delta t}$, where $\hat{u}(\xi,t)$ solves FNS. This actually works for either the two- or three-dimensional case. After this transformation the FNS equations (4) become the δ-FNS equations

$$\frac{\partial \hat{v}_k}{\partial t} + (\delta + \nu|\xi|^2)\hat{v}_k = \frac{-i}{(\sqrt{2\pi})^n}\xi_j(\hat{v}_k * \hat{v}_j)e^{\delta t} - i\xi_k\hat{p}e^{-\delta t} + \hat{f}_k e^{-\delta t} \tag{17}$$

The two sets of equations, FNS and δ-FNS are compared in light of this change. The viscosity term in the Navier-Stokes equations was the source of the clock in the branching process, (i.e. the holding time parameter $\nu|\xi|^2$). The transformation changes this to a new clock $\delta + \nu|\xi|^2$; it also changes the multiplier $m(\xi)$ to the new (time dependent) multiplier

$$m'(\xi,t) = \frac{2h*h(\xi)|\xi|e^{\delta(t-s)}}{h(\xi)(\delta+\nu|\xi|^2)(2\pi)^{n/2}} \tag{18}$$

permitting a choice of a majorizing kernel $h(\xi)$ making $m'(\xi,t) \leq 1$.

The probabilistic version of δ-FNS is obtained directly from equation (10) with the transformation $\chi'(\xi,t) = \hat{v}(\xi,t)/h(\xi) = \hat{u}(\xi,t)e^{-\delta t}/h(\xi) = \chi(\xi,t)e^{-\delta t}$. This gives the integral equation

$$\chi'(\xi,t) = e^{-(\delta+\nu|\xi|^2)t}\chi'_0(\xi) + \int_0^t (\delta+\nu|\xi|^2)e^{-(\delta+\nu|\xi|^2)s}\,\{\cdots\}\,ds \tag{19}$$

$$\{\cdots\} = \frac{1}{2}m'(\xi,t)\int_{R^n}\chi'(\eta,t-s)\otimes_\xi \chi'(\xi-\eta,t-s)K_\xi(\eta,\xi-\eta)d\eta + \frac{1}{2}\phi'(\xi,t-s)$$

where $\phi'(\xi,t) = 2\hat{f}(\xi,t)e^{-\delta t}/[(\delta+\nu|\xi|^2)h(\xi)]$. We obtain for this analogue of (10) a similar solution representation $\chi'(\xi,t) = \mathbb{E}\chi'(\tau'(\xi,t))$. The representation method is made to work over a finite time interval $0 \leq t \leq T_*$ where the factor $e^{\delta(t-s)}$ in $m'(\xi,t)$ is bounded by $e^{\delta T_*}$. The bound $m'(\xi,t) \leq 1$ is achieved by finding an $h(\xi) > 0$ such that

$$M'_{\delta,T_*}(\xi) \stackrel{\text{def}}{=} \frac{2h*h(\xi)|\xi|e^{\delta T_*}}{h(\xi)(\delta+\nu|\xi|^2)(2\pi)^{n/2}} \leq 1 \quad \forall \xi \in \mathbb{R}^n. \tag{20}$$

Here ν is intrinsic to Navier-Stokes and is fixed. The parameters δ and T_* are chosen along with $h(\xi)$ so that this inequality holds. This may done by taking the majorizing kernel $h(\xi)$ to be one of the n-dimensional analogues of the Cauchy

density. These are the radially symmetric functions

$$h(\xi) = C_n \frac{\beta}{(\beta^2 + |\xi|^2)^{(n+1)/2}} \tag{21}$$

that share many properties of the one-dimensional Cauchy densities. Here $\beta > 0$ is a scale parameter and $C_n = \Gamma(\frac{n+1}{2})\pi^{-(n+1)/2}$ is the normalization constant. For dimension $n = 2$, Feller [2] calls this, and not the product of two Cauchy densities, the bivariate Cauchy density. The scale parameters add under convolution:

$$h * h(\xi) = C_n \frac{2\beta}{(4\beta^2 + |\xi|^2)^{(n+1)/2}}. \tag{22}$$

A proof of this using Fourier transforms on $\mathbb{R}^n$ is given in [5, p. 50]. Using this $h(\xi)$, the desired inequality (20) becomes

$$\left(\frac{\beta^2 + |\xi|^2}{4\beta^2 + |\xi|^2}\right)^{(n+1)/2} \leq \frac{(\delta + \nu|\xi|^2)(2\pi)^{n/2}}{4|\xi|e^{\delta T_*}} \qquad \forall \xi \in \mathbb{R}^n. \tag{23}$$

The supremum (over ξ) of the l.h.s. is 1, for any β, and the infimum (over ξ) of the r.h.s. occurs at $|\xi| = \sqrt{\delta/\nu}$, assuming the value $\sqrt{\nu\delta}(2\pi)^{n/2}/(2e^{\delta T_*})$. By taking

$$\delta = \frac{4e}{\nu(2\pi)^n}, \qquad T_* = \frac{\nu(2\pi)^n}{8e}, \tag{24}$$

the infimum of the right side of (23) becomes 1, and the inequality is attained:

$$\left(\frac{\beta^2 + |\xi|^2}{4\beta^2 + |\xi|^2}\right)^{(n+1)/2} \leq 1 = \frac{\sqrt{\nu\delta}(2\pi)^{n/2}}{2e^{\delta T_*}} \leq \frac{(\delta + \nu|\xi|^2)(2\pi)^{n/2}}{4|\xi|e^{\delta T_*}} \qquad \forall \xi \in \mathbb{R}^n. \tag{25}$$

With this bound $m'(\xi, t) \leq 1$, we may proceed as before and obtain the solution representation $\chi'(\xi, t) = \mathbb{E}\chi'(\tau'(\xi, t))$ for (19) over the finite time interval $0 \leq t \leq T_*$, that is similar to the representation $\chi(\xi, t) = \mathbb{E}\chi(\tau(\xi, t))$ for (10). The underlying stochastic models for the FNS and the δ-FNS are basically the same, except that the transformation $\hat{v}(\xi, t) = \hat{u}(\xi, t)e^{-\delta t}$ introduces a δ term that changes the mean particle lifetimes from $1/\nu|\xi|^2$ to $1/(\delta + \nu|\xi|^2)$, and new particles are born with a different distribution that depends on the majorizing kernel (in two dimensions)

$$h(\xi) = \frac{1}{2\pi} \frac{\beta}{(\beta^2 + |\xi|^2)^{3/2}}. \tag{26}$$

Note that there is no restriction on the scale parameter β. The times functional $\chi'(\tau'(\xi, t))$ is almost the same as $\chi(\tau(\xi, t))$ except that the multiplier $m'(\xi, t)$ is time dependent. The initial data and forcing must satisfy $|\chi'_0(\xi)| \leq 1$ and $|\phi'(\xi, t)| \leq 1$. Then $\chi'(\xi, t) = \mathbb{E}\chi'(\tau'(\xi, t))$ is bounded by 1 and uniquely solves (19) among solutions that remain bounded by 1 over the time interval $0 \leq t \leq T_*$.

References

[1] Bhattacharya, R.N., Chen, L., Dobson, S., Guenther, R.B., Orum, J.C., Ossiander, M., Thomann, E., and Waymire, E.C. (2002). Majorizing kernels and stochastic cascades with applications to incompressible Navier-Stokes equations. In preparation.

[2] Feller, W. (1966). *An introduction to probability theory and its applications. Vol. II.* John Wiley & Sons Inc.

[3] Le Jan, Y. and Sznitman, A.S. (1997). Stochastic cascades and 3-dimensional Navier-Stokes equations. *Probab. Theory Related Fields* **109**, 343-366.

[4] Mathieu, J. and Scott, J. (2000) *An introduction to turbulent flow.* Cambridge University Press, Cambridge.

[5] Strichartz, R.S. (1994). *A guide to distribution theory and Fourier transforms.* CRC Press, Boca Raton, FL.

ON ESTIMATION THEORY FOR MULTIPLICATIVE CASCADES

MINA OSSIANDER
Department of Mathematics, Oregon State University

1 Introduction

The notion of multiplicative cascade was introduced into the statistical theory of turbulence by A.N. Kolmogorov (1941), (1962) as a phenomenological framework intended to accommodate the intermittency and large fluctuations observed in turbulent fluid flows. The basic idea is that energy is redistributed from larger to smaller scales via a splitting mechanism involving random multiplicative factors known as *cascade generators*. Primarily owing to the scaling structure of this class of models, applications have been extended to a wide variety of other naturally occurring phenomena such as rainfall, internet packet traffic, market prices, etc. which exhibit intermittent and highly variable behavior in space and time. The probability distribution of the cascade generators represents a hidden parameter which is reflected in the fine scale limiting behavior of certain scaling exponents calculated from a single sample realization. This paper gives a sketch of the underlying statistical theory for estimation of the distribution of the generators and describes related problems in turbulence and precipitation.

2 An Overview of the Theoretical Foundations

In this section we provide precise statements of the basic results underlying current statistical theory. Proofs of results given in this section may be found in Ossiander and Waymire (2000).

Let $b \geq 2$ be a natural number and let $\mathbf{T}$ denote the product space

$$\mathbf{T} = \{0, 1, 2, \ldots, b-1\}^{\mathbf{N}} \tag{2.1}$$

equipped with the metric $\rho(s,t) = b^{-|s \wedge t|}, s, t \in \mathbf{T}$, where $\mathbf{N}$ denotes the set of natural numbers and $|s \wedge t| = \inf\{n \geq 0 : s_{n+1} \neq t_{n+1}\}, s = (s_1, s_2, \ldots), t = (t_1, t_2, \ldots) \in \mathbf{T}$. Denote the corresponding Borel sigmafield on $\mathbf{T}$ by $\mathcal{B}(\mathbf{T})$. For $t = (t_1, t_2, \ldots) \in \mathbf{T}$ let $t|n = (t_1, t_2, \ldots, t_n)$. If points $t \in \mathbf{T}$ are viewed as paths through a b-ary tree then $v = t|n$ denotes the nth generation vertex along t and we write $|v| = n$.

For $s \in \mathbf{T}, n \in \mathbf{N}$, denote the closed ball of radius $r = b^{-n}$ centered at s by

$$\Delta_n(s) \equiv \Delta_n(s|n) = B_{b^{-n}}(s) = \{t \in \mathbf{T} : t_i = s_i, i \leq n\}. \tag{2.2}$$

The normalized Haar measure λ on $\mathbf{T}$, viewed as a countable product of cyclic groups of order b, is specified by

$$\lambda(\Delta_n(s)) = b^{-n}, s \in \mathbf{T}, n \geq 1. \tag{2.3}$$

The *cascade generators* are given by a denumerable family of i.i.d. non-negative mean one random variables $\{W_v : v \in \{0, 1, \ldots, b-1\}^n, n \geq 1\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{F}_n, n \geq 1$, denote the filtration defined by

$$\mathcal{F}_n = \sigma\{W_v : |v| \leq n\}, n \geq 1. \tag{2.4}$$

The cascade generators define a sequence of random measures $\{\lambda_n : n \geq 1\}$ on $(\mathbf{T}, \mathcal{B}(\mathbf{T}))$ for $n \geq 1$ via

$$\frac{d\lambda_n}{d\lambda}(t) = Q_n(t) = \prod_{i=0}^{n} W_{t|i} = W_\emptyset \prod_{i=1}^{n} W_{t|i}, t \in \mathbf{T}, \tag{2.5}$$

where $W_\emptyset$, referred to as the *cascade initiator,* is an a.s. positive random variable independent of $\mathcal{F}_n, n \geq 1$.

One may easily check that for any bounded Borel measurable function $f : \mathbf{T} \to \mathbf{R}$, the sequence of random variables $\{\int_{\mathbf{T}} f d\lambda_n\}_{n=1}^{\infty}$ is an L_1-bounded martingale with respect to $\mathcal{F}_n$, and thus has an a.s. limit as $n \to \infty$. This leads to a random measure λ_∞ on $(\mathbf{T}, \mathcal{B}(\mathbf{T}))$ such that

$$P(\lambda_n \Rightarrow \lambda_\infty \, as \, n \to \infty) = 1, \tag{2.6}$$

where $\Rightarrow$ denotes vague convergence; e.g. see Kahane and Peyrière (1976). Indeed, for any countable family Φ of bounded Borel measurable functions, cf. Kahane (1989),

$$P(\lim_{n\to\infty} \int_{\mathbf{T}} f(t)\lambda_n(dt) = \int_{\mathbf{T}} f(t)\lambda_\infty(dt), f \in \Phi) = 1. \tag{2.7}$$

The random measure λ_∞ is known as the *multiplicative cascade measure.* The following basic structure theorem for λ_∞ is also well-known; see Kahane and Peyrière (1976). First let

$$\chi_b(h) = \log_b \mathbf{E}[W^h \mathbf{1}[W > 0]] - (h-1), \tag{2.8}$$

where W is a generic cascade generator distributed as W_v for $v \neq \emptyset$. The structure function $\chi_b(h)$ is defined for all real numbers h but may be infinite, with the conventions that $0^0 = 0, 0 \cdot \infty = 0$. Notice that χ_b is a modified version of the cumulant generating function of $\ln W$. The use of the indicator function $\mathbf{1}[W > 0]$ allows incorporation of the case $h < 0$ into the general theory.

Theorem 2.1 *(Kahane and Peyrière (1976))*
(i.) (Nondegeneracy) $\mathbf{E}\lambda_\infty(\mathbf{T}) > 0$ *iff* $\chi_b'(1-) < 0$.

(ii.) (Convergence of moments) $\mathbf{E}\lambda_\infty^h(\mathbf{T}) < \infty$ *for* $0 \le h \le 1$, *and, if* $h_c := \sup\{h \ge 1 : \chi_b(h) \le 0\} > 1$, *then* $\mathbf{E}\lambda_\infty^h(\mathbf{T}) < \infty$ *for* $1 < h < h_c$.
(iii.) (Support dimension) If $\lambda_\infty(\mathbf{T}) > 0$, *then* λ_∞ *is a.s. supported on a subset of* $\mathbf{T}$ *with Hausdorff dimension* $-\chi_b'(1-)$.

Notice that the last part of Theorem 1.1 delineates an important intermittency parameter associated with the cascade measure λ_∞, $-\chi_b'(1-)$, the a.s. Hausdorff dimension of the subset of $\mathbf{T}$ that supports λ_∞; see Waymire and Williams (1995) for a proof in this generality. Two strongly consistent estimators of this parameter are given in Section 3.

In most applications the probability distribution of the cascade generators is not known apriori. However, one may have data in the form of a single sample realization of the cascade measure of pixels at some prescribed fine scale of resolution. The following result shows that, asymptotically, such data consistently determines the distribution of the cascade generators.

Theorem 2.2 *(Ossiander and Waymire (2000)) Assume that* $\chi_b'(1-) < 0$. *If* $\mathbf{E}[W^h \mathbf{1}[W > 0]]$ *exists and is finite for* h *belonging to some neighborhood of* 0, *then* $\{\lambda_\infty(\Delta_n(v)) : v \in \{0, 1, \ldots, b-1\}^n, n \ge 0\}$ *uniquely determines the distribution of the cascade generator* W.

Throughout the remainder of the paper we will restrict our consideration to cascade generators for which

$$\chi_b'(1-) < 0, \tag{2.9}$$

so that $\mathbf{E}\lambda_\infty(\mathbf{T}) > 0$; cf Theorem 2.1. A basic family of statistics which we consider are the *nth-scale sample moments* defined by

$$M_n(h) = \sum_{|v|=n} \lambda_\infty^h(\Delta_n(v)), h \in \mathbf{R}. \tag{2.10}$$

These are a natural family of functionals which are computable from observing the multiplicative cascade on pixels, or balls, $\Delta_n(v)$. Two different estimators of $\chi_b(h)$, $\widehat{\tau}_n(h)$ and $\widetilde{\tau}_n(h)$, can be defined in terms of the $M_n(h)$'s as follows:

$$\widehat{\tau}_n(h) = n^{-1} \log_b M_n(h) \tag{2.11}$$

and

$$\widetilde{\tau}_n(h) = \log_b(M_{n+1}(h)/M_n(h)). \tag{2.12}$$

It is convenient to introduce a class of random measures $\lambda_\infty(h; dt), h \in \mathbf{R}$, which we refer to as *h-cascades,* and define via the *h-cascade generators*

$$W_v(h) = \frac{W_v^h}{\mathbf{E}W_v^h}, h \in \mathbf{R}. \tag{2.13}$$

The h-cascades are related to the original cascades via

$$\frac{\lambda_n^h(\Delta_n(v))}{b^{n\chi_b(h)}} = \lambda_n(h;\Delta_n(v)), \tag{2.14}$$

where the sequence of *nth level h-cascades,* $n = 1, 2, \ldots$ is defined by

$$\frac{d\lambda_n(h;\cdot)}{d\lambda}(t) \equiv Q_n(h;t) = \prod_{i=0}^{n} W_{t|i}(h), t \in \mathbf{T}. \tag{2.15}$$

The following proposition points to the usefulness of h-cascades in this context.

Proposition 2.1 *For* $h \in \mathbf{R}, n \geq 1,$ *one has*

$$\frac{M_n(h)}{b^{n\chi_b(h)}} = \sum_{|v|=n} Z_\infty^h(v)\lambda_n(h;\Delta_n(v)) = \int_{\mathbf{T}} Z_\infty^h(t|n)\lambda_n(h;dt)$$

where a.s.

$$Z_\infty(v) = \lim_{N\to\infty} \sum_{|u|=N-n} \prod_{i=1}^{N-n} W_{v*(u_1\ldots u_i)} b^{-(N-n)},$$

*and * denotes the concatenation*

$$(v_1, \ldots, v_n) * (u_1, \ldots, u_N) = (v_1, \ldots, v_n, u_1, \ldots, u_N).$$

Notice that this proposition gives an implicit decomposition of the limiting cascade measures. This decomposition is crucial in reaching an understanding of the properties of λ_∞ and functionals thereof. There is one important family of cascade generators for which analysis of this decomposition gives the exact distribution of the cascade measure itself. Fix $b \geq 2$, and take W/b to have a Beta [3] distribution with parameters r and $(b-1)r$ for some $r > 0$. Then the distribution of Z_∞ is that of a Gamma r.v. with shape parameter br and scale parameter $(br)^{-1}$. In particular an exact simulation of the limit cascade may be achieved for models in this family.

The next proposition results from an application of Theorem 2.1 to describe the limiting behavior of the h-cascades. The structure function of the $W_v(h)$'s is given by

$$\chi_{b,h}(r) = \chi_b(hr) - r\chi_b(h).$$

This yields

$$\chi'_{b,h}(1) = h\chi'_b(h) - \chi_b(h).$$

It is easy to check that, as a function of h, $\chi'_{b,h}(1)$ is convex. In light of Theorem 2.1, this gives the following.

[3] In the physics literature the term beta-model is commonly used to describe the cascade model generated with weighted Bernoulli r.v.'s that corresponds to a Galton-Watson branching process. The physics terminology is at odds with the terminology used in the statistics literature, and thus the reader is duly cautioned.

Proposition 2.2 *Assume that* $\chi_b'(1-) < 0$ *and let*

$$H_c^+ = \sup\{h \geq 1 : h\chi_b'(h) - \chi_b(h) < 0\}$$

and

$$H_c^- = \inf\{h \leq 0 : h\chi_b'(h) - \chi_b(h) < 0\}$$

Then $H_c^- \leq 0 < 1 \leq H_c^+$, *with* $h\chi_b'(h) - \chi_b(h) < 0$ *for all* $H_c^- < h < H_c^+$. *Furthermore, for* $h \in [0,1] \cup (H_c^-, H_c^+)$, $\lambda_n(h;\mathbf{T}) \to \lambda_\infty(h;\mathbf{T})$ $P-a.s.$, *where* $\mathbf{E}\lambda_\infty(h;\mathbf{T}) = 1$ *and for* $h \in (H_c^+, h_c)$, $\lambda_n(h;\mathbf{T}) \to 0$ $P-a.s.$ *as* $n \to \infty$.

A discussion of the structure and relationship of the support sets in $\mathbf{T}$ of the limiting h-cascades can be found in Ossiander (2000).

The following theorem is the basis of our derivation of the a.s. convergence of both $\widehat{\tau}_n(h)$ and $\widetilde{\tau}_n(h)$. A generalization of the first part of this theorem is given in Section 3.

Theorem 2.3 *(Ossiander and Waymire (2000))*
(i.) For $h \in [0,1] \cup (H_c^-, H_c^+)$,

$$\frac{M_n(h)}{b^{n\chi_b(h)}} \to \lambda_\infty(h,\mathbf{T})\mathbf{E}\lambda_\infty^h(\mathbf{T})$$

$P-a.s.$ *as* $n \to \infty$.
(ii.) For $h \in (H_c^+, h_c)$,

$$\frac{M_n(h)}{b^{n\chi_b(h)}} \to 0$$

$P-a.s.$ *as* $n \to \infty$.

Convergence of the estimators $\widehat{\tau}_n(h)$ and $\widetilde{\tau}_n(h)$ to the structure function $\chi_b(h)$ for h inside the critical interval (H_c^-, H_c^+) is delineated in Corollaries 2.1 and 2.2 and Corollary 2.3 respectively.

Corollary 2.1 *For any* $h \in [0,1] \cup (H_c^-, H_c^+)$, *the following hold* $P-a.s.$ *as* $n \to \infty$ *on the set* $[\lambda_\infty(\mathbf{T}) > 0]$*:*
(i.) $(\log_b M_n(h) - n\chi_b(h)) \to \log_b \lambda_\infty(h,\mathbf{T}) + \log_b \mathbf{E}\lambda_\infty^h(\mathbf{T})$
and
(ii.) $\widehat{\tau}_n(h) \to \chi_b(h)$.

Corollary 2.2 *On the set* $[\lambda_\infty(\mathbf{T}) > 0]$,

$$\{\widehat{\tau}_n(h) : h \in [0,1] \cup (H_c^-, H_c^+)\} \to \{\chi_b(h) : h \in [0,1] \cup (H_c^-, H_c^+)\}$$

$P-a.s.$ *as* $n \to \infty$.

Corollary 2.3 *On the set* $[\lambda_\infty(\mathbf{T}) > 0]$ *one has* $P-$*a.s. that*

$$\{\widetilde{\tau}_n(h) : h \in [0,1] \cup (H_c^-, H_c^+)\} \to \{\chi_b(h) : h \in [0,1] \cup (H_c^-, H_c^+)\}$$

as $n \to \infty$.

Although both $\widehat{\tau}_n(h)$ and $\widetilde{\tau}_n(h)$ converge to $\chi_b(h)$ for h within the critical regime, in practice $\widetilde{\tau}_n(h)$ is a more useful estimate of $\chi_b(h)$ for moderate values of n. This can be seen from the following heuristic calculations motivated by (i.) of Corollary 2.1:

$$\begin{aligned} \widehat{\tau}_n(h) - \chi_b(h) \quad &= n^{-1} \log_b M_n(h) - \chi_b(h) \\ &\approx n^{-1}(\log_b \lambda_\infty(h, \mathbf{T}) + \log_b \mathbf{E}\lambda_\infty^h(\mathbf{T})). \end{aligned}$$

The term $n^{-1}(\log_b \lambda_\infty(h, \mathbf{T}) + \log_b \mathbf{E}\lambda_\infty^h(\mathbf{T}))$ can be thought of as an asymptotically negligible bias term. On the other hand, $\widetilde{\tau}_n(h)$ is given by a weighted differencing of $\widehat{\tau}_n(h)$ in a way that decreases the bias term:

$$\widetilde{\tau}_n(h) - \chi_b(h) = (n+1)\widehat{\tau}_{n+1}(h) - n\widehat{\tau}_n(h) \approx 0. \tag{2.16}$$

The following theorem reveals that the limiting behavior of $\widehat{\tau}_n(h) = n^{-1} \log_b M_n(h)$, viewed as a function of h, is different outside the set $[0,1] \cup (H_c^-, H_c^+)$; i.e. when the low frequency h-cascade $\lambda_n(h, \mathbf{T})$ dies out a.s. with respect to P. It also makes clear that for estimation purposes the interval (H_c^-, H_c^+) is a true critical interval. Indeed, it shows that for h outside this interval $\widehat{\tau}_n(h)$ estimates a linear extension of $\chi_b(h)$ rather than $\chi_b(h)$ itself. Weaker versions of this result appear in Lovejoy and Schertzer (1991), Holley and Waymire (1992), Collet and Koukiou (1992), Franchi (1995), Molchan (1996), and Troutman and Vecchia (1999).

Theorem 2.4 *(Ossiander and Waymire (2000)) Let*

$$\overline{\chi}_b(h) = \begin{cases} h\chi_b'(H_c^-) & \text{if } h \le H_c^- < 0 \\ \chi_b(h) & \text{if } h \in (H_c^-, H_c^+) \cup [0,1] \\ h\chi_b'(H_c^+) & \text{if } h \ge H_c^+. \end{cases}$$

If $H_c^+ < h_c$ *and* $H_c^- < 0$, *with* $\mathbf{E}W^h\mathbf{1}[W > 0] < \infty$ *for some* $h < H_c^-$, *then on* $A = [\lambda_\infty(\mathbf{T}) > 0]$,

$$\{\widehat{\tau}_n(h) : h \in \mathbf{R}\} \to \{\overline{\chi}_b(h) : h \in \mathbf{R}\}$$

$P-$*a.s. as* $n \to \infty$. *If* $H_c^- = 0$ *and* $H_c^+ < h_c$, *then on* A,

$$\{\widehat{\tau}_n(h) : h \ge 0\} \to \{\overline{\chi}_b(h) : h \ge 0\}$$

$P-$*a.s. as* $n \to \infty$.

Some important open problems remain. For example the limiting behavior of $\widetilde{\tau}_n(h)$ for h outside the critical region $[H_c^-, H_c^+]$ is poorly understood at best. In addition, the lack of satisfactory inversion theory makes it difficult to use the structure function estimators to provide estimates of the distribution of the cascade generators.

Martingale central limit theory may be exploited to obtain asymptotic error distributions for the estimators $\widehat{\tau}_n(h)$ and $\widetilde{\tau}_n(h)$ for h within the scaled critical interval $(H_c^-/2, H_c^+/2)$. The central limit theorem for the estimator $\widetilde{\tau}_n(h)$ is given in Corollary 2.5. The key result for these error distributions may be stated as follows.

For each $n \geq 1$, let $\{X_n(v) : |v| = n\}$ be a collection of independent random variables which are also independent of $\mathcal{F}_n$. Define

$$S_n(h) = \sum_{|v|=n} X_n(v)\lambda_n(h; \Delta_n(v)). \tag{2.17}$$

Also let

$$R_n(h) = \frac{S_n(h)}{(\sum_{|v|=n} \lambda_n^2(h; \Delta_n(v)))^{\frac{1}{2}}}. \tag{2.18}$$

Theorem 2.5 *(Ossiander and Waymire (2000))*
If $\mathbf{E}X_n^2(v) = 1$ *and* $\mathbf{E}X_n(v) = 0$ *for each* v, *and if*

$$\sup_n \sup_{|v|=n} \mathbf{E}|X_n(v)|^{2(1+\delta)} < \infty$$

for some $\delta > 0$, *then for* $h \in (H_c^-/2, H_c^+/2)$,

$$\lim_{n\to\infty} \mathbf{E}[e^{izR_n(h)}|\mathcal{F}_n] = \mathbf{1}[\lambda_\infty(\mathbf{T}) = 0] + e^{-\frac{1}{2}z^2}\mathbf{1}[\lambda_\infty(\mathbf{T}) > 0], \tag{2.19}$$

with the convention that $R_n(h) = 0$ *if* $\lambda_n(\mathbf{T}) = 0$.

Corollary 2.4 *For* $h \in (H_c^-/2, H_c^+/2)$,

$$R_n(h) \to^d \eta N_h$$

where $\eta = \mathbf{1}[\lambda_\infty(\mathbf{T}) = 0]$ *and* N_h *is an independent standard normal random variable.*

Note: it can also be shown that the N_h's are independent for different values of h.

The estimator $\widetilde{\tau}_n(h)$ of $\chi_b(h)$ is obtained by differencing the logarithms of the $h-$th sample moments at scales of resolution $n+1$ and n; namely $\widetilde{\tau}_n(h) = \log_b(M_{n+1}(h)/M_n(h))$. In view of Corollary 2.3 we have asymptotic consistency of this estimator for $h \in (H_c^-, H_c^+)$. The following gives an observable normalization of this estimator,

which allows computation of asymptotically exact confidence intervals for $\chi_b(h)$ for observation of a single realization of the random cascade.

Define

$$V_n^2(h) = \sum_{|v|=n} \left(\frac{\lambda_\infty^h(\Delta_n(v))}{M_n(h)} - \sum_{i=0}^{b-1} \frac{\lambda_\infty^h(\Delta_{n+1}(v*i))}{M_{n+1}(h)}\right)^2. \tag{2.20}$$

The following corollary then gives a central limit theorem for a completely observable statistic whose asymptotic distribution does not depend on the distributions of the unobservable generator variables W or the unknown distribution of the cascade itself, $\lambda_\infty(\mathbf{T})$. The independence in h of the N_h's noted above indicates that the errors in this estimator of $\chi_b(h)$, namely $\widetilde{\tau}_n(h)$, are asymptotically independent.

Corollary 2.5 *For $h \in (H_c^-/2, H_c^+/2)$,*

$$\frac{\widetilde{\tau}_n(h) - \chi_b(h)}{V_n(h)} \to^d (\log b)^{-1} \eta N_h. \tag{2.21}$$

3 Dimension Estimates: Some Recent Results

The intermittency of the cascade measure is reflected in the Hausdorff dimension of the supporting set of the measure. This important geometric parameter is given by $-\chi_b'(1^-) = 1 - \mathbf{E}W \log_b W$, the derivative of the structure function at $h = 1$, whenever the cascade survives. Two natural estimators for the Hausdorff dimension are

$$\widehat{D}_n = -(n\lambda_\infty(\mathbf{T}))^{-1} \sum_{|v|=n} \lambda_\infty(\Delta_n(v)) \log_b \lambda_\infty(\Delta_n(v)), \tag{3.1}$$

defined for $n \geq 1$, and

$$\widetilde{D}_n = (\lambda_\infty(\mathbf{T}))^{-1} \quad \Big(\sum_{|v|=n} \lambda_\infty(\Delta_n(v)) \log_b \lambda_\infty(\Delta_n(v)) - \sum_{|v|=n+1} \lambda_\infty(\Delta_{n+1}(v)) \log_b \lambda_\infty(\Delta_{n+1}(v))\Big), \tag{3.2}$$

defined for $n \geq 0$. Notice that $\widehat{D}_n = -\widehat{\tau}_n'(1)/\lambda_\infty(\mathbf{T})$ and $\widetilde{D}_n = -\widetilde{\tau}_n'(1)/\lambda_\infty(\mathbf{T})$. $\widehat{D}_n$ has been used by physicists as an estimator of the Hausdorff dimension of the support set of a measure; see Chhabra and Jensen(1989). $\widetilde{D}_n$ was introduced by Ossiander and Waymire (2001). The following theorem shows that both $\widehat{D}_n$ and $\widetilde{D}_n$ are strongly consistent estimators of $-\chi_b'(1)$.

Theorem 3.1 *(Ossiander and Waymire (2001))) If $h_c > 1$, then both*
(i.) $\widetilde{D}_n \to -\chi_b'(1)$
and
(ii.) $\widehat{D}_n \to -\chi_b'(1)$
$P-a.s.$ as $n \to \infty$.

A central limit theorem for the estimator $\widetilde{D}_n$ is also obtainable. The observable normalization is given by

$$\widetilde{V}_n^2 = \sum_{|v|=n} (\lambda_\infty(\Delta_n(v)) \log_b \lambda_\infty(\Delta_n(v)) \tag{3.3}$$
$$- \sum_{i=0}^{b-1} \lambda_\infty(\Delta_{n+1}(v*i)) \log_b \lambda_\infty(\Delta_{n+1}(v*i))$$
$$- \lambda_\infty(\Delta_n(v))\widetilde{D}_n)^2.$$

Theorem 3.2 *(Central Limit Theorem for $\widetilde{D}_n$) If $H_c^+ > 2$, then*

$$\frac{(\widetilde{D}_n + \chi_b'(1))\lambda_\infty(\mathbf{T})}{\widetilde{V}_n} \to^d \eta N,$$

where $\eta = \mathbf{1}[\lambda_\infty(\mathbf{T}) = 0]$ and N is an independent standard normal random variable.

The proofs of both Theorem 4.1 and 4.2 above can be found in Ossiander and Waymire (2001). Both depend on the following generalization of Theorem 2.3. It can be thought of a strong law of large numbers for a collection of identically distributed r.v.'s $\{X(v)\}$ with the random weights $\lambda_n(\Delta_n(v))$ playing the role of the usual deterministic weights b^{-n}.

Theorem 3.3 *Suppose that $\{X(v) : |v| = n, n \geq 1\}$ is a collection of identically distributed random variables defined on $(\Omega, \mathcal{F}, P)$ with $\mathbf{E}|X(v_0)|^{1+\epsilon} < \infty$ for some $\epsilon > 0$ and, for each $n \geq 1$, $\{X(v) : |v| = n\}$ is a collection of independent random variables which is also independent of $\mathcal{F}_n$. Then for $h \in [0,1] \cup (H_c^-, H_c^+)$,*

$$\sum_{|v|=n} X(v)\lambda_n(h; \Delta_n(v)) \to \lambda_\infty(h; \mathbf{T})\mathbf{E}X(v_0)$$

$P-$ a.s. as $n \to \infty$.

The consistent statistical estimation of other parameters of the fine scale structure, e.g. other points of the singularity spectrum and multifractal dimensions, remains as an undeveloped problem.

4 Lognormal Versus log Poisson Generators; A Statistical Problem in Turbulence

This section outlines a major outstanding problem for statistics in the physical sciences, namely statistical inference for cascade models of energy dissipation in turbulence. The research sketched below focuses on the multiscaling exponent data

and theory available in the physics literature; alternative approaches are possible which are not reviewed here, see for example Jouault, Greiner, Lipa (2000) and Barndorff-Nielsen, Jensen, and Sorensen (1990).

The *energy dissipation rate* ϵ defined by

$$\epsilon(x) = \frac{\nu}{2} \sum_{i,j=1}^{3} (\frac{\partial u_i}{\partial x_j})^2, \quad x \in R^3, \tag{4.1}$$

is computed in terms of the fluid velocity $u = (u_1, u_2, u_3)$ as the local rate of decay of kinetic energy $\frac{d}{dt}\frac{1}{2}\int_V |u(x)|^2 dx$ from incompressible Navier-Stokes equation in a region V with viscosity parameter $\nu > 0$.

One begins with the assumption that the multiplicative cascade with i.i.d. non-negative mean one generators is a valid statistical model for the turbulent redistribution of energy in the statistical model of the random dissipation field $\epsilon(dx)$ over an appropriate range of length scales, referred to as the *Kolmogorov inertial range*. The Kolmogorov inertial range is an interval of length scales from the largest length scale at which energy enters the system down to the smallest length scale at which energy is dissipated by fluid viscosity. Actual observations of ϵ are one-dimensional cross sections wherein (4.1) is replaced by the surrogate measurement $15\nu(\frac{\partial u_1}{\partial x_1})^2$. As a result Jouault, Greiner, and Lipa (2000) have effectively argued that i.i.d. mean one generators provide the appropriate model for measurements of energy dissipation rates from the point of view of conservation laws. In particular, taking one-dimensional cuts through the three dimensional energy dissipation field makes the measurements non-conservative in an almost sure sense. The statistical model with i.i.d. mean one generators provides conservation on average.

Kolmogorov's lognormal hypothesis leads to a quadratic structure function $\chi_b(h)$ as follows. A lognormal cascade generator can be written as $W = e^{\sigma Z - \frac{\sigma^2}{2}}$, where Z has a standard normal distribution. This gives the quadratic structure function

$$\chi_b(h) = \frac{\sigma^2}{2\ln b} h^2 - (\frac{\sigma^2}{2\ln b} + 1)h + 1 \tag{4.2}$$

with critical values given by the roots

$$H_c^+ = \frac{\sqrt{2\ln b}}{\sigma} \quad \text{and} \quad H_c^- = -\frac{\sqrt{2\ln b}}{\sigma} \tag{4.3}$$

of $h\chi_b'(h) - \chi_b(h)$. Early analysis of turbulence data revealed a departure from quadratic multiscaling exponents, which is now understood to be remarkably adjusted by the linear correction $\overline{\chi}_b(h)$ as depicted in Figure 1. The data points depicted here are from Anselmet et al. (1984). This effect had already been anticipated by preliminary calculations in the physics literature; see Lovejoy and Schertzer (1991) and Molchan (1997). Here the parameterizing ratio $\sigma^2/2\ln b$ is taken to be

.1 as suggested by Anselmet et al. This gives $\chi_b(h) = .1h^2 - 1.1h + 1$, $H_c^+ = \sqrt{10}$, and $\chi'_b(1) = -.9$.

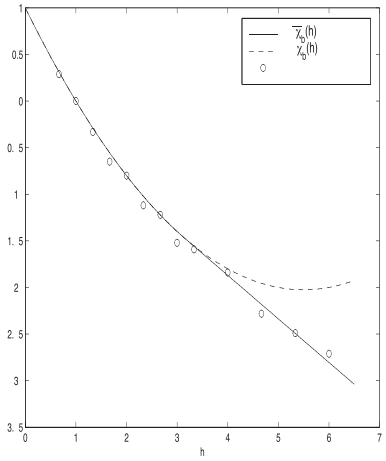


Figure 1. Graph of the linearly corrected lognormal structure function with the Anselmet turbulence data superimposed. Here $H_c^+ = \sqrt{10}$.

Largely prompted by discrepancies between the observed data and the quadratic structure function as illustrated in Figure 1, various adhoc alternatives to the lognormal hypothesis have been considered in the physics literature; see Frisch (Figure 8.8, p.132;1995). The log Poisson distribution surfaced as an alternate hypothesis as a somewhat indirect consequence of an analysis by She and Lèvêsque (1994), Dubrulle (1994), and She and Waymire (1994,1995). Specifically, She and Lèvêsque (1994) obtain the following second order, linear, nonhomogeneous difference equation for the scaling exponents $\tau(h) = -\chi_b(h) - (h-1)$:

$$\tau(h+2) - (1+\beta)\tau(h+1) + \beta\tau(h) + \frac{2}{3}(1-\beta) = 0, \tag{4.4}$$

where $\beta = \frac{2}{3}$, and $\tau(0) = \tau(1) = 0$ as a consequence of the following log-convexity hypothesis on the structure of the size-biased moments $\epsilon_l^{(h)} := E\epsilon_l^{h+1}/E\epsilon_l^h$ of energy dissipation:

$$\epsilon_l^{(h+1)} = A_h(\epsilon_l^{(h)})^\beta(\epsilon_l^{(\infty)})^{1-\beta}, \quad 0 < \beta < 1. \tag{4.5}$$

According to She (personal communication) this hypothesis was formulated in response to observations made in numerical simulations of Navier-Stokes equations. It is straightforward to check that the structure function specified above corresponds to one deriving from a log Poisson generator. That is, distributionally $W = b^{2/3}\beta^Y$, with Y being a Poisson r.v. with parameter $2\ln b/3(1-\beta) > 0$. This gives

$$\chi_b(h) = (2\beta^h - (1-\beta)h + 1 - 3\beta)/3(1-\beta). \tag{4.6}$$

It is interesting to note that for $\beta = 2/3$ as specified above, $\chi'_b(1) = -.874$. This gives a value for the support dimension that is quite close that given by Kolmogorov's

lognormal model. A comparison of the Anselmet data to the She and Lèvêsque log Possion structure function with and without the linear correction is given in Figure 2.

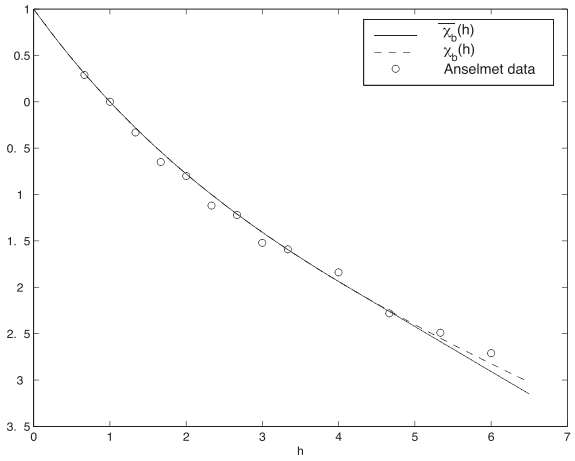


Figure 2. Graph of the linearly corrected log Poisson structure function of She and Lèvêsque with the Anselmet turbulence data superimposed. Here $H_c^+ \approx 4.14$.

The approach of She and Lèvêsque has been generalized by others. For example Chen and Cao (1995) describe a family of models with structure functions of the form $\chi(h) = (\beta^h - \beta)/(1 - \beta)$ for $0 < \beta < 1$. This family of structure functions again corresponds to log Poisson generators. In particular Chen and Cao suggest taking $\beta = 7/9$, leading to $\chi'(1) = -.8796$. A comparison of the Anselmet data to the Chen and Cao structure function with and without the linear correction is given in Figure 3.

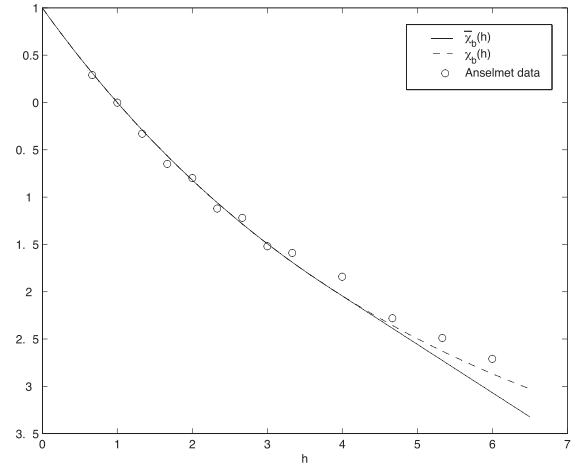


Figure 3. Graph of the linearly corrected log Poisson structure function of Chen and Cao with the Anselmet turbulence data superimposed. Here $H_c^+ \approx 3.87$.

Obviously if the $\widehat{\tau}_n(h)$'s as observed by Anselmet et al. gave consistent estimates of the structure function $\chi_b(h)$ for all values of h, then the need for extensive statistical theory to test distributional hypotheses concerning cascade generators would hardly be justified. However, the problem of distinguishing between the proposed models based on an understanding of the convergence properties of $\widehat{\tau}_n(h)$ as illustrated in Figures 1 through 3 requires an understanding of the error distributions of the observed statistics. It should be noted that in the literature the reported data is often that of velocity exponents $\zeta(h)$ defined by $E(u_1(x+\lambda) - u_1(x))^h \sim \lambda^{\zeta(h)}$ from which $\tau(h)$, defined by $E\epsilon^h(dx) \sim (dx)^{\tau(h)}$, is obtained by the assumed relationship based on dimensional arguments that $\zeta(h) = \frac{h}{3} + \tau(\frac{h}{3})$. Distinguishing between proposed models will entail an analysis relying on more detailed tabulation of measurements of the dissipation rates at small length scales.

5 Cascade Models and Temporal Rainfall

Although the standard canonical cascade model described above captures the qualitative structure of rainfall intensity over space at a fixed point in time (c.f. Gupta and Waymire (1993)), there are some well-known inconsistencies that become manifest when using it as a model of rainfall intensity over time at a fixed point in space. These are illustrated in Olsson (1998), Menabde and Sivapalan (2000), and Rupp et al. (2001). In particular, it seems that in order to adequately fit a multiplicative-type model to measurements of rainfall over time, the distribution of the cascade generators $W_{t|n}$ must be allowed to depend on the length of the time interval Δ_n. This is seen by looking at ratios of rainfall measurement of nested pixels at adjacent time scales $b^{-(n-1)}$ and b^{-n}. Assuming a multiplicative cascade structure, these ratios can be simplified as follows:

$$\frac{\lambda_\infty(\Delta_n(t))}{\lambda_\infty(\Delta_{n-1}(t))} = \frac{b^{-n}\prod_{j=1}^{n} W_{t|j} \times Z_\infty(t|n)}{b^{-(n-1)}\prod_{j=1}^{n-1} W_{t|j} \times Z_\infty(t|(n-1))} = \frac{W_{t|n} Z_\infty(t|n)}{b Z_\infty(t|(n-1))}. \tag{5.1}$$

Under the standard canonial cascade model, these ratios should have the same distribution at each scale of resolution. However, histograms of these ratios in Olsson, Menabde and Sivapalan, and Rupp et al. all show that the distributions of these ratios change systematically as the time scale changes. In particular, in both Olsson and Rupp et al. it appears that the distribution of the ratios becomes more concentrated about the fixed value b^{-1} as the scale of resolution decreases to 0. This is illustrated in Figure 4, which presents histograms from Rupp et al. of the ratios of rainfall measurements at different sites close to Corvallis, Oregon.

This indicates that the splitting of rainfall between adjacent pixels is on average more uniform at fine scales of resolution compared to that at coarse scales and reflects a process which is locally smoother and more correlated than a canonical multiplicative cascade. It should be added that in the data generating Figure 4, the

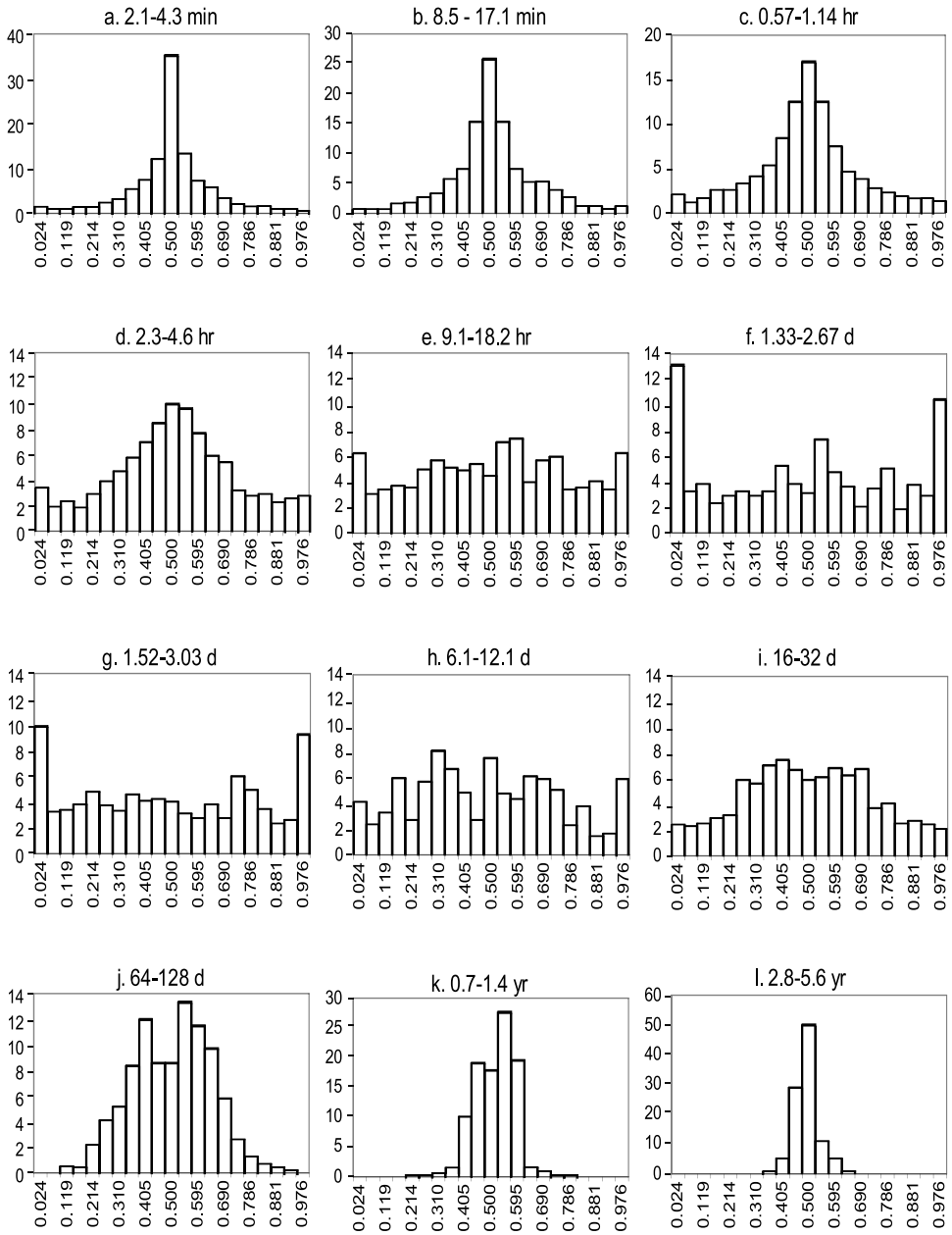


Figure 4. Relative frequency histograms of the cascade splitting ratio (5.1) at different cascade levels. (Data Sources; a-e, g: Johnson Creek tipping bucket; f: Corvallis hourly; h-i: Corvallis daily.)

seasonal periodicity of rainfall is handled via a statistical homogenization technique; a Fourier series representation of mean rainfall is used to normalize time, having the effect of expanding time during the rainy season and contracting time during the dry season so that the expected re-scaled rate of rainfall is constant. The ratios

presented in Figure 4 were originally calculated both with and without the time homogenization. The same local smoothing appeared in both cases. This preliminary work can be found in Rupp et al. (2001), where a scale-dependent version of the canonical cascade model is developed in order to better fit and simulate rainfall data.

The above considerations illustrate a long-standing need for realistic stochastic models of space-time precipitation fields. Promising attempts toward extending the spatial models to the space-time setting can be traced to the pioneering efforts of LeCam (1961) which are widely utilized in hydrologic applications for rain on large scales of bands and frontal systems. Temporal extensions of cascades initiated by Gupta and Over(1996) demonstrate this to be a promising research direction in contending with the high intensity fine scale structure in space and time. Others who have recently worked in this vein include Jothityangkoon et al. (2000) and Deidda (2000).

References

[1] Anselmet,F., Y. Gagne, E.J. Hopfinger, and R.A. Antonia (1984) High-order velocity structure functions in turbulent shear flows, *J. Fluid Mech.* **140**, 63-89

[2] Barndorff-Nielsen, O.E., J.L. Jensen, and M. Sorensen (1990) Parametric modeling of turbulence, *Phil. Trans. R. Soc. Lond. A* **332** 439-455.

[3] Chen,S. and N. Cao (1995) Inertial range scaling in turbulence, *Phys. Rev. E* **52**, R5757-R5759

[4] Chhabra,A. and R.V.Jensen. (1989) Direct determination of the $f(\alpha)$ singularity spectrum. *Phys. Rev. Let.,* **62** 1327-1330.

[5] Collet,P. and F. Koukiou. (1992) Large deviations for multiplicative cascades. *Commun. Math. Phys.,* **147** 329-342.

[6] Deidda, R., R. Benzi, and F. Siccardi. (1999) Multifractal modeling of anomalous scaling laws in rainfall, *Water Resources Research*, **35**, 1853-1867.

[7] Dubrulle, B. (1994) Intermittancy in fully developed turbulence: logPoisson statistics and generalized scale invariance. *Phys. Rev. Lett.* **73** 959-962.

[8] Franchi, J. (1995) Chaos multiplicatif: un traitment simple et complet de la fonction de partition. *Lecture Notes in Math: Sem. de Prob.* **1613** Springer-Verlag, Berlin.

[9] Frisch, U. (1995) *Turbulence: The Legacy of A.N. Kolmogorov.* Cambridge University Press, New York.

[10] Gupta, V.K. and E. Waymire. (1993) A statistical analysis of mesoscale rainfall as a random cascade. *Jour. Appld. Meteor.* **32** (2) 251-267.

[11] Holley, R. and E. Waymire. (1992) Multifractal dimensions and scaling exponents for strongly bounded random cascades. *Ann. Applied Probab.* **2** 819-845.

[12] Jothityangkoon, C., M. Sivapalan, and N.R. Viney, (2000) Tests of a space-time model of daily rainfall in southwestern Australia based on nonhomogeneous random cascades. *Water Resources Research*, **36**, 267-284.

[13] Jouault,B., M. Greiner, P. Lipa Fix-point multiplier distributions in discrete turbulent cascade models, *Physica D* **135** 154-174.

[14] Kahane, J.P. and J. Peyrière. (1976) Sur certaines martingales de B. Mandelbrot. *Adv. in Math.* **22** 131-145.

[15] Kolmogorov, A.N. (1941) The local structure of turbulence in incompressible viscous fluid for very large Reynolds number. *Dokl. Akad. Nauk SSSR.* **30** 9-13.

[16] Kolmogorov, A.N. (1962) A refinement of previous hypothesis concerning the local structure of turbulence in a viscous incompressible fluid at high Reynold number. *J. Fluid Mech.* **13** 82-85.

[17] Le Cam, L. (1961) A stochastic description of precipitation *Proc. 4th Berkeley Sympos. Math. Statist. and Prob.*, **III**, 165-186.

[18] Lovejoy, S. and D. Schertzer. (1991) Universal hard multifractal turbulence:theory and observations. *Nonlinear Dynamics of Structures.* (Eds R.Z. Sagdeev, U. Frisch, F. Hussain, S.s. Moisdeev, N.S. Erokhin.) World Scientific Press.

[19] Menabde, M., and M. Sivapalan, (2000) Modeling of rainfall time series and extremes using bounded random cascades and Levy-stable distributions, *Water Resources Research*, **36**, 3293-3300.

[20] Molchan, G. M. (1996) Scaling exponents and multifractal dimensions for independent random cascades. *Comm. Math. Phys.* **179** 681-702.

[21] Molchan, G. M. (1997) Turbulent cascades: limitations and a statistical test of the lognormal hypothesis. *Phys. Fluids* **9** 2387-2396.

[22] Olsson, J., (1998) Evaluation of a scaling cascade model for temporal rainfall disaggregation, *Hydrology and Earth System Sciences*, **2**, 19-30.

[23] Ossiander, M.E. (2000) Support fragmentation for multiplicative cascade measures. *Progress in Probability, Proceedings of High Dimensional Probability II, Seattle 1999.* (Eds. E. Gine, D. M. Mason, J. A. Wellner.) Birkhauser, Boston. (In press.)

[24] Ossiander, M.E. and E.C. Waymire (2000) Statistical estimation theory for multiplicative cascades, *Ann. Stat.* **28**, 1533-1560.

[25] Ossiander, M.E. and E.C. Waymire (2002) On estimation theory for multiplicative cascades, *To appear in Sankhya.*

[26] Over, T. and V. Gupta (1994) Statistical analysis of mesoscale rainfall: dependence of a random cascade generator on large-scale forcing, *J. of Applied Meteorology* **33** 1526-1542.

[27] Rupp, D.E., R.F. Keim, M. Ossiander, M. Brugnach, J.S. Selker, A scale-dependent random cascade model of temporal rainfall with periodicity, *preprint.*

[28] She, Z.S. and E. Lèvêsque. (1994) Universal scaling laws in fully developed turbulence. *Phys. Rev. Lett.* **72** 336.

[29] She, Z.S. and E. Waymire. (1994) LogPoisson statistics in fully developed turbulence. *MSRI Series* **No. 046-94** Berkeley, CA.

[30] She, Z.S. and E. Waymire. (1995) Quantized energy cascade and logPoisson statistics in fully developed turbulence. *Phys. Rev. Lett.* **74(2)** 262-265.

[31] Troutman, B. and A.V. Vecchia. (1999) Estimation of Renyi exponents in random cascades. *Bernoulli.* **5** 191-207.

[32] Waymire, E. and S.C. Williams.(1995) Multiplicative cascades: Dimension spectra and dependence *Jour. Fourier Anal. and Appl.* **Special Edition** 589-609.

PARTIAL DIFFERENTIAL EQUATIONS AND MULTIPLICATIVE PROCESSES

ENRIQUE THOMANN
Oregon State University

1 Introduction

In this paper a survey of recently developed methods for representing solutions of linear, semilinear and quasilinear parabolic differential equations as an expected value of a random variable defined on a branching process that incorporates the nonlinearities of the equation under consideration is presented. The solution representation is obtained by considering the Fourier transform of the equations and using its parabolic character to introduce a natural clock that controls the branching process. The approach is general enough to be applied to equations that have a polynomial nonlinearity. Examples include the Kolmogorov, Petrovskii and Piskunov equation (KPP), Burgers equation and the incompressible Navier-Stokes equations. The method developed here is an extension of a branching process developed in LeJan-Sznitman [10] to study the Navier Stokes equations. Further relations of this approach with McKean [11] branching Brownian motion developed for the KPP equations and Kato [9] abstract Picard iteration for the Navier-Stokes equations are also presented in detail. Finally, the use of the this branching process to obtain existence, uniqueness and regularity properties of solutions of these nonlinear equations is developed. The results presented in this paper are joint work with R. Bhattacharya, L. Chen, S. Dobson, J. C. Orum, M. Ossiander and E. Waymire (see Bhattacharya et al. [2]).

The organization of this paper is as follows. In Section 2 linear equations are considered and a branching processes is used to provide an alternative to the Feynman-Kac formula. In Section 3, solutions of the Navier-Stokes equations are represented as expected values of a functional defined along an appropriate branching process. This constitute the major motivation for the methods developed in this paper. A detailed construction of the branching process is also included in this section. In Section 4, the properties and examples of majorazing kernels introduced in section 3 are presented. In Section 5, the representation of solutions of the Navier-Stokes equations obtained in section 4 is related to the abstract Picard iteration scheme developed by Kato [9]. Also, a similar relation is established for the KPP equation relating the methods presented in this paper with the more classical branching process introduced by McKean [11] (see also Bramson [3] and Friedlin [7].) Further conclusion and remarks are made in the section 6.

2 Linear equations: Feynman-Kac Formula

A simple example that illustrates the use of branching processes in the study of solutions of linear parabolic differential equations is given by considering the initial value problem of the heat equation on the real line with a scalar potential whose Fourier transform is a complex measure. This provides an alternative to the Feynman-Kac formula. To make the presentation simple, consider the initial value problem with a $\cos(x)$ potential written as

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + (\cos(x) + \frac{1}{2})u - \frac{1}{2}u, \qquad u(x,0) = u_0(x) \tag{2.1}$$

where a is a positive real number.

As usual, define the Fourier transform of an integral function in R^d by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{R^d} e^{-ix\xi} f(x) dx. \tag{2.2}$$

A simple calculation gives

$$\begin{aligned} \hat{u}(\xi,t) &= \hat{u}_0(\xi) e^{-\lambda(\xi)t} \\ +(2\pi)^{d/2} &\int_0^t e^{-\lambda(\xi)s} \left[\frac{1}{2}\hat{u}(\xi+1,t-s) + \frac{1}{2}\hat{u}(\xi-1,t-s) + \frac{1}{2}\hat{u}(\xi,t-s)\right] ds, \end{aligned} \tag{2.3}$$

where $\lambda(\xi) = a|\xi|^2 + 1/2$.

Consider now random trees τ_θ rooted at a vertex θ of type ξ which after an exponential length of time S_θ with parameter $\lambda(\xi_\theta)$ is replaced by a vertex of type $\xi_{<1>}$ chosen with distribution

$$\mathbf{P}_\xi(\xi_{<1>} = \xi + 1) = \mathbf{P}_\xi(\xi_{<1>} = \xi) = \mathbf{P}_\xi(\xi_{<1>} = \xi - 1) = \frac{1}{3}.$$

Define the recursive functional

$$\curlywedge(\tau_\theta(\xi,t)) = \begin{cases} \hat{u}_0(\xi) & \text{if } S_\theta \geq t \\ m(\xi)\curlywedge(\tau_{<1>}(\xi_{<1>}, t - S_\theta)) & \text{if } S_\theta < t \end{cases} \tag{2.4}$$

where $\tau_{<1>}$ denotes the random tree re-rooted at $\xi_{<1>}$ and

$$m(\xi) = \frac{3(2\pi)^{d/2}}{2\lambda(\xi)}.$$

Thus, using the strong Markov property of the branching process it follows that

$$v(\xi,t) = \mathbf{E}_\xi\left(\curlywedge(\tau_\theta(\xi,t))\right)$$

satisfies (2.3) provided the expected value is finite.

In general, Ito [8] showed that the Feynman-Kac formula applies to scalar potential $c(x)$ the Fourier transform of which is a complex measure. In this context, a similar result can be obtained using a branching process analogue to (2.4) to obtain (see [5] for details)

Proposition 2.1 *Assume $\hat{c}(\xi)$ is a complex measure and $u_0 \in L^1(R)$. The Fourier transform of the solution of the IVP*

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + c(x)u, \quad t > 0, \quad u(x,0) = u_0(x)$$

is given by

$$\hat{u}(\xi,t) = \mathbf{E}\left[\curlywedge(\tau_\theta(\xi,t))\right]$$

where $\curlywedge$ is the recursive functional defined by (2.4) for which the distribution of types is determined by $\hat{c}(\xi)$.

3 Nonlinear equations: Navier-Stokes

The study of existence, uniqueness and regularity of solutions to 3d incompressible Navier-Stokes (NS) equations remains one of the outstanding open problems in applied mathematics (see e.g. Fefferman [6]). Following LeJan and Sznitman [10] work, it is possible to obtain a representation of solutions of the Navier-Stokes equations using a branching process and use this representation to study properties of the solutions. In this section, the construction of the branching process is presented in detail providing a motivation for the introduction of majorizing kernels as a tool for establising properties of solutions of the Navier Stokes equations. It should be remarked that the branching process is not intrinsically defined by the Navier Stokes equations in contrast with the case of the linear problem presented in the previous section (cf Proposition 2.1.) Instead, the majorizing kernels introduced in this section serve the dual purpose of determining the distribution of types in the branching process and providing the means for establishing the existence of solutions to the Navier Stokes equations (cf. Proposition 3.1 and Theorem 3.1.)

Recall that the Navier- Stokes equations in R^3 are given by

$$\frac{\partial \mathbf{u}}{\partial t} = \nu\Delta\mathbf{u} - \nabla\cdot(\mathbf{u}\otimes\mathbf{u}) + g(x,t) - \nabla p.$$

$$\nabla\cdot\mathbf{u} = 0$$

where $\mathbf{u}$ is the velocity of the fluid, Δ denotes the 3d (componentwise) Laplacian, $\nu > 0$ is the viscosity, p the pressure and g represents known external forces. Also, $\mathbf{u}\otimes\mathbf{u}$ denotes the matrix with entries u_iu_j and $\nabla\cdot(\mathbf{u}\otimes\mathbf{u})$ is the vector with components $\sum_j \partial u_j u_i/\partial x_j$. In terms of the Fourier transform of the unknown velocity $\hat{\mathbf{u}}$, the equations can be written as

$$\begin{aligned}\hat{\mathbf{u}}(\xi,t) &= e^{-\nu t|\xi|^2}\hat{\mathbf{u}}_0(\xi) + \int_0^t e^{-\nu|\xi|^2 s}\Big\{|\xi|(2\pi)^{-\frac{3}{2}} \\ &\int_{\mathbf{R}^3} \hat{\mathbf{u}}(\eta,t-s)\odot_\xi \hat{\mathbf{u}}(\xi-\eta,t-s)d\eta + \hat{g}(\xi,t-s)\Big\}ds, \qquad \text{(FNS)}\end{aligned}$$

where for complex vectors w, z

$$w \odot_\xi z = -i(e_\xi \cdot z)\Pi_{\xi^\perp} w, \qquad e_\xi = \frac{\xi}{|\xi|}, \tag{3.1}$$

and $\Pi_{\xi^\perp} w = I - e_\xi \otimes e_\xi$ is the projection of w orthogonal to ξ which eliminates the unknown pressure.

For $\xi \neq 0$, LeJan and Sznitman [10] rescale the equation (FNS) to normalize the integrating factor $e^{-\nu|\xi|^2 s}$ to the exponential probability density $\nu|\xi|^2 e^{-\nu|\xi|^2 s}$. The resulting equation is precisely the form for a branching random walk recursion for $\frac{\hat{u}(\xi,t)}{\nu|\xi|^2}$ resulting in the representation of the solution of the (FNS) equation as an expected value. Furthermore, if $|\hat{g}(\xi,t)| \leq \nu^2(2\pi^3)^{-1/2}$ and $|\hat{\mathbf{u}}_0(\xi)| \leq \nu(2/\pi^3)^{1/2}|\xi|^{-2}$ they show global existence and uniqueness of the solution to the (FNS) equation with $|\hat{\mathbf{u}}(\xi,t)| \leq \nu(2/\pi^3)^{1/2}|\xi|^{-2}, 0 \leq t \leq T, \xi \neq 0$.

To extend this approach, introduce non-negative measurable functions h such that

$$h * h(\xi) \leq B|\xi| h(\xi), \quad \xi \neq 0, B > 0. \tag{3.2}$$

Refer to such a function h as an *FNS-admissible*[4] *majorizing kernel*, or simply as a majorizing kernel. Note that if h is a majorizing kernel with constant B then $\frac{h}{B}$ is a majorizing kernel with unit constant. Also, if $h(\xi)$ is a majorizing kernel then so is $ce^{a\cdot\xi}h(\xi)$ for arbitrary fixed vector a and positive scalar c. To avoid unnecessary technicalities regarding their supports, this section is restricted to positive majorizing kernels $h(\xi)$ defined for $\xi \neq 0$. However, examples of majorizing kernels with support W_h a proper subset of $R^3 - \{0\}$ are given in the next section and the construction of the branching process given below can be extended to cover this case as well. Formulated in this way, the results of LeJan and Sznitman [10] show that

$$h_0(\xi) = \frac{1}{|\xi|^2} \quad \text{and} h_1(\xi) = \frac{e^{-|\xi|}}{|\xi|} \tag{3.3}$$

are fully supporter majorizing kernels satisfying equality in (3.2).

Given a majorizing kernel consider the Fourier transformed equation (FNS) rescaled by factors of the form $\frac{1}{h(\xi)}$, for $\xi \neq 0$. Namely, consider

$$\chi(\xi,t) = e^{-\nu t|\xi|^2}\chi_0(\xi) \quad + \quad \int_0^t \nu|\xi|^2 e^{-\nu|\xi|^2 s} \qquad (\text{FNS})_h$$
$$\{\frac{1}{2}m(\xi)\int_{\mathbf{R}^3} \chi(\eta, t-s) \odot_\xi \chi(\xi-\eta, t-s)\ \frac{h(\eta)h(\xi-\eta)}{h*h(\xi)}\ d\eta + \frac{1}{2}\varphi(\xi, t-s)\}ds,$$

[4]The method is more generally applicable to kernels that are defined on other subsets of R^3, and other equations having nonlinearities reflected in a multilinear form. This is currently being developed by the authors in a more comprehensive monograph.

where

$$m(\xi) = \frac{2h * h(\xi)}{\nu(2\pi)^{\frac{3}{2}}|\xi|h(\xi)}, \quad \chi(\xi,t) = \frac{\hat{u}(\xi,t)}{h(\xi)}, \quad \varphi(\xi,t) = \frac{2\hat{g}(\xi,t)}{\nu|\xi|^2 h(\xi)}.$$

Notice that for each fixed ξ with $h * h(\xi) \neq 0$, the convolution $h * h(\xi)$ simply normalizes the product $h(\eta_1)h(\eta_2)$ to be a probability kernel on the set $\eta_1 + \eta_2 = \xi$. It is then possible to show the existence of globally defined solutions of the (FNS) equations, the regularity of which depends on the particular majorizing kernel.

A stochastic model consistent with $(\mathrm{FNS})_h$ is obtained by consideration of a multitype branching random walk of nonzero Fourier wavenumbers ξ, thought of as particle *types,* as follows: A particle of type $\xi \neq 0$ initially at the root θ holds for an exponentially distributed length of time S_θ with holding time parameter $\lambda(\xi) = \nu|\xi|^2$; i.e. $ES_\theta = \frac{1}{\nu|\xi|^2}$. When this exponential clock rings, a coin κ_θ is tossed and either with probability $\frac{1}{2}$ the event $[\kappa_\theta = 0]$ occurs and the particle is terminated, or with probability $\frac{1}{2}, [\kappa_\theta = 1]$ and the particle is replaced by two offspring particles of types η_1, η_2 selected from the set $\eta_1 + \eta_2 = \xi$ according to the probability kernel

$$K_\xi(\eta_1, \eta_2) = \frac{h(\eta_1)h(\eta_2)}{h * h(\xi)}, \quad \eta_1 + \eta_2 = \xi. \tag{3.5}$$

This process is repeated independently for each of the particle types η_1, η_2 rooted at the vertices $< 1 >, < 2 >$, respectively.

In order to complete the interpretation of $(\mathrm{FNS})_h$ as an expected value, denote by $\mathcal{V}$ the vertex set of a complete binary tree rooted at θ

$$\mathcal{V} = \cup_{j=0}^{\infty}\{1,2\}^j = \{\theta, < 1 >, < 2 >, < 11 >, \dots\}, \tag{3.6}$$

where $\{1,2\}^0 = \{\theta\}$. Also let $\partial\mathcal{V} = \prod_{j=0}^{\infty}\{1,2\} = \{1,2\}^{\mathbf{N}}$.

For $\mathbf{v} = (v_1, v_2, \dots, v_k) \in \mathcal{V}$, let $|\mathbf{v}| = k$, $|\theta| = 0$. For $\mathbf{v} = (v_1, v_2, \dots) \in \partial\mathcal{V}$, and $j = 0, 1, 2 \dots$, let $\mathbf{v}|j = (v_1, \dots, v_j)$, $\mathbf{v}|0 = \theta$. That is, $\theta = \mathbf{v}|0, \mathbf{v}|1, \mathbf{v}|2, \dots$ may be viewed as a path through vertices of the tree determined by the vertex path $\mathbf{v} \in \partial\mathcal{V}$. For a fixed but arbitrary initial particle type (wavenumber) $\xi_\theta = \xi \neq 0$ of the root vertex θ, and time $t > 0$, define a *random tree*

$$\tau_\theta(\xi,t) = \{\mathbf{v} \in \mathcal{V} : \sum_{j=0}^{|\mathbf{v}|} S_{\mathbf{v}|j} \leq t, \prod_{j=0}^{|\mathbf{v}|-1} \kappa_{\mathbf{v}|j} = 1, \kappa_{\mathbf{v}} = 0\}$$
$$\cup\{\mathbf{v} \in \mathcal{V} : \sum_{j=0}^{|\mathbf{v}|-1} S_{\mathbf{v}|j} \leq t < \sum_{j=0}^{|\mathbf{v}|} S_{\mathbf{v}|j}, \prod_{j=0}^{|\mathbf{v}|-1} \kappa_{\mathbf{v}|j} = 1\}.$$

At a *clock ring* defined by $\sum_{j=0}^{|\mathbf{v}|-1} S_{\mathbf{v}|j}$, the vertex $\mathbf{v}$ is born and is replaced by either 0 or 2 new particles at time $\sum_{j=0}^{|\mathbf{v}|} S_{\mathbf{v}|j}$. Now, recalling (3.4), for given initial data and forcings $\chi_0(\xi)$ and $\varphi(\xi,t)$, $\xi \neq 0, t \geq 0$, define a functional $\chi(\chi_0, \varphi; \tau_\theta(\xi,t))$ by the following stochastic recursion:

$$
\begin{aligned}
&\curlywedge(\chi_0,\varphi;\tau_\theta(\xi,t))\\
=&\begin{cases}
\chi_0(\xi), & \text{if } S_\theta \geq t\\
\varphi(t-S_\theta,\xi), & \text{if } S_\theta < t, \kappa_\theta = 0,\\
m(\xi_\theta)\curlywedge(\chi_0,\varphi;\tau_{<1>}(\eta_1,t-S_\theta))\odot_{\xi_\theta}\curlywedge(\chi_0,\varphi;\tau_{<2>}(\eta_2,t-S_\theta)) & \text{else}
\end{cases}
\end{aligned}
$$

where $\eta_1+\eta_2=\xi_\theta$ are distributed according to $K_{\xi_\theta}(d\eta_1,d\eta_2)$ and $\tau_{<1>},\tau_{<2>}$ are the trees defined by re-rooting at the vertices $<1>,<2>$ of new types η_1,η_2, respectively. Standard results on critical branching show that this recursion will terminate in finite time with probability one. In particular there can be no explosion of the branching random walk in finite time. Thus $\curlywedge(\chi_0,\varphi;\tau_\theta(\xi,t))$ is a finite random variable for each time t and wavenumber ξ obtained as a product of m's, χ_0's, and φ's evaluated at the nodes of a multitype branching random walk. Moreover, decomposing the functional $\curlywedge$ in terms of the events $[S_\theta \geq t],[S_\theta < t,\kappa_\theta = 0]$ and $[S_\theta < t,\kappa_\theta = 1]$, one may check the following consequence of the strong Markov property.

Proposition 3.1 *If* $E|\curlywedge(\chi_0,\varphi;\tau_\theta(\xi,t))|<\infty$, *for each* $\xi\neq 0$, *then*

$$\chi(\xi,t)=E\curlywedge(\chi_0,\varphi;\tau_\theta(\xi,t))$$

solves $(\mathrm{FNS})_h$.

If $m(\xi)\leq 1$, $|\varphi(\xi,t)|\leq 1$ and $|\chi_0(\xi)|\leq 1$, then the finite number of factors appearing in the product functional $|\curlywedge(\chi_0,\varphi;\tau_\theta(\xi,t))|$ are bounded by 1, and consequently $|\chi(\xi,t)|\leq 1$ for all ξ and t. With this a motivation, define a Banach space with a norm that depends on a (fully supported) majorizing kernel $h(\xi)$ by

$$\mathcal{F}_{h,T}=\{v\in\mathcal{S}' : |v\|_{h,T}=\sup_{\xi\neq 0,0\leq t\leq T}\frac{|\hat{v}(\xi,t)|}{h(\xi)}<\infty\}. \tag{3.7}$$

In the case $h(\xi)=|\xi|^{-2}$ this is the Besov type space introduced by Cannone and Planchon [4]. Also, the result of LeJan and Sznitman [10] is obtained as a consequence of the following general theorem considering the same majorizing kernel.

Theorem 3.1 *Let* $h(\xi)$ *be a majorizing kernel with constant* $B=\frac{\nu}{2}(2\pi)^{\frac{3}{2}}$. *Fix* $T>0$. *Suppose that* $|\hat{u}_0(\xi)|\leq h(\xi), |\hat{g}(\xi,t)|\leq\frac{\nu}{2}(2\pi)^{\frac{3}{2}}|\xi|^2h(\xi),\quad \xi\neq 0, 0\leq t\leq T$. *Then (FNS) has a unique solution in the unit ball centered at* 0 *in the space* $\mathcal{F}_{h,T}$. *obtained*

4 NS-Majorizing Kernels

As illustrated by Theorem 3.1, a particular role of majorizing kernels is to construct various sufficient bounds on moduli of Fourier transforms of forcings and/or initial data to obtain global existence, uniqueness and regularity of solutions to the

Navier-Stokes equations. Thus the theory utilizes majorizing kernels of two types. "Large majorizing kernels" are used to identify existence and uniqueness conditions for mild global solutions, while "small majorizing kernels" may be used to track regularity of the global solution. A second role is in the construction of probabilistic representations. In this section, further examples of majorizing kernels in R^3 are presented as well as examples of initial data and forcings that satisfy the bounds required for global existence of solutions.

The problem of identifying majorizing kernels is intimately tied to dimensionality. While the majorizing kernel need not be integrable, it is required that the convolution $h * h(\xi)$ be finite for each $\xi \in \mathbf{R}^3 \backslash \{0\}$. The following Propositions 4.1-4.3 provide further examples that extend those already identified in (3.3) of majorizing kernels which are accordingly smaller or larger than these. Proofs and more extensive treatment of the majorizing problem will be given in Bhattacharya et al. [2]

Proposition 4.1 *Each of the following defines a fully supported majorizing kernel:*

$$h_\beta^{(\alpha)}(\xi) = \frac{e^{-\alpha|\xi|^\beta}}{|\xi|^{2-\beta}}, \quad \xi \neq 0, \quad 0 \leq \beta \leq 1, \alpha > 0.$$

Furthermore, if $u \in \mathcal{F}_h$ is a solution of the Navier-Stokes equation, then u is $\mathcal{C}^\infty$ in the space variables.

Using results of Aronszajn and Smith [1] one may also show that certain Bessel kernels and similar transforms provide further interesting majorizing kernels as in the following proposition.

Proposition 4.2 *Each of the following functions defines a radially symmetric majorizing kernel in R^3.*
(a) For $1 \leq \alpha \leq 2$,

$$h(\xi) = \int_0^\infty \frac{1}{t^{\frac{3}{2}}} e^{-\frac{|\xi|^2}{t}} t^{\frac{\alpha-2}{2}} e^{-t} dt$$

(b) For $0 \leq \beta \leq 1$,

$$h(\xi) = \int_0^\infty \frac{1}{t^{\frac{3}{2}}} e^{-\frac{|\xi|^2}{t}} e^{-t^\beta} t^{\frac{\beta-1}{2}} dt.$$

Examples of non-radial majorizing kernels may be obtained as follows.

Proposition 4.3 *Each of the following defines a majorizing kernel h in R^3.*
(a) For dimensions $n \geq 3$ and $\xi \in \{\xi \in R^n : \sum_{j=1}^n \delta_{\xi_j,0} < \frac{n+1}{2}\}$,

$$h(\xi) = \int_0^\infty \frac{t^n}{\prod_{j=1}^n (t^2 + \xi_j^2)} dt.$$

(b) Suppose that $k_1, \dots, k_m$ is a partition of $n \geq 1$, and h_j is a non-negative function on R^{k_j} such that

$$h_j * h_j(\xi_j) \leq |\xi_j|^{\theta_j} h_j(\xi_j), \quad \xi_j \in R^{k_j} \backslash \{0\},$$

where $\theta_j > 0$, $\sum_j \theta_j = 1$. Then define

$$h(\xi) = \prod_{j=1}^{m} h_j(\xi_j), \xi = (\xi_1, \xi_2, \dots, \xi_m), \ \xi_j \in R^{k_j} \backslash \{0\}.$$

In view of the role of majorizing kernels in providing bounds on the Fourier transformed forcings and/or initial data, the theory contains a dual problem which is to identify classes of divergence free vector fields in physical space which are so dominated.

Proposition 4.4 *Let $\mathcal{M}$ denote the space of finite signed measures on $\mathbf{R}^3$ with total variation norm $||\ ||$. Let $0 < \beta \leq 1$ and denote the "Fourier transformed Bessel kernel" of order β by $G_\beta(x) = (1 + |x|^2)^{-\frac{1+\beta}{2}}$. Then for each $g = G_\beta * \mu, \mu \in \mathcal{M}$, one has for any $\alpha > 0$,*

$$|\hat{g}(\xi)| \leq C_\beta^{(\alpha)} h_\beta^{(\alpha)}(\xi) ||\mu||, \quad \xi \neq 0,$$

for a constant $C_\beta^{(\alpha)} > 0$ such that $C_\beta^{(\alpha)} = C_\beta^{(1)}$, for $0 < \beta < 1, \alpha > 0$. In particular, if $v \in L^1$ is a divergence free vector field then $g = G_\beta * v$ is also a divergence free vector field whose Fourier transform is dominated by $h_\beta^{(\alpha)}$.

The following are further classes of vector fields whose Fourier transform is dominated by $h_\beta^{(\alpha)}$.

Proposition 4.5 *Let $\gamma_j(t), t > 0, j = 1, 2, 3$, be measurable functions such that $\int_0^\infty e^{-|x|^2 t} |\gamma_j(t)| dt < \infty, x \in \mathbf{R}^3, j = 1, 2, 3$. Define a radially symmetric vector field with components $v_j, j = 1, 2, 3$, by*

$$v_j(x) = \int_0^\infty e^{-|x|^2 t} \gamma_j(t) dt, x \in \mathbf{R}^3.$$

Let u be the divergence free projection of v. Then,
(a) If $|\gamma(t)| \leq c t^{-\frac{1}{2}}$ then $|\hat{u}_j(\xi)| \leq c' h_0(\xi)$ for some $c' > 0, j = 1, 2, 3$.
(b) If $|\gamma(t)| \leq c e^{-\alpha t}$ then $|\hat{u}_j(\xi)| \leq c' h_1^{(\alpha)}(\xi)$ for some $c' > 0, j = 1, 2, 3$.
(c) For arbitrary $\epsilon > 0$ there is a smooth probability density function R_ϵ supported on $[-\epsilon, \epsilon]^3$ such that

$$|\hat{R}_\epsilon(\xi)| \leq c(\beta) \exp\{-|\xi|^\beta\}, \xi \in \mathbf{R}^3, c(\beta) 0.$$

Let v be any divergence-free integrable vector field such that $|\hat{v}(\xi)| \leq c h_0(\xi), \ \xi \neq 0$. Then the componentwise perturbation

$$u = R_\epsilon * v$$

is a divergence-free infinitely differentiable vector field such that $|\hat{u}_j(\xi)| \leq c' h_\beta^{(\alpha)}(\xi)$, for $\alpha = \epsilon^\beta$ and some $c' > 0, j = 1, 2, 3$.

5 Successive Iterations of a Contraction Map

A standard method for establishing solutions of nonlinear partial differential equations is furnished by a Picard iteration in which at each stage a linear problem is solved and a contraction map argument is used to obtain convergence of the sequence so obtained. For example, in the context of Navier Stokes, Kato [9] succeded in establishing existence of solutions for initial data $\mathbf{u}_0 \in L^3(R^3)$. In this section, the representation of the Fourier transform of the solution in the form of an expected value obtained in the previous sections is shown to correspond to the Fourier transform of the iterates obtained using a Picard iteration scheme. A similar result holds for the representation of the solutions of the KPP equation obtained by McKean [11].

First consider the following iteration scheme (Kato [9]) applied to the Navier Stokes equations. Write (FNS) as

$$\hat{\mathbf{u}}(\xi,t) = e^{-\nu|\xi|^2 t}\hat{\mathbf{u}}_0(\xi) + \hat{B}(\hat{\mathbf{u}},\hat{\mathbf{u}})(\xi,t)$$
$$+ \int_0^t e^{-\nu|\xi|^2 s}\hat{g}(\xi,t-s)ds := \mathcal{Q}[\hat{\mathbf{u}};\hat{\mathbf{u}}_0,\hat{g}](\xi,t) \tag{5.1}$$

where

$$\hat{B}(\hat{\mathbf{u}},\hat{\mathbf{v}})(\xi,t) := \int_0^t e^{-\nu|\xi|^2 s}|\xi|(2\pi)^{-\frac{3}{2}}$$
$$\int \Big\{\frac{1}{2}\hat{\mathbf{u}}(\xi-\eta,t-s)\odot_\xi \hat{\mathbf{v}}(\eta,t-s) + \frac{1}{2}\hat{\mathbf{v}}(\xi-\eta,t-s)\odot_\xi \hat{\mathbf{u}}(\eta,t-s)\Big\}d\eta d\{ \tag{5.2}$$

Define

$$\hat{\mathbf{u}}_{n+1}(\xi,t) = \mathcal{Q}[\hat{\mathbf{u}}_n;\hat{\mathbf{u}}_0,\hat{g}](\xi,t), \tag{5.3}$$

where $\hat{\mathbf{u}}_1(\xi,t) = \mathcal{Q}[\mathbf{u}^{(0)};\hat{\mathbf{u}}_0,\hat{g}](\xi,t)$, for $\mathbf{u}^{(0)}(\xi,t) = e^{-\nu|\xi|^2 t}\hat{\mathbf{u}}_0(\xi)$.

The relation between the iteration scheme and the expected value representation of the solution obtained in Section 3 is established in the following proposition. Define the replacement time of a vertex $\mathbf{v}$ by

$$R_{\mathbf{v}} = \sum_{k=0}^{|\mathbf{v}|} S_{\mathbf{v}|k}$$

Introduce

$$A_n(\xi,t) = \{\tau_\theta(\xi,t) : |\mathbf{v}| \le n \ \forall\ \mathbf{v}\in\tau \text{ and } \ R_{\mathbf{v}} \ge t \ \forall\ \mathbf{v}\in\tau, |\mathbf{v}| = n\}$$

and let $\mathbf{1}[n;\xi,t]$ the indicator of the event $A_n(\xi,t)$.

Proposition 5.1 *Let*

$$\begin{aligned}\mathbf{v}_k(\xi,t) &= h(\xi)\chi_k(\xi,t)\\ &= h(\xi)\mathbf{E}_{(\xi,t)}\{\mathbf{1}[k;\xi,t]\chi(\hat{\mathbf{u}}_0,\hat{g},\tau_\theta(\xi,t))\}.\end{aligned}$$

Then $\mathbf{v}_k(\xi,t)=\hat{\mathbf{u}}_k(\xi,t)$.

A consequence of the proposition is that the convergence of the iteration scheme (5.3) and the existence of the expected value in Proposition 3.1 are essentially equivalent. Specifically, from Theorem 3.1 it follows that if $|m(\xi)|\leq 1$, $|\varphi(\xi,t)|\leq 1$ and $|\chi_0(\xi)|\leq 1$ in (3.4), then $|\chi(\xi,t)|\leq 1$ for all ξ and t. A direct proof of the convergence of the iteration scheme based on contraction properties of the nonlinear operator $\mathcal{Q}$ evidently requires that these inequalities be strict. Nonetheless, replacing the unit ball by a ball of radius $0<\rho<1$ in the statement of Theorem 3.1, then the iteration scheme (5.3) converges in the indicated norm. Specifically, one has the following theorem.

Theorem 5.1 *Let h be a fully supported majorizing kernel. Assume that there exists $r,\rho>0$ such that for all $\xi\neq 0, t\geq 0$, $|\hat{u}_0(\xi)|\leq \rho h(\xi)$, $h*h(\xi)\leq r\frac{\nu}{2}(2\pi)^{\frac{3}{2}}|\xi|h(\xi)$, and $|\hat{g}(\xi,t)|\leq \rho\frac{\nu}{2}|\xi|^2h(\xi)$. If $r\rho<1$, then $\mathcal{Q}$ is a strict contraction on the ball of radius ρ in $\mathcal{F}_{h,T}$.*

Scaling the majorizing constant to $B=\frac{r\nu}{2}(2\pi)^{\frac{3}{2}}$ in Theorem 3.1, one sees that the case $r\rho=1$ is covered there.

Similar considerations apply to the KPP equations given by

$$\frac{\partial u}{\partial t}=a\frac{\partial^2 u}{\partial x^2}+u^2-u,\quad u(x,0)=u_0(x).$$

Indeed, following McKean [11] the solution of this initial value problem is given by

$$u(x,t)=\mathbf{E}_x\left[\prod[u_0(B_{\mathbf{v}}(t))]\right] \tag{5.4}$$

where $B_{\mathbf{v}}(t)$ is the location of a branching Brownian motion defined recursively as follows. Start a standard Brownian path, $X_\theta(t)$, and let T_θ be an exponential random variable with parameter 1 independent of this Brownian motion. If $T_\theta\geq t$, set $B_\theta(t)=x+X_\theta(t)$. Else, start two independent Brownian paths $X_{<1>}$ and $X_{<2>}$ each with its own independent exponential time $T_{<1>}$ and $T_{<2>}$ respectively and iterate on this process. Let

$$\gamma_\theta(x,t)=\{\mathbf{v}\in\mathcal{V}: R_{\mathbf{v}}=\sum_{j=0}^{|\mathbf{v}|-1}T_{\mathbf{v}|j}<t\leq\sum_{j=0}^{|\mathbf{v}|}T_{\mathbf{v}|j}\}$$

Then for $\mathbf{v} \in \gamma_\theta(x,t)$,

$$B_{\mathbf{v}}(t) = x + \sum_{j=0}^{|\mathbf{v}|-1} X_{\mathbf{v}|j}(T_{\mathbf{v}|j}) + X_{\mathbf{v}}(t - R_{\mathbf{v}}).$$

Finally, let $M(\gamma_\theta(x,t)) = \max\{|\mathbf{v}| : \mathbf{v} \in \gamma_\theta\}$ and let $\mathbf{1}[k;x,t]$ the indicator of the event $[M(\gamma_\theta(\xi,t)) \leq k]$.

Let

$$u_k(x,t) = \mathbf{E}_x \left[\prod [u_0(B_{\mathbf{v}}(t))]\, \mathbf{1}[k : x,t] \right]. \tag{5.5}$$

It follows that

$$u(x,t) = \lim_{k\to\infty} u_k(x,t)$$

On the other hand, consideration of the Fourier transform of the KPP equation leads after a simple integration to the integral equation

$$\hat{u}(\xi,t) = e^{-\lambda(\xi)t}\hat{u}_0(\xi) + \int_0^t e^{-\lambda(\xi)s} \int \hat{u}(\eta, t-s)\hat{u}(\xi-\eta, t-s)d\eta ds \tag{5.6}$$

where

$$\lambda(\xi) = 1 + a|\xi|^2.$$

Proceeding as done with the Navier Stokes equations, scale (5.6) by $1/h(\xi)$ to obtain

$$\begin{aligned} \chi(\xi,t) &= e^{-\lambda(\xi)t}\chi_0(\xi) + \int_0^t e^{-\lambda(\xi)s}\lambda(\xi)m(\xi) \\ &\qquad \int \chi(\eta,t-s)\chi(\xi-\eta,t-s)\frac{h(\eta)h(\xi-\eta)}{h*h(\xi)}d\eta ds. \end{aligned} \tag{5.7}$$

where

$$m(\xi) = \frac{(h*h)(\xi)}{h(\xi)\lambda(\xi)},\ \chi(\xi,t) = \frac{\hat{u}(\xi,t)}{h(\xi)}.$$

Define the recursive functional

$$\curlywedge(\tau_\theta(\xi,t)) = \begin{cases} \chi_0(\xi), & \text{if } S_\theta \geq t \\ \varphi(\xi, t-S_\theta), & \text{if } S_\theta < t, \kappa_\theta = 0, \\ m(\xi_\theta)\curlywedge(\tau_{<1>}(\xi_{<1>}, t-S_\theta))\curlywedge(\tau_{<2>}(\xi_{<2>}, t-S_\theta)) & \text{else} \end{cases}$$

where $\tau_\theta(\xi,t)$ denotes the random tree defined in section 3, $\tau_{<1>}, \tau_{<2>}$ are re-rooted trees at vertices of types $\xi_{<1>}, \xi_{<2>}$ respectively and the distribution of types is given on $\eta_1 + \eta_2 = \xi_\theta$ by

$$K_\xi(\eta_1,\eta_2) = \frac{h(\eta_1)h(\eta_2)}{h*h(\xi)}, \quad \eta_1+\eta_2 = \xi.$$

Note that the only difference with the recursive functional corresponding to the Navier-Stokes equations is the node operation which for the KPP equations is standard multiplication.

Using the strong Markov property it follows that the solution of (5.6) is given by

$$\hat{u}(\xi,t) = h(\xi)\mathbf{E}\left[\chi(\tau_\theta(\xi,t))\right].$$

provided the expected value is finite. The analogue of a majorizing kernel for the KPP equation is given by

$$(h * h)(\xi) \leq B(1 + a|\xi|^2)h(\xi).$$

It is simple to check that Cauchy distributions,

$$h(\xi) = \frac{\alpha}{\pi}\frac{1}{\alpha^2 + \xi^2}$$

are majorizing kernels.

Finally, the relation with the McKean [11] representation of the solutions is furnished by the following proposition which is identical to proposition 5.1. Recall that $A_n(\xi,t)$ defined above Proposition 5.1 denotes the event that all vertex on a tree rooted at ξ are of length less than or equal to n and those vertex of exactly lenght n are replaced after time t. As in that proposition, let $\mathbf{1}[k;\xi,t]$ denote the indicator of $A_k(\xi,t)$.

Proposition 5.2 *Let $\hat{u}_k(\xi,t)$ denote the Fourier transform of the function $u_k(x,t)$ defined in (5.5) and let*

$$\begin{aligned}\mathbf{v}_k(\xi,t) &= h(\xi)\chi_k(\xi,t)\\ &= h(\xi)\mathbf{E}_{(\xi,t)}\{\mathbf{1}[k;\xi,t]\chi(\hat{\mathbf{u}}_0,\hat{g},\tau_\theta(\xi,t))\}.\end{aligned}$$

Then $\mathbf{v}_k(\xi,t) = \hat{\mathbf{u}}_k(\xi,t)$.

6 Conclusions and Remarks

The introduction and identification of majorizing kernels provides a way in which one may obtain existence and uniqueness of mild solutions of Navier-Stokes equations and track regularity of initial data to solutions.

The same methods may be applied to the Fourier coefficients in the case of periodic initial data and forcings. In fact the identification of majorizing kernels is somewhat simpler here due to the fact that on the integer lattice the origin need not be a singularity of the majorizing kernel. One may use a lattice version of Proposition 4.3 to construct fully supported majorizing kernels on the integer lattice in all dimensions $n \geq 2$. In the case $n = 1$ one also obtains cascade representations of

solutions to Burger's equation by these techniques. In this case, majorizing kernels supported on the positive half-line, $h(\xi) = \mathbf{1}[\xi > 0]$, also appear naturally and yield an existence/uniqueness theory for complex-valued solutions in Hardy spaces H^p.

The majorization may be augmented by introducing linear perturbations of the form $-\delta u$, $\delta > 0$, into the equations to obtain a cascade representation with non-stationary branching random walk and multiplication factors. More generally, the same iterative arguments may be applied with time-dependent majorizing kernels $h(\xi, t)$ with the rather obvious changes in the definitions. This may permit a way to develop a local existence and uniqueness theory. For an application of this approach see the paper of J. C. Orum [12] in this volume.

The notion of majorizing kernel as described in this announcement exploits simple contractive rates and/or sufficient bounds on the stochastic times functional $\chi(\tau_\theta(\xi, t))$. In fact, the essential property of the majorizing kernel is the finiteness of the convolution $h * h(\xi)$. In particular, it may be possible to obtain significantly sharper results by more detailed analysis of the branching random walk product.

References

[1] Aronszajn, N., K.T. Smith (1961): Theory of Bessel potentials I, *Ann. Inst. Fourier (Grenoble)* **11** 385-475

[2] Bhattacharya R., L. Chen, S. Dobson, R. Guenther, J. C. Orum, M. Ossiander, E. Thomann, E. Waymire (2002): Majorizing Kernels and Stochastic Cascades With Applications to the Incompressible Navier-Stokes Equations, (in preparation).

[3] Bramson, M. (1978) Maximal displacement of branching Brownian motion. Comm. Pure and Applied Math Vol 31, 531-582.

[4] Cannone, M. and F. Planchon (2000): On the regularity of the bilinear term for solutions to the incompressible Navier-Stokes equations *Revista Matemática Iberoamericana* **16** 1-16.

[5] OSU Applied Mathematics/Probability Research Group; L. Chen, S. Dobson, R. Guenther, J. C. Orum, M. Ossiander, E. Thomann, E. Waymire (2002): On Ito's Complex Measure Condition for a Feynman-Kac Formula, (in preparation).

[6] Fefferman, Ch. (2000). Exitence and Smoothness of the Navier-Stokes Equation. Clay Mathematics Institute `http://www.ClayMath.org/prizeproblems/index.htm`

[7] Friedlin, M. (1985) *Function Integration and Partial Differential Equations.* Annals of Mathematical Studies Vol 109. Princeton Univ Press.

[8] Ito, K. (1967) Generalized uniform complex measures in the Hilbertian metric spce with their application to the Feynman path integral. Proc. Fifth Berkeley Symposium on Mathematical Statistics and Probability. Univ. California Berkeley. Vol II, part 1, 145-161.

[9] Kato, T. (1984): Strong L^p solutions of the Navier-Stokes equations in $\mathbf{R}^m$ with applications to weak solutions, *Math. Z.* **187** 471-480.

[10] LeJan, Y. and A.S. Sznitman (1997): Stochastic cascades and 3-dimensional Navier-Stokes equations, *Prob. Theory and Rel. Fields* **109** 343-366.

[11] McKean, H. (1975): Applications of Brownian motion to the equation of Kolmogorov, Petrovskii and Piskunov. Comm. Pure and Applied Math. Vol 28, 323-331

[12] Orum, J. C. (2001). Stochastic Cascades and 2d Fourier Navier-Stokes Equations. This volume

RIVER FLOW MASS EXPONENTS WITH FRACTAL CHANNEL NETWORKS AND RAINFALL

BRENT M. TROUTMAN
U.S. Geological Survey, Denver, CO 80225

A classic problem in hydrologic science is prediction of river flow properties given knowledge of rainfall and drainage basin properties. Although many approaches have been used for flow prediction, recent significant advances have been made using the framework of scaling invariance. It is by now well known that both river networks, representing the primary mechanism for transport of water over the surface of large basins, and rainfall exhibit certain forms of scaling invariance (see, e.g., Rodriguez-Iturbe and Rinaldo (1997), Gupta and Waymire (1990), Over and Gupta (1994), and the papers in Barndorff-Nielsen et al.(1998)) . Given that both rainfall and the network obey certain scaling laws, an important ongoing research problem is to understand what type of structure this induces on the resulting flow through the network, both spatially and in time. We shall describe the properties of the network, of the rainfall, and of the resulting flow by *mass exponents*, or Rényi exponents, which characterize the power-law behavior of moments as a function of scale. One result to be discussed in this talk is an expression relating the temporal flow mass exponent for the instantaneous response function of a basin to a spatial mass exponent for rainfall and a mass exponent describing network branching. This result reveals that in some cases flow reflects the network scaling, and in other cases flow reflects the rainfall scaling, with degree of spatial rainfall variability determining which of the two constituent processes dominates flow.

Drainage basins are modeled here by recursive replacement trees, which are a special case of iterated function systems, and which have received much attention recently as a model which reproduces self-similar branching behavior seen in actual river networks (Tokunaga (1966), Newman et al. (1997), Peckham (1995), Veitzer (1999), Veitzer and Gupta (2000)). In modeling channel networks, a distinction has usually been made between "interior" and "exterior" edges in the drainage tree, as these tend to have different geometric and physical properties in actual basins. We therefore in what follows use the recursive replacement algorithm as developed by Veitzer (1999) which maintains this distinction. We begin by defining two rooted labelled trees the *interior generator* and the *exterior generator*. One distinguished node in each generator is labelled the root, and in the interior generator there is also a second distinguished node. Denote by c the distance (i.e., number of edges) between

the two distinguished nodes in the interior generator, and we assume throughout that $c \geq 2$. The edges of these generators are labeled exterior if adjacent to a node of degree one other than one of the distinguished nodes, and edges are labeled interior otherwise. We then define the sequence of recursive replacement trees as follows. We begin with the rooted labelled tree t_0 consisting of two nodes connected by an edge which is taken to be either "interior" or "exterior." The sequence of trees obtained by the recursive replacement process will be conditioned on this initial designation. For $n \geq 0$, let tree t_n be given, with each edge being either interior or exterior. Each interior edge in t_n is replaced by the interior generator after identifying the downstream node with the root and the upstream node with the second distinguished node. Similarly, each exterior edge is replaced by the exterior generator, with the downstream node being identified with the root. Tree t_{n+1} is tree tn with all edges so replaced, and the designation of edges in t_{n+1} as either interior or exterior is identical to the designation in the corresponding replacement generators.

Let $n(\alpha', \alpha), \alpha', \alpha \in \{I, E\}$, where I stands for "interior" and E stands for "exterior," be the number of type α edges in the type α' generator, and we can label the edges in the generators with an index taking values in the set $\{1, 2, \cdots, n(\alpha', \alpha)\}$. We shall take α_0 to be the type designation given to the initial edge in tree t_0. Edges in the nth generation tree tn may be labelled with a length $2n$ sequence of elements $(\alpha_1, \upsilon_1; \cdots, \alpha_n, \upsilon_n)$, with the $\alpha_i \in \{I, E\}$ designating the sequence of edge types in the recursive replacement process, and with $\upsilon_i \in \{1, 2, \cdots, n(\alpha_{i-1}, \alpha_i)\}, i \geq 1$ identifying edges . We shall denote by Γ the "drainage basin" resulting from carrying out the replacement process indefinitely. Specifically, Γ is defined to be the set of infinite paths $\gamma = (\gamma_1, \gamma_2, \cdots)$ where $\gamma_i = (\alpha_i, \upsilon_i)$. Equivalently the set Γ may be identified with the "boundary" of the infinite tree obtained by the replacement process. We shall denote the curtailment of the infinite sequence γ (i.e., the edges in t_n) by $(\gamma|n) = (\gamma_1, \cdots, \gamma_n)$, and we shall denote the subset of Γ for which the first n elements of the path γ are fixed and equal to $(\gamma|n)$ by $\Gamma_{\gamma|n}$. The branching number b of the infinite tree is defined to be the maximum eigenvalue of the 2 x 2 matrix $\{n(\alpha', \alpha)\}$.

For fixed distance j, the width of the rooted tree t_n is the number of edges with downstream node at distance j from the root. Considered as a function of j, the *width function* is proportional to the probability mass function of distance from the root of a randomly chosen edge. Let $d_{n,\alpha_0}(\gamma|n)$ denote the distance of edge $(\gamma|n)$ from the outlet in tree t_n and define distance to the outlet in Γ as $d_{\alpha_0}(\gamma) = \lim_{n\to\infty} c^{-n} d_{n,\alpha_0}(\gamma|n)$. The subset of $\Delta := [0, \infty)$ representing possible flow distances, i.e. $d_{\alpha_0}(\Gamma)$, is a finite interval $[0, \sigma_{\alpha_0}]$. Consider sequences of non-negative integers $\delta = \{\delta_0, \delta_1, \cdots\}$, with $\delta_i \in \{0, 1, \cdots, c-1\}, i \geq 1$ and $\delta_0 \in \{0, 1 \cdots, c^* - 1\}$ where $c^* \geq c$ is 1 plus the greatest integer less than $\max(\sigma_{\alpha_0})$. For given curtailment $(\delta|m) = (\delta_0, \delta_1, \delta_2, \cdots, \delta_m)$ we define $\Delta_{\delta|m}$ to be the subset of Δ for which the first $m+1$ elements in the path δ are $(\delta_0, \delta_1, \delta_2, \cdots, \delta_m)$. Let the probability measure ρ_{α_0} on (Borel) subsets of the real line be defined such that $\rho_{\alpha_0}[0, x)$ is the limiting

fraction of edges $(\gamma|n)$ in tree t_n for which $c^{-n}d_{n,\alpha_0}(\gamma|n) < x$. Also, let $\chi_{net}(h)$, the mass exponent for the width function, be the limit

$$\frac{\log \sum_{\delta|m} \rho^h_{\alpha_0}(\Delta_{\delta|m})}{m \log c} \to \chi_{\text{net}}(h), \quad m \to \infty.$$

It is shown by Troutman and Over (2001) that, for positive integers h, this limit is given by

$$\chi_{\text{net}}(h) = \frac{\log \omega(h) - h \log b}{\log c}.$$

where $\omega(h)$ is the maximum eigenvalue of a matrix $A(h)$ that may be expressed in terms of elementary generator characteristics.

We assume that we are interested in modeling flow response to an instantaneous burst of rainfall over the basin, so that only spatial (not temporal) characteristics of rainfall are of interest. Spatial variability of rainfall is modeled by a multiplicative random cascade (e.g., Kahane and Peyriere (1976), Holley and Waymire (1992), Waymire and Williams (1996), Troutman and Vecchia (1999), Ossiander and Waymire (2000)). Rainfall has been widely modeled with multifractals in a variety of applications in recent years (e.g., Gupta and Waymire (1990, 1993), Over (1995), Over and Gupta (1994), Lovejoy and Schertzer (1990), Olsson (1995), Menabde et al. (1997), Deidda (1999), and Carsteanu et al. (1999)). We begin by making the drainage basin Γ a metric space by defining the metric $D(\gamma, \gamma') = \exp(-|\gamma \wedge \gamma'|)$ where $|\gamma \wedge \gamma'|$ denotes the number of edges the two paths γ and γ' have in common. With each finite sequence $(\gamma|n)$ we associate a nonnegative random variable $W_{\gamma|n}$ defined on a probability space $(\Omega, \mathcal{F}, P)$; these random variables constitute the *cascade generator process*. We shall make the assumption throughout that the $\{W_{\gamma|n}\}$ are independent and identically distributed with mean one. Define

$$\mu_{n,\alpha_0}(B) = \int_B \prod_{i=1}^{n} [\sum_{\gamma|i} W_{\gamma|i} I_{\Gamma_{\gamma|i}}(\gamma)] \lambda_{\alpha_0}(d\gamma),$$

where B is a (Borel) subset of Γ, and λ_{α_0}, representing the rainfall measure for spatially uniform rainfall, is defined on subsets $\Gamma_{\gamma|n}$ by

$$\lambda_{\alpha_0}(\Gamma_{\gamma|n}) = \prod_{i=1}^{n} \phi(\alpha_{i-1}, \alpha_i),$$

where $\phi(\alpha_0, \alpha)$ is the limiting fraction of points in a tree with initial generator type α_0 that drain into a type α edge of this initial tree. Troutman and Over (2001) give an expression for ϕ in terms of $\{n(\alpha', \alpha)\}$. Using martingale theory, it may be shown (e.g., Waymire and Williams (1996)) that this sequence of measures converges almost surely to a random measure μ_{α_0}. Denote the limiting total mass by $Z_{\infty,\alpha_0} = \mu_{\alpha_0}(\Gamma)$.

For the special case for which the number of edges in interior and exterior generators is identical, the common value is b and the drainage basin Γ may be taken to be a b-ary tree. The resulting model is the usual Mandelbrot cascade, and properties (nondegeneracy and moment existence) of the limiting total mass for this special case were given by Kahane and Peyriere (1976). The model we are considering, with different interior and exterior generators, is somewhat more complicated, but it is straightforward to show that the theory for the Mandelbrot cascade generalizes readily, and that many of the same results carry through. First, $EZ_{\infty,\alpha_0} > 0$ if and only if $\chi'_{rain}(1-) < 0$, where

$$\chi_{rain}(h) = \log_b EW^h - (h-1).$$

Also, Z_{∞,α_0} has a finite moment of order $h > 1$ if and only if

$$h < h_c = \sup\{h \geq 1 : \chi_{rain}(h) < 0\}.$$

Note that h_c does not depend on α_0. Also, we define the mass exponent, when it exists, as the limit

$$\tau_{rain}(h) := \lim_{n\to\infty} \frac{\log \sum_{\gamma|n} \mu^h_{\alpha_0}(\Gamma_{\gamma|n})}{n \log b}.$$

A generalization of a result for the Mandelbrot cascade given in Holley and Waymire (1992) is as follows: If $2h < h_c$ and $EW^{2h}/(EW^h)^2 < b$, then with probability one this limit exists and is given by $\tau_{rain}(h) = \chi_{rain}(h)$.

The temporal flow measure at the outlet of Γ with initial generator α_0 is defined by $\pi_{\alpha_0} = \mu_{\alpha_0} d^{-1}_{\alpha_0}$. The assumption here is thus that flow of water, which has been deposited instantaneously as rainfall over the basin with no losses, to the outlet occurs at constant velocity of unity, allowing distance to the outlet to be used as a surrogate for time. Let h be a positive integer, $r = \frac{\log b}{\log c}$, and define

$$\chi_{flow}(h) = \max[\chi_{net}(h), r\chi_{rain}(h)].$$

It is proven in Troutman and Over (2001) that if $2h < h_c$ and $r\chi_{rain}(2h) < 2\chi_{flow}(h)$ then with probability one

$$\tau_{flow}(h) := \lim_{m\to\infty} \frac{\log \sum_{\delta|m} \pi^h_{\alpha_0}(\Delta_{\delta|m})}{m \log c} = \chi_{flow}(h).$$

The key element in this theorem is the definition of the flow mass exponent $\chi_{flow}(h)$. It is the maximum of two terms, the network mass exponent and the rainfall mass exponent (scaled by r), indicating that for a given order h, the flow moment scaling will reflect only the more dominant, as measured by magnitude of the mass exponent, of the two constituent processes.

We illustrate this result assuming that rainfall is modeled as a beta-lognormal cascade (Over (1995)). The distribution of the rainfall cascade generator W is given

by $P[W = 0] = 1 - b^{-\beta}$ and $P[W = W^+] = b^{-\beta}$ where $0 \leq \beta < 1$ and W^+, the positive part of W, is a log-normal random variable with variance σ^2. The figure shows which term in the definition of $\chi_{flow}(h)$ is the maximum for various values of the parameters σ^2 and β in the beta-lognormal model and for a Peano network. Spatial variability of rainfall increases as σ^2 increases and as β increases, with the limiting case $\sigma^2 = \beta = 0$ yielding spatially uniform rainfall. One can see in the figure the increasing tendency for flow to reflect rainfall scaling as σ^2 and β increase. The blank regions in the figure indicate values of h for which $h > h_c$.

An important special case of the beta-lognormal rainfall model is the so-called beta model, for which $\sigma^2 = 0$; for this model, the positive part of the cascade generator W is degenerate at b^β and $\chi_{rain}(h) = (\beta - 1)(h - 1)$. It may be shown that for a certain class of networks and beta rainfall there exists a critical value β_c of the parameter β, given by $\beta_c = \log \tilde{n}_{max} / \log b$ where $\tilde{n}_{max}$ is the maximum generator width function, such that

$$\chi_{flow}(h) = \begin{cases} \chi_{net}(h) & \text{if } \beta < \beta_c, \ h \text{ sufficiently large} \\ r\chi_{rain}(h) & \text{if } \beta > \beta_c \end{cases} .$$

β is an intermittency parameter, and this result shows how the magnitude of this parameter determines the relative influence of rainfall and network properties on flow. This extends a result of Gupta et al. (1996).

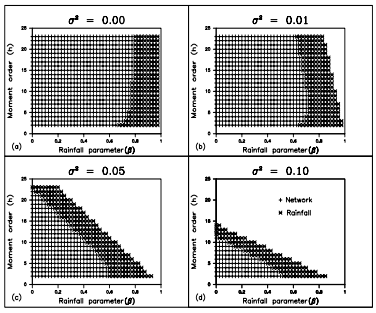


Figure 1: Plots showing maximal term (network or rainfall) in determination of flow exponent for selected values of h and beta-lognormal rainfall parameters.

References

[1] Barndorff-Nielsen, O.E., V.K. Gupta, V. Perez-Abreu, and E. Waymire, eds. Stochastic methods in hydrology: Rain, Landforms, and Floods. vol. 7 of Advanced Series on Statistical Science and Probability. Singapore: World Scientific, 1998.

[2] Carsteanu, A., V. Venugopal, and E. Foufoula-Georgiou. Event-specific multiplicative cascade models and an application to rainfall. J. Geophys. Res. 1999; 104(D24): 31611-31622.

[3] Deidda, R. Multifractal analysis and simulation of rainfall fields in space. Phys. Chem. Earth (B) 1999; 24: 73-78.

[4] Gupta, V.K., S.L. Castro, and T.M. Over. Scaling exponents of spatial peak flows from rainfall and river network geometry. J. Hydrol. 1996; 187: 81-104.

[5] Gupta, V.K. and Waymire, E.C. Multiscaling properties of spatial rainfall and river flow distributions. J. Geophys. Res. 1990; 95: 1999-2009.

[6] Gupta, V.K. and Waymire, E.C. A statistical analysis of mesoscale rainfall as a random cascade, J. Applied Meteor. 1993; 32: 251-267.

[7] Holley, R. and E. Waymire. Multifractal dimensions and scaling exponents for strongly bounded random cascades, Ann. Appl. Probab. 1992; 2: 819-845.

[8] Kahane, J.P. and J. Peyriere. Sur certaines martingales de Benoit Mandelbrot. Adv. in Math., 1976; 22: 131-145.

[9] Lovejoy, S. and D. Schertzer. Multifractals, universality classes, and satellite and radar measurements of cloud and rain fields. J. Geophys. Res. 1990; 95: 2021-2034.

[10] Menabde, M., A. Seed, D. Harris, and G. Austin. Self-similar random fields and rainfall simulation. J. Geophys. Res. 1997; 102(D12): 13,509-13,515.

[11] Newman, W.I., D.L. Turcotte, and A.M. Gabrielov. Fractal trees with side branching. Fractals 1997; 5: 603-614.

[12] Olsson, J. Limits and characteristics of the multifractal behavior of high-resolution rainfall time series. Nonlinear Processes Geophys. 1995; 2: 23-29.

[13] Ossiander, M, and E. Waymire. Statistical estimation for multiplicative cascades . Ann. Statist. 2000; 28: 1533-1560.

[14] Over, T.M. Modeling Space-Time Rainfall at the Mesoscale Using Random Cascades. Ph.D. Dissertation, Program in Geophysics, University of Colorado, 1995.

[15] Over, T.M. and V.K Gupta. Statistical analysis of mesoscale rainfall: Dependence of random cascade generator of the large-scale forcing. J. Appl. Meteor. 1994; 33: 1526-1542. Peckham, S.D. New results for self-similar trees with applications to river networks. Water Resour. Res. 1995; 31: 1023-1030.

[16] Peckham, S.D. New results for self-similar trees with applications to river networks. Water Resources Res. 1995; 31: 1023-1030.

[17] Rodriguez-Iturbe, I. and A. Rinaldo. Fractal River Basins, Chance and Self-Organization, Cambridge: Cambridge University Press, 1997.

[18] Tokunaga, E. The composition of drainage networks in Toyohira river basin and valuation of Horton's first law. Geophysical Bulletin of Hakkaido University 1966; 15: 1-19.

[19] Troutman, B.M. and T.M. Over. River flow mass exponents with fractal channel networks and rainfall. Adv. Water Resources 2001; 24: 967-989.

[20] Troutman, B.M. and A.V. Vecchia. Estimation of Renyi exponents in random cascades. Bernoulli 1999; 5(2): 191-207.

[21] Veitzer, S. A theoretical framework for understanding river networks: Connecting topology, process, and geometry across many scales, Ph.D. dissertation, Dept. of Physics, University of Colorado, 1999.

[22] Veitzer, S. and V.K. Gupta. Random self-similar river networks and derivations of generalized Horton laws in terms of statistical simple scaling. Water Resources Res. 2000; 36(4): 1033-1048.

[23] Waymire, E. and S.C. Williams. A cascade decomposition theory with applications to Markov and exchangeable cascades. Trans. Am. Math. Soc. 1996; 348: 585-632.

GRADIENT-DIRECTED DIFFUSIONS AND RIVER NETWORK MODELS

DEE WINN
Oregon State University

Empirical scaling and self-similarity laws that describe statistical and geometric features of river basins have been known for most of the century due to pioneering work of Hack [1], Horton [2], and others; e.g. see Jarvis and Waldenberg [3], Rinaldo and Rodriguez-Iturbe [4], Gupta and Waymire [5] for more comprehensive discussions in the context of river basin hydrology. While these geomorphologic relations do not explain the physical mechanisms that form river networks, they do provide dominant regularities which serve to guide the search for such physical laws. In this direction a number of river network models have been formulated on a discrete lattice in an effort to quantify, simulate and gain insight into the precise nature of these empirical observations. Some basic examples which motivated the theory presented below originated from Scheidegger [6], Leheney and Nagel [7], Pastor-Satorras and Rothman [8], and Troutman and Karlinger [9].

The most basic physical notion in river formation is that gravity drives water flow downhill toward an outlet to the ocean. In this paper we consider a general class of stochastic-dynamic network models based on this fundamental principal which we refer to as *gradient-directed diffusions.* Two points of view are possible in this regard. First, one may attempt to model the high degree of variability in the landscape as a random surface and then view the flow of water as a deterministic evolution due to gravity, erosion, and centrifugal forces. Most models based on Digital Elevation Maps (DEM's) employ this convention. The theoretical framework for such models is that of walks on random environments as investigated by Solomon [10], Hughes [11], Pemantle [12], and others. Such an approach in this context puts the focus on adequately representing detailed fluctuations in the basin surface. Alternatively, one may view the paths taken by water as randomly fluctuating curves on a mean field (deterministic) picture of the landscape surface in which gravity and the hill gradient impose a mean drift on the direction of flow. This latter point of view appears to be more readily amenable to data types furnished by remote sensing and DEMs since one may construct an average surface from the given data and then focus on modelling the statistics of the flow paths. We show that this framework includes various lattice models previously considered in the literature.

The essential idea behind gradient-directed diffusion models is that water flows

downhill in the direction of steepest descent with random fluctuation due to physical factors. Fluctuations in flow due to geomorphological surface effects will be viewed as random perturbations of flow on this smooth surface. We view the average surface as a C^1 surface and model the flow of water both as discreet processes on lattices and as stochastic diffusions. Transitions occur only in downhill directions with steepest paths having the greatest probability. In the discreet case, we see many models currently in the literature may be viewed as gradient models on a special surface. Further, the diffusive models are obtained as weak limits of appropriately scaled discreet gradient processes. Interestingly, as with Pastor-Satorras and Rothman [8], we see separate scalings in the downhill and tangential directions.

We will view H as a C^1 function. Fluctuations in flow due to geomorphological surface effects will be viewed as random perturbations of flow on this smooth surface. We wish to trace the flow path of water that begins at the point $(\mathbf{x}, H(\mathbf{x}))$. To do this, we will begin at the point $\mathbf{x} = (x_1, x_2)^T$ that is not a local maximum point on the surface, and construct a random graph in $\mathbf{R}^2$, beginning at $\mathbf{x}$, which is the projection of the surface flow path. Using ideas similar to Pastor-Satorras and Rothmanin a small time interval h the water will flow $\delta_{//}(h)$ units from a location $\mathbf{x}$ in the direction of the downward surface gradient $D^{//}(\mathbf{x}) := -\nabla H(\mathbf{x})$, with a random fluctuation in the tangential direction at most $\delta_{\perp}(h)$ units. The tangential direction is with respect to the level curve at $\mathbf{x}$ that gives a right-hand orientation when rotated into $D^{//}(\mathbf{x})$. Thus, the tangent vector is $D^{\perp}(\mathbf{x}) = (-\partial_2 H(\mathbf{x}), \partial_1 H(\mathbf{x}))^T$, where ∂_i is the corresponding partial derivative $\partial / \partial x_i$. All height gradients in this paper are evaluated at $\mathbf{x}$, so the dependence will often be omitted. As with Pastor-Satorras and Rothman we will observe separate scalings of displacement in the two directions.

We allow transitions along a semi-ellipse centered at $\mathbf{x}$ whose vertical axis is in the direction of $D^{//}$ and whose horizontal axis is parallel to $D^{\perp}$. This ensures that infinitesimally water flows only in the downhill direction. To be explicit, for a fixed time step $h > 0$, transitions occur by

$$\mathbf{x} \longrightarrow \mathbf{x} + T(\Theta, h) \cdot (-\nabla H)$$

where $\Theta = \Theta(\mathbf{x})$ is a random variable with distribution $F_h(d\theta) \equiv F_h(\mathbf{x}, d\theta)$ on $[0, \pi]$ and

$$T(\theta, h) = \begin{pmatrix} \delta_{//}(h) \sin\theta & \delta_{\perp}(h) \cos\theta \\ -\delta_{\perp}(h) \cos\theta & \delta_{//}(h) \sin\theta \end{pmatrix}.$$

Here $\delta_{\perp}$ and $\delta_{//}$ are the lengths of the axes on the ellipse, such that $\|(\delta_{\perp}, \delta_{//})^T\| = o(1)$ as $h \to 0$. For symmetric Θ this makes the proper scaling, as $h \to 0$, diffusive in the tangent direction and deterministic in the downward gradient direction. More precisely, denote the h time scale drift $\mathbf{b}^h(\mathbf{x})$ and diffusion coefficient $a^h(\mathbf{x})$. Assume that Θ is symmetric with respect to $\pi/2$. This ensures that $\int_{[0,\pi]} \cos\theta \, F_h(d\theta) = 0$

and $\int_{[0,\pi]} \sin\theta\, \cos\theta\, F_h(d\theta) = 0$. Then

$$\begin{aligned} h \cdot \mathbf{b}^h(\mathbf{x}) &= \int_{[0,\pi]} \begin{pmatrix} \delta_{//}(h)\sin\theta & \delta_{\perp}(h)\cos\theta \\ -\delta_{\perp}(h)\cos\theta & \delta_{//}(h)\sin\theta \end{pmatrix} (-\nabla H(\mathbf{x}))\, F_h(d\theta) \\ &= \delta_{//} \int_{[0,\pi]} \sin\theta\, F_h(d\theta) \cdot (-\nabla H). \end{aligned}$$

Then, for $\delta_{//} \sim h$ as $h \to 0$,

$$\mathbf{b}(\mathbf{x}) = \lim_{h \searrow 0} \mathbf{b}^h(\mathbf{x})/h = \left(\lim_{h \searrow 0} \int_{[0,\pi]} \sin\theta\, F_h(d\theta) \right) \cdot -\nabla H.$$

Likewise, for the diffusion coefficient,

$$\begin{aligned} & h \cdot a^h(\mathbf{x}) \\ &= \int_{[0,\pi]} T(\theta,h) \cdot \begin{pmatrix} -\partial_1 H // - \partial_2 H \end{pmatrix} \cdot \begin{pmatrix} -\partial_1 H & -\partial_2 H \end{pmatrix} \cdot (T(\theta,h))^T\, F_h(d\theta) \\ &= \int_0^{\pi} \begin{pmatrix} \delta_{\perp}^2 (\partial_2 H)^2 \cos^2\theta + o(h) & \delta_{\perp}^2 (\partial_1 H)(\partial_2 H)\cos^2\theta + o(h) \\ \delta_{\perp}^2 (\partial_1 H)(\partial_2 H)\cos^2\theta + o(h) & \delta_{\perp}^2 (\partial_1 H)^2 \cos^2\theta + o(h) \end{pmatrix} F_h(d\theta) \end{aligned}$$

by the symmetry of F_h. Hence, getting a non-trivial limit requires $\delta_{\perp} \sim \sqrt{h}$. In this case, letting $F(d\theta) \equiv F(x, d\theta)$ denote the weak limit of $F_h(x, d\theta)$ as $h \to 0$,

$$a(\mathbf{x}) = \lim_{h \searrow 0} a^h(\mathbf{x})/h = \frac{\int_0^{\pi} \cos^2\theta\, F(d\theta)}{\|\nabla H\|} \cdot \begin{pmatrix} (\partial_2 H)^2 & -\partial_1 H \partial_2 H \\ -\partial_1 H \partial_2 H & (\partial_1 H)^2 \end{pmatrix}.$$

Without the integral stretch factor, a is idempotent which makes the dispersion matrix easily computable. Defining

$$\beta(\mathbf{x}) = \int_{[0,\pi]} \sin\theta\, F(d\theta); \quad \alpha(\mathbf{x}) = \sqrt{\int_0^{\pi} \cos^2\theta\, F(d\theta)},$$

the flow paths may then be modeled as a diffusion $\mathbf{X}(t)$ in $\mathbf{R}^2$ satisfying the stochastic differential equation

$$d\mathbf{X} = \mathbf{b}(\mathbf{X}(t))\, dt + \sqrt{a}(\mathbf{X}(t))\, d\mathbf{B}(t)$$

where $\mathbf{B}(t)$ is standard two-dimensional Brownian motion and

$$\mathbf{b}(\mathbf{x}) = -\beta \nabla H; \quad \sqrt{a}(\mathbf{x}) = \frac{\alpha}{\|\nabla H\|} \cdot \begin{pmatrix} (\partial_2 H)^2 & -\partial_1 H \partial_2 H \\ -\partial_1 H \partial_2 H & (\partial_1 H)^2 \end{pmatrix}.$$

One may note that the eigenvalues of $a(\mathbf{x})$ are 0 and $\|\nabla H\|$ with corresponding eigenvectors $D^{//}$ and $D^{\perp}$. That is, the diffusion acts deterministically downhill

(it is singular in that direction), with diffusive fluctuations in the tangent directions $\pm D^{\perp}(\mathbf{x})$. Since erosion reinforces flow paths, we anticipate the singularity of the matrix to begin dominating the flow paths in temporal models. However such considerations will be postponed to future work.

Let us now consider various models for flow paths on lattices from this point of view. We are given a lattice that is a subset of Ξ,

$$\mathcal{L} = \{(x_1, x_2) \in \Xi : (x_1, x_2) = (m_i, m_j),\ i, j \in \mathbf{Z}\}$$

and the coordinates are oriented so that the landscape is tilted toward the x-axis, $D_2^{//} < 0$. Assume $\delta_{\perp}(1)$ and $\delta_{//}(1) = 1$, and that for each $h \leq 1$, F_h is an atomic measure on the refined lattice

$$\mathcal{L}(h) = \left\{(x_1, x_2) \in \Xi : (x_1, x_2) = (\delta_{\perp} m_i, \delta_{//} m_j),\ i, j \in \mathbf{Z}\right\}.$$

$F_h(\mathbf{x}, d\theta)$ will be concentrated on angles that allow transitions to the set of neighboring lattice sites, $N(\mathbf{x})$. We would like $F_h(\mathbf{x}, d\theta)$ to be a function of the derivative of the hillslope in the direction θ. That is, for $\mathbf{y} = \mathbf{x} + T(h, \Theta) \cdot D^{//}$, the probability that $\mathbf{x}$ flows to $\mathbf{y}$ is given by

$$\frac{\mathbf{1}(H(\mathbf{y}) - H(\mathbf{x}) \geq 0) \cdot \exp(H(\mathbf{y}) - H(\mathbf{x}))}{\sum_{\mathbf{z} \in N(\mathbf{x})} \mathbf{1}(H(\mathbf{z}) - H(\mathbf{x}) \geq 0) \cdot \exp(H(\mathbf{z}) - H(\mathbf{x}))}.$$

For now the symmetry assumption in the diffusion computation rules out a general surface gradient.

The Scheidegger model arises easily in this context. Here one considers the triangular lattice defined by

$$\mathcal{T}_m = \{(x_1, x_2) \in \Xi : (x_1, x_2) = (mi, mj),\ i, j \in \mathbf{Z} \text{ and } i + j \text{ is even}\}.$$

If the surface is an inclined plane, $H(\mathbf{x}) = I x_2$ where $I > 0$, then the resulting probabilities for site-to-site flow are simply Bernoulli trials. Thus, the Scheidegger model may be viewed as a special case of the gradient model where the surface is featureless. To compute area statistics, Nguyen [13] constructs boundary networks on the dual lattice

$$\mathcal{T}_m^* = \{(x_1, x_2) \in \Xi : (x_1, x_2) = (mi, mj),\ i, j \in \mathbf{Z} \text{ and } i + j \text{ is odd}\}.$$

In this framework the flow paths converge to a two-dimensional process $(W(t), -t)$, which is the graph of one-dimensional Brownian motion rotated $\pi/2$. There is a one-to-one correspondence between sites and bonds on $\mathcal{T}$ and $\mathcal{T}^*$, so that the boundary processes converge to $(B(t), t)$. With this device Nguyen [13] is able to compute basin area as the area contained between two independent Brownian motions conditioned to coalesce at a given "time", where here the time is actually the *length* of the longest

channel in the basin. As a result of the diffusive width scaling by $t^{\frac{1}{2}}$, for the channel length t the area under the curve is of the order $t^{\frac{1}{2}}t = t^{\frac{3}{2}}$. This a Hack law scaling with exponent $\frac{2}{3}$; see Troutman and Karlinger [14], Ossiander, Waymire, and Zhang [15] for some related calculations on the width of the network in this context.

In summary, we have obtained a new river network model as a coalescing network of flow paths defined by *gradient-directed diffusions*. The idea is to infinitesimally trace the path that water takes when placed at a point on a given surface. Due to gravity and fine scale surface irregularities the path is assumed to follow the downhill gradient with random fluctuation in the tangential direction. We have shown that for such models, the flow paths may be viewed as diffusions that drift in the downhill direction with tangential random fluctuations. The second major physical dynamic is that of erosion, which may be accommodated in this framework as a reinforcement mechanism in the flow paths along the lines of Pastor-Satorras and Rothman [8]. We plan to pursue these temporal effects in future work.

References

[1] J.Hack, U.S. Geol. Survey Professional Paper 294B, B1 (1957).

[2] R. Horton, Geol. Soc. America Bull. **56,** 275 (1945).

[3] R. Jarvis and M. Waldenberg, *River Networks* (Hutchison Ross Publishing Company, Stroudsburg, PA, 1984).

[4] I. Rodriguez-Iturbe and A. Rinaldo *Fractal River Basins* (Cambridge-University Press, NY, 1997).

[5] V.K. Gupta, E.Waymire, *Scale Dependence and Scale Invariance in Hydrology,* ed. G. Sposito, Cambridge University Press, Cambridge, UK (1998)

[6] A. Scheidegger, *Theoretical Geomorphology, 3rd ed.* (Springer-Verlag, Berlin, 1991), chs. 4-5.

[7] R. Leheney and S. Nagel, Phys. Rev. Lett. **71,** 1470 (1993).

[8] R. Pastor-Satorras and D. Rothman, Journal of Statistical Physics, **93,** 477 (1998).

[9] B. Troutman and M. Karlinger, Water Resources Res. **30**(7), 2325-2334 (1994).

[10] F. Solomon, Annals of Probability, **3,** 1 (1975).

[11] B. Hughes, *Random Walks and Random Environments vol. 2*, (Clarendon Press, Oxford, 1996).

[12] R. Pemantle, *Random processes with Reinforcement*, MIT thesis, Boston, (1988).

[13] B. Nguyen, Journal of Applied Probability, **27,** 269 (1990).

[14] B. Troutman and M. Karlinger, J. Applied Probability, **22** 836-849, 1984.

[15] M. Ossiander, E. Waymire, and Q. Zhang, Ann. Applied Probability, **7** 972-995, 1997.