Random Trees and Spatial Branching Processes

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Concentrated Advanced Course on

Lévy Processes and Branching Processes

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Contents

Preface

In these notes, we present a number of recent results concerning discrete and continuous random trees, and spatial branching processes. In the last chapter we also briefly discuss connections with topics such as partial differential equations or infinite particle systems. Obviously, we did not aim at an exhaustive account, and we give special attention to the quadratic branching case, which is the most important one in applications. The case of a general branching mechanism is however discussed in our presentation of superprocesses in Chapter 3, and in the connections with Lévy processes presented in Chapter 5 (more about these connections will be found in the forthcoming monograph [19]). Our first objective was to give a thorough presentation of both the coding of continuous trees (whose prototype is Aldous' CRT) and the construction of the associated spatial branching processes (superprocesses, Brownian snake). In Chapters 2 to 4, we emphasize explicit calculations: Marginal distributions of continuous random trees or moments of superprocesses or the Brownian snake. On the other hand, in Chapter 5 we give a more probabilistic point of view relying on certain deep properties of spectrally positive Lévy processes. In the first five chapters, complete proofs are provided, with a few minor exceptions. On the opposite, Chapter 6, which discusses connections with other topics, contains no proofs (including them would have required many more pages).

Chapter 1 discusses scaling limits of Galton-Watson trees whose offspring distribution is critical with finite variance. We give a detailed proof of the fact that the rescaled contour processes of a sequence of independent Galton-Watson trees converge in distribution, in a functional sense, towards reflected Brownian motion. Our approach is taken from [19], where the same result is proved in a much greater generality. Aldous [2] gives another version (more delicate to prove) of essentially the same result, by considering the Galton-Watson tree conditioned to have a (fixed) large population size.

The results of Chapter 1 are not used in the remainder of the notes, but they strongly suggest that continuous random trees can be coded (in the sense of the contour process) by Brownian excursions. This coding is explained at the beginning of Chapter 2 (which is essentially Chapter III of [26]). The main goal of Chapter 2 is then to give explicit formulas for finite-dimensional marginals of continuous trees coded by Brownian excursions. In the case of the Brownian excursion conditioned to have duration 1, one recovers the marginal distributions of Aldous' continuum random tree [1],[2].

Chapter 3 is a brief introduction to superprocesses (measure-valued branching processes). Starting from branching particle systems where both the number of particles and the branching rate tend to infinity, a simple analysis of the Laplace functionals of transition kernels yields certain semigroups in the space of finite measures. Superprocesses are then defined to be the Markov processes corresponding to these semigroups. We emphasize expressions for Laplace functionals, in the spirit of Dynkin's work, and we use moments to derive some simple path properties in the quadratic branching case. The presentation of Chapter 3 is taken from [26]. More information about superprocesses may be found in [9], [10], [16], or [30].

Chapter 4 is devoted to the Brownian snake approach [24],[26]. This approach exploits the idea that the genealogical structure of superprocesses with quadratic branching mechanism is described by continuous random trees, which are themselves coded by Brownian excursions in the sense explained in Chapter 2. In the Brownian snake construction of superprocesses, the genealogical structure is first prescribed by a Poisson collection of Brownian excursions, and the spatial motions of "individual particles" are then constructed in a way compatible with this genealogical structure. Chapter 4 also gives a few applications of the Brownian snake construction. In particular, the Brownian snake yields a simple approach to the random measure known as ISE (Integrated Super-Brownian Excursion), which has appeared in several recent papers discussing asymptotics for models of statistical mechanics [11], [23].

In Chapter 5, we extend the snake approach to the case of a general branching mechanism. The basic idea is the same as in the Brownian snake approach of Chapter 4. However the genealogy of the superprocess is no longer coded by Brownian excursions, but instead by a certain functional of a spectrally positive Lévy process whose Laplace exponent is precisely the given branching mechanism. This construction is taken from [27], [28] (applications are given in [19]). However the approach presented here is less computational (although a little less general) than the one given in [28], which relied on explicit moment calculations in the spirit of Chapter 4 of the present work.

Finally, Chapter 6 discusses connections with other topics such as partial differential equations, infinite particle systems or scaling limits of lattice trees.

Chapter 1

Galton-Watson trees

1.1 Preliminaries

Our goal in this chapter is to study the convergence in distribution of rescaled Galton-Watson trees, under the assumption that the associated offspring distribution is critical with finite variance. To give a precise meaning to the convergence of trees, we will code Galton-Watson trees by a discrete height process, and we will establish the convergence of these (rescaled) discrete processes to reflected Brownian motion. We will also prove that similar convergences hold when the discrete height processes are replaced by the contour processes of the trees.

Let us introduce the assumptions that will be in force throughout this chapter. We start from an offspring distribution μ, that is a probability measure $(\mu(k), k = 0, 1, \ldots)$ on the nonnegative integers. We make the following two basic assumptions:

(i) (critical branching) $$\sum_{k=0}^{\infty} k\,\mu(k) = 1.$$

(ii) (finite variance) $$0 < \sigma^2 := \sum_{k=0}^{\infty} k^2\,\mu(k) - 1 < \infty.$$

The criticality assumption means that the mean number of children of an individual is equal to 1. The condition $\sigma^2 > 0$ is needed to exclude the trivial case where μ is the Dirac measure at 1.

The μ-Galton-Watson tree is then the genealogical tree of a population that evolves according to the following simple rules. At generation 0, the population starts with only one individual, called the ancestor. Then each individual in the population has, independently of the others, a random number of children distributed according to the offspring distribution μ.

Under our assumptions on μ, the population becomes extinct after a finite number of generations, and so the genealogical tree is finite a.s. It will be convenient to view the μ-genealogical tree $\mathcal{T}$ as a (random) finite subset of

$$\bigcup_{n=0}^{\infty} \mathbb{N}^n$$

where $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}^0 = \{\emptyset\}$ by convention. Here $\emptyset$ corresponds to the ancestor, the children of the ancestor are labelled $1, 2, \ldots$, the children of 12 are labelled $11, 12, \ldots$ and so on (cf Fig. 1). In this special representation, it is implicit that the children of each individual are ordered, a fact that plays an important role in what follows. We can also view the μ-Galton-Watson tree as a random element of the set of all finite rooted ordered trees.

Our main interest is in studying the law of the μ-Galton-Watson tree **conditioned** on the event that this tree is large in some sense. One could use several methods to make the conditioning precise. For instance, it would be natural to condition the tree to have exactly n vertices or individuals (this makes sense under mild assumptions on μ) and then to let n tend to infinity. For mathematical convenience, we will adopt a different point of view and prove a limit theorem for a sequence of independent μ-Galton-Watson trees. It turns out that this limit theorem really gives information about the "large" trees in the sequence and thus answers our original question in a satisfactory way.

How can we make sense of the convergence of rescaled random trees ? In this work, we will code Galton-Watson trees by random functions and then use the well-known notions of weak convergence of random processes. This approach has the advantage of avoiding any additional formalism (topology on discrete or continuous trees) and is still efficient for applications. We start from a sequence $\mathcal{T}_1, \mathcal{T}_2, \ldots$ of independent μ-Galton-Watson trees. We attach to this sequence two discrete-time integer-valued random processes. The first one, called the **contour process** is especially easy to visualize (cf Fig.1). We imagine the displacement of a particle that starts at time 0 from the ancestor of the tree $\mathcal{T}_1$ and then moves on this tree according to the following rules. The particle jumps from an individual to its first not yet visited child, if any, and if none to the father of the individual. Eventually, the particle comes back to the ancestor after having visited all individuals of the tree, and it then jumps to the ancestor of the next tree. The value C_n of the contour process at time n is the generation of the individual visited at step n in this evolution. In some sense, the graph of the function $n \to C_n$ draws the contour of the tree.

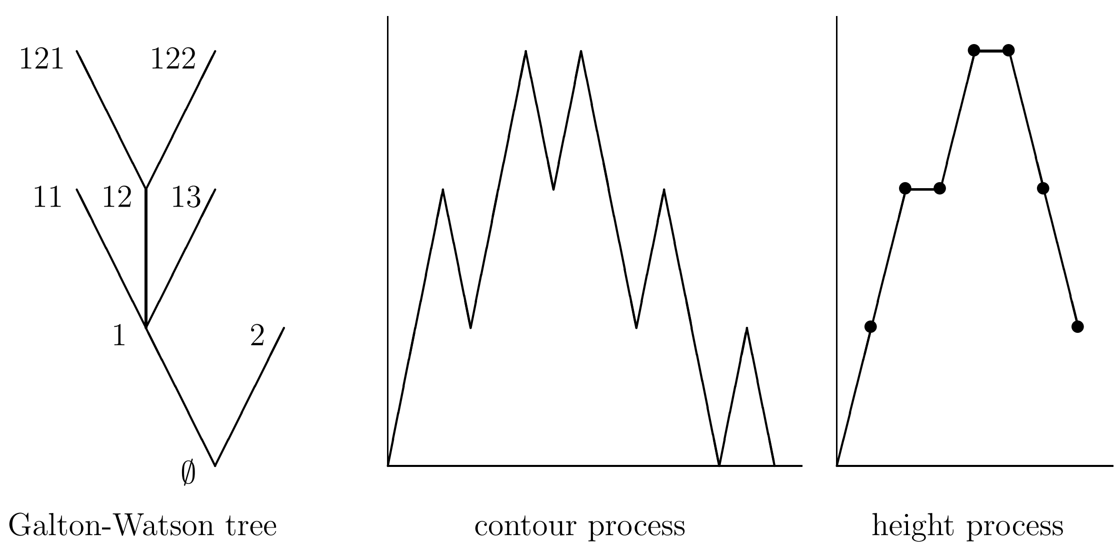

Figure 1

The **height process** H_n is defined in a slightly more complicated way, but is mathematically more tractable. Write

$$\mathcal{T}_1 = \{u_0^1, u_1^1, \dots, u_{n_1-1}^1\}$$

for the individuals of the tree $\mathcal{T}_1$ listed in lexicographical order (thus $u_0^1 = \emptyset$, $u_1^1 = 1$, etc.). Then, for $n \in \{0, 1, \dots, n_1 - 1\}$, we let $H_n = |u_n^1|$ be the generation of individual u_n^1. Similarly, if

$$\mathcal{T}_2 = \{u_0^2, u_1^2, \dots, u_{n_2-1}^2\}$$

are the individuals of the tree $\mathcal{T}_2$ listed in genealogical order, we set $H_n = |u_{n-n_1}^2|$ for $n \in \{n_1, n_1 + 1, \dots, n_1 + n_2 - 1\}$, and so on. In other words, we have a particle that visits the different vertices of the sequence of trees one tree after another and in lexicographical order for each tree (thus each vertex is visited exactly once) and we let H_n be the generation of the vertex visited at step n. Fig. 1 shows the contour process and the height process for a single Galton-Watson tree.

It is elementary to verify that both the height process and the contour process characterize the sequence $\mathcal{T}_1, \mathcal{T}_2, \dots$ and in this sense provide a coding of the sequence of trees.

We start with a simple lemma that is crucial for our approach. Recall that if ν is a probability distribution on the integers, a discrete-time process $(S_n, n \geq 0)$ is called a random walk with jump distribution ν if it can be written as

$$S_n = Y_1 + Y_2 + \cdots + Y_n$$

where the variables $Y_1, Y_2, \dots$ are independent with distribution ν.

Lemma 1.1.1 *Let $\mathcal{T}_1, \mathcal{T}_2, \dots$ be a sequence of independent μ-Galton-Watson trees, and let $(H_n, n \geq 0)$ be the associated height process. There exists a random walk S_n with jump distribution $\nu(k) = \mu(k+1)$, for $k = -1, 0, 1, 2, \dots$, such that for every $n \geq 0$,*

$$H_n = \operatorname{Card}\{k \in \{0, 1, \dots, n-1\} : S_k = \inf_{k \leq j \leq n} S_j\}. \tag{1.1}$$

It would be cumbersome to give a detailed proof of this lemma, and we only explain the idea. As previously, write n_1, resp. $n_2, n_3, \dots$ for the number of individuals in the tree $\mathcal{T}_1$, resp. $\mathcal{T}_2, \mathcal{T}_3, \dots$. Recall that H_n is the generation of the individual visited at time n for a particle that visits the different vertices of the sequence of trees one tree after another and in lexicographical order for each tree. Write R_n for the quantity equal to the number of younger brothers (younger refers to the order on the tree) of the individual visited at time n plus the number of younger brothers of his father, plus the number of younger brothers of his grandfather etc. Then the random walk that appears in the lemma may be defined by

$$S_n = R_n - (j-1) \qquad \text{if } n_1 + \cdots n_{j-1} \leq n < n_1 + \cdots + n_j.$$

To verify that S is a random walk with jump distribution ν, note that because of the lexicographical order of visits, we have at time n no information on the fact that the individual visited at that time has children or not. If he has say $k \geq 1$ children, which occurs with probability $\mu(k)$, then the individual visited at time $n+1$ will be the first of these children, and our definitions give $R_{n+1} = R_n + (k-1)$ and $S_{n+1} = S_n + (k-1)$. On the other hand if he has no child, which occurs with probability $\mu(0)$, then the individual visited at time $n+1$ is the first of the brothers counted in the definition of R_n (or the ancestor of the next tree if $R_n = 0$) and we easily see that $S_{n+1} = S_n - 1$. We thus get exactly the transition mechanism of the random walk with jump distribution ν.

Let us finally explain formula (1.1). From our definition of R_n and S_n, it is easy to see that the condition $n < \inf\{j > k : S_j < S_k\}$ holds iff the individual visited at time n is a descendant of the individual visited at time k (more precisely, $\inf\{j > k : S_j < S_k\}$ is the time of the first visit after k of an individual that is not a descendant of individual k). Put in a different way, the condition $S_k = \inf_{k \le j \le n} S_j$ holds iff the individual visited at time k is an ascendant of the individual visited at time n. It is now clear that the right-hand side of (1.1) just counts the number of ascendants of the individual visited at time n, that is the generation of this individual.

The representation (1.1) easily leads to the following useful property of the height process.

Lemma 1.1.2 *Let τ be a stopping time of the filtration $(\mathcal{F}_n)$ generated by the random walk S. Then the process*

$$\Big(H_{\tau+n} - \inf_{\tau \le k \le \tau+n} H_k, n \ge 0\Big)$$

is independent of $\mathcal{F}_\tau$ and has the same distribution as $(H_n, n \ge 0)$.

Proof. Using (1.1) and considering the first time after τ where the random walk S attains its minimum over $[\tau, \tau+n]$, one easily gets

$$\inf_{\tau \le k \le \tau+n} H_k = \mathrm{Card}\,\{k \in \{0, 1, \ldots, \tau - 1\} : S_k = \inf_{k \le j \le \tau+n} S_j\}.$$

Hence,

$$\begin{aligned} H_{\tau+n} - \inf_{\tau \le k \le \tau+n} H_k &= \mathrm{Card}\,\{k \in \{\tau, \ldots, \tau+n-1\} : S_k = \inf_{k \le j \le \tau+n} S_j\} \\ &= \mathrm{Card}\,\{k \in \{0, \ldots, n-1\} : S^\tau_k = \inf_{k \le j \le n} S^\tau_j\}, \end{aligned}$$

where S^τ denotes the shifted random walk $S^\tau_n = S_{\tau+n} - S_\tau$. Since S^τ is independent of $\mathcal{F}_\tau$ and has the same distribution as S, the desired result follows from the previous formula and Lemma 1.1.1. $\square$

1.2 The basic limit theorem

We will now state and prove the main result of this chapter. Recall the assumptions on the offspring distribution μ formulated in the previous section. By definition, a reflected Brownian motion (started at the origin) is the absolute value of a standard linear Brownian motion started at the origin. The notation $[x]$ refers to the integer part of x.

Theorem 1.2.1 *Let $\mathcal{T}_1, \mathcal{T}_2, \dots$ be a sequence of independent μ-Galton-Watson trees, and let $(H_n, n \geq 0)$ be the associated height process. Then*

$$(\frac{1}{\sqrt{p}} H_{[pt]}, t \geq 0) \xrightarrow[p\to\infty]{\text{(d)}} (\frac{2}{\sigma}\,\beta_t, t \geq 0)$$

where β is a reflected Brownian motion. The convergence holds in the sense of weak convergence on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$.

The proof of Theorem 1.2.1 consists of two separate steps. In the first one, we obtain the weak convergence of finite-dimensional marginals and in the second one we prove tightness.

First step. Let $S = (S_n, n \geq 0)$ be as in Lemma 1.1.1. Note that the jump distribution ν has mean 0 and finite variance σ^2, and thus the random walk S is recurrent. We also introduce the notation

$$\begin{aligned} M_n &= \sup_{0\leq k\leq n} S_k \\ I_n &= \inf_{0\leq k\leq n} S_k\,. \end{aligned}$$

Donsker's invariance theorem gives

$$(\frac{1}{\sqrt{p}} S_{[pt]}, t \geq 0) \xrightarrow[p\to\infty]{\text{(d)}} (\sigma\, B_t, t \geq 0) \tag{1.2}$$

where B is a standard linear Brownian motion started at the origin.

For every $n \geq 0$, introduce the time-reversed random walk $\widehat{S}^n$ defined by

$$\widehat{S}^n_k = S_n - S_{(n-k)^+}$$

and note that $(\widehat{S}^n_k, 0 \leq k \leq n)$ has the same distribution as $(S_n, 0 \leq k \leq n)$. ¿From formula (1.1), we have

$$H_n = \text{Card}\,\{k \in \{0, 1, \dots, n-1\} : S_k = \inf_{k\leq j\leq n} S_j\} = \Phi_n(\widehat{S}^n),$$

where for any discrete trajectory $\omega = (\omega(0), \omega(1), \dots)$, we have set

$$\Phi_n(\omega) = \text{Card}\,\{k \in \{1, \dots, n\} : \omega(k) = \sup_{0\leq j\leq k} \omega(j)\}.$$

We also set

$$K_n = \Phi_n(S) = \text{Card}\,\{k \in \{1, \dots, n\} : S_k = M_k\}.$$

The following lemma is standard.

Lemma 1.2.2 *Define a sequence of stopping times T_j, $j = 0, 1, \dots$ inductively by setting $T_0 = 0$ and for every $j \geq 1$,*

$$T_j = \inf\{n > T_{j-1} : S_n = M_n\}.$$

Then the random variables $S_{T_j} - S_{T_{j-1}}$, $j = 1, 2, \dots$ are independent and identically distributed, with distribution

$$P[S_{T_1} = k] = \nu([k, \infty))\ , \quad k \geq 0.$$

Note that the distribution of S_{T_1} has a finite first moment:

$$E[S_{T_1}] = \sum_{k=0}^{\infty} k\,\nu([k,\infty)) = \sum_{j=0}^{\infty} \frac{j(j+1)}{2}\nu(j) = \frac{\sigma^2}{2}.$$

The next lemma is the key to the first part of the proof.

Lemma 1.2.3 *We have*

$$\frac{H_n}{S_n - I_n} \xrightarrow[n\to\infty]{\text{(P)}} \frac{2}{\sigma^2},$$

where the notation $\overset{\text{(P)}}{\to}$ *means convergence in probability.*

Proof. From our definitions, we have

$$M_n = \sum_{T_k \le n} (S_{T_k} - S_{T_{k-1}}) = \sum_{k=1}^{K_n} (S_{T_k} - S_{T_{k-1}}).$$

Using Lemma 1.2.2 and the law of large numbers, we get

$$\frac{M_n}{K_n} \xrightarrow[n\to\infty]{\text{(a.s.)}} E[S_{T_1}] = \frac{\sigma^2}{2}.$$

By replacing S with the time-reversed walk $\widehat{S}^n$ we see that for every n, the pair (M_n, K_n) has the same distribution as $(S_n - I_n, H_n)$. Hence the previous convergence entails

$$\frac{S_n - I_n}{H_n} \xrightarrow[n\to\infty]{\text{(P)}} \frac{\sigma^2}{2},$$

and the lemma follows. □

¿From (1.2), we have for every choice of $0 \le t_1 \le t_2 \le \cdots \le t_m$,

$$\frac{1}{\sqrt{p}}\Big(S_{[pt_1]} - I_{[pt_1]}, \ldots, S_{[pt_m]} - I_{[pt_m]}\Big) \xrightarrow[p\to\infty]{\text{(d)}} \sigma\Big(B_{t_1} - \inf_{0\le s\le t_1} B_s, \ldots, B_{t_m} - \inf_{0\le s\le t_m} B_s\Big).$$

Therefore it follows from Lemma 1.2.3 that

$$\frac{1}{\sqrt{p}}\Big(H_{[pt_1]}, \ldots, H_{[pt_m]}\Big) \xrightarrow[p\to\infty]{\text{(d)}} \frac{2}{\sigma}\Big(B_{t_1} - \inf_{0\le s\le t_1} B_s, \ldots, B_{t_m} - \inf_{0\le s\le t_m} B_s\Big).$$

However, a famous theorem of Lévy states that the process

$$\beta_t = B_t - \inf_{0\le s\le t} B_s$$

is a reflected Brownian motion. This completes the proof of the convergence of finite-dimensional marginals in Theorem 1.2.1.

Second step. To simplify notation, set

$$H_t^{(p)} = \frac{1}{\sqrt{p}} H_{[pt]}.$$

We have to prove the tightness of the laws of the processes $H^{(p)}$ in the set of all probability measures on the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$. By standard results (see e.g. Corollary 3.7.4 in [20]), it is enough to verify the following two properties:

(i) For every $t \geq 0$ and $\eta > 0$, there exists a constant $K \geq 0$ such that

$$\liminf_{p\to\infty} \; P[H_t^{(p)} \leq K] \geq 1 - \eta.$$

(ii) For every $T > 0$ and $\delta > 0$,

$$\lim_{n\to\infty} \; \limsup_{p\to\infty} \; P\Big[\sup_{1\leq i\leq 2^n} \; \sup_{t\in[(i-1)2^{-n}T, i2^{-n}T]} |H_t^{(p)} - H_{(i-1)2^{-n}T}^{(p)}| > \delta\Big] = 0.$$

Property (i) is immediate from the convergence of finite-dimensional marginals. Thus the real problem is to prove (ii). We fix $\delta > 0$ and $T > 0$ and first observe that

$$P\Big[\sup_{1\leq i\leq 2^n} \; \sup_{t\in[(i-1)2^{-n}T, i2^{-n}T]} |H_t^{(p)} - H_{(i-1)2^{-n}T}^{(p)}| > \delta\Big] \leq A_1(n,p) + A_2(n,p) + A_3(n,p) \quad (1.3)$$

where

$$\begin{aligned}
A_1(n,p) &= P\Big[\sup_{1\leq i\leq 2^n} |H_{i2^{-n}T}^{(p)} - H_{(i-1)2^{-n}T}^{(p)}| > \frac{\delta}{5}\Big] \\
A_2(n,p) &= P\Big[\sup_{t\in[(i-1)2^{-n}T, i2^nT]} H_t^{(p)} > H_{(i-1)2^{-n}T}^{(p)} + \frac{4\delta}{5} \text{ for some } 1 \leq i \leq 2^n\Big] \\
A_3(n,p) &= P\Big[\inf_{t\in[(i-1)2^{-n}T, i2^nT]} H_t^{(p)} < H_{i2^nT}^{(p)} - \frac{4\delta}{5} \text{ for some } 1 \leq i \leq 2^n\Big]
\end{aligned}$$

The term A_1 is easy to bound. By the convergence of finite-dimensional marginals, we have

$$\limsup_{p\to\infty} A_1(n,p) \leq P\Big[\sup_{1\leq i\leq 2^n} \frac{2}{\sigma} |\beta_{i2^{-n}T} - \beta_{(i-1)2^{-n}T}| \geq \frac{\delta}{5}\Big]$$

and the path continuity of the process β ensures that the right-hand side tends to 0 as $n \to \infty$.

To bound the terms A_2 and A_3, we introduce the stopping times $\tau_k^{(p)}$, $k \geq 0$ defined by induction as follows:

$$\begin{aligned}
&\tau_0^{(p)} = 0 \\
&\tau_{k+1}^{(p)} = \inf\{t \geq \tau_k^{(p)} : H_t^{(p)} > \inf_{\tau_k^{(p)} \leq r \leq t} H_r^{(p)} + \frac{\delta}{5}\}.
\end{aligned}$$

Let $i \in \{1, \ldots, 2^n\}$ be such that

$$\sup_{t \in [(i-1)2^{-n}T, i2^nT]} H_t^{(p)} > H_{(i-1)2^{-n}T}^{(p)} + \frac{4\delta}{5}. \tag{1.4}$$

Then it is clear that the interval $[(i-1)2^{-n}T, i2^nT]$ must contain at least one of the random times $\tau_k^{(p)}$, $k \geq 0$. Let $\tau_j^{(p)}$ be the first such time. By construction we have

$$\sup_{t \in [(i-1)2^{-n}T, \tau_j^{(p)})} H_t^{(p)} \leq H_{(i-1)2^{-n}T}^{(p)} + \frac{\delta}{5},$$

and since the positive jumps of $H^{(p)}$ are of size $\frac{1}{\sqrt{p}}$, we get also

$$H_{\tau_j^{(p)}}^{(p)} \leq H_{(i-1)2^{-n}T}^{(p)} + \frac{\delta}{5} + \frac{1}{\sqrt{p}} < H_{(i-1)2^{-n}T}^{(p)} + \frac{2\delta}{5}$$

provided that $p > (5/\delta)^2$. From (1.4), we have then

$$\sup_{t \in [\tau_j^{(p)}, i2^nT]} H_t^{(p)} > H_{\tau_j^{(p)}}^{(p)} + \frac{\delta}{5},$$

which implies that $\tau_{j+1}^{(p)} \leq i2^{-n}T$. Summarizing, we get for $p > (5/\delta)^2$

$$A_2(n,p) \leq P\Big[\tau_k^{(p)} < T \text{ and } \tau_{k+1}^{(p)} - \tau_k^{(p)} < 2^{-n}T \text{ for some } k \geq 0\Big]. \tag{1.5}$$

A similar argument gives exactly the same bound for the quantity $A_3(n,p)$.

The following lemma is directly inspired from [20] p.134-135.

Lemma 1.2.4 *For every $x > 0$ and $p \geq 1$, set*

$$G_p(x) = P\Big[\tau_k^{(p)} < T \textit{ and } \tau_{k+1}^{(p)} - \tau_k^{(p)} < x \textit{ for some } k \geq 0\Big]$$

and

$$F_p(x) = \sup_{k \geq 0} P\Big[\tau_k^{(p)} < T \textit{ and } \tau_{k+1}^{(p)} - \tau_k^{(p)} < x\Big].$$

Then, for every integer $L \geq 1$,

$$G_p(x) \leq L\,F_p(x) + L\,e^T \int_0^\infty dy\, e^{-Ly}\, F_p(y).$$

Proof. For every integer $L \geq 1$, we have

$$\begin{aligned} G_p(x) &\leq \sum_{k=0}^{L-1} P[\tau_k^{(p)} < T \text{ and } \tau_{k+1}^{(p)} - \tau_k^{(p)} < x] + P[\tau_L^{(p)} < T] \\ &\leq LF_p(x) + e^T E\Big[1_{\{\tau_L^{(p)} < T\}} . \exp\Big(-\sum_{k=0}^{L-1}(\tau_{k+1}^{(p)} - \tau_k^{(p)})\Big)\Big] \\ &\leq LF_p(x) + e^T \prod_{k=0}^{L-1} E\Big[1_{\{\tau_L^{(p)} < T\}} \exp(-L(\tau_{k+1}^{(p)} - \tau_k^{(p)}))\Big]^{1/L}. \end{aligned}$$

Then observe that for every $k \in \{0, 1, \ldots, L-1\}$,

$$\begin{aligned} E\Big[1_{\{\tau_L^{(p)}<T\}} \exp(-L(\tau_{k+1}^{(p)} - \tau_k^{(p)}))\Big] &\leq E\Big[1_{\{\tau_k^{(p)}<T\}} \int_{\tau_{k+1}^{(p)}-\tau_k^{(p)}}^{\infty} dy\, Le^{-Ly}\Big] \\ &\leq \int_0^\infty dy\, Le^{-Ly}\, F_p(y). \end{aligned}$$

The desired result follows. □

Thanks to Lemma 1.2.4, the limiting behavior of the right-hand side of (1.5) will be reduced to that of the function $F_p(x)$. To handle $F_p(x)$, we use the next lemma.

Lemma 1.2.5 *The random variables $\tau_{k+1}^{(p)} - \tau_k^{(p)}$ are independent and identically distributed. Furthermore,*

$$\lim_{x\downarrow 0} \Big(\limsup_{p\to\infty} P[\tau_1^{(p)} \leq x] \Big) = 0.$$

Proof. The first assertion is a straightforward consequence of Lemma 1.1.2. Let us turn to the second assertion. To simplify notation, we write $\delta' = \delta/5$. For every $\eta > 0$, set

$$T_\eta^{(p)} = \inf\{t \geq 0 : \frac{1}{\sqrt{p}} S_{[pt]} < -\eta\}.$$

Then,

$$P[\tau_1^{(p)} \leq x] = P\Big[\sup_{s\leq x} H_s^{(p)} > \delta'\Big] \leq P\Big[\sup_{s\leq T_\eta^{(p)}} H_s^{(p)} > \delta'\Big] + P[T_\eta^{(p)} < x].$$

On one hand, by (1.2)

$$\limsup_{p\to\infty} P[T_\eta^{(p)} < x] \leq \limsup_{p\to\infty} P[\inf_{t\leq x} \frac{1}{\sqrt{p}} S_{[pt]} \leq -\eta] \leq P[\inf_{t\leq x} \beta_t \leq -\eta],$$

and the right-hand side goes to zero as $x \downarrow 0$, for any choice of $\eta > 0$. On the other hand, the construction of the height process shows that the quantity

$$\sup_{s\leq T_\eta^{(p)}} H_s^{(p)}$$

is distributed as $(M_p - 1)/\sqrt{p}$, where M_p is the extinction time of a Galton-Watson process with offspring distribution μ, started at $[\eta\sqrt{p}] + 1$. Write g for the generating function of μ and g_k for the k-th iterate of g. It follows that

$$P\Big[\sup_{s\leq T_\eta^{(p)}} H_s^{(p)} > \delta'\Big] = 1 - g_{[\delta'\sqrt{p}]+1}(0)^{[\eta\sqrt{p}]+1}.$$

A classical result of the theory of branching processes [4] states that, as $k \to \infty$,

$$g_k(0) = 1 - \frac{2}{\sigma^2 k} + o(\frac{1}{k}).$$

It follows that

$$\lim_{\eta\to 0}\left(\liminf_{p\to\infty} g_{[\delta'\sqrt{p}]+1}(0)^{[\eta\sqrt{p}]+1}\right) = 1$$

$andthus \lim_{\eta\to 0}\left(\limsup_{p\to\infty} P\left[\sup_{s\le T_\eta^{(p)}} H_s^{(p)} > \delta'\right]\right) = 0. Thesecondassertionofthelemmanowfollows.$

□

We can now complete the proof of Theorem 1.2.1. Set:

$$F(x) = \limsup_{p\to\infty} F_p(x)\ , \quad G(x) = \limsup_{p\to\infty} G_p(x).$$

Lemma 1.2.5 immediately shows that $F(x)\downarrow 0$ as $x\downarrow 0$. On the other hand, we get from Lemma 1.2.4 that for every integer $L\ge 1$,

$$G(x) \le L\,F(x) + L\,e^T \int_0^\infty dy\, e^{-Ly}\,F(y).$$

It follows that we have also $G(x)\downarrow 0$ as $x\downarrow 0$. By (1.5), this gives

$$\lim_{n\to\infty}\left(\limsup_{p\to\infty} A_2(n,p)\right) = 0,$$

and the same property holds for $A_3(n,p)$. This completes step 2 and the proof of Theorem 1.2.1. □

1.3 Convergence of contour processes

In this section, we show that the limit theorem obtained in the previous section for rescaled discrete height processes can be formulated as well in terms of the contour processes of the Galton-Watson trees. The proof relies on simple connections between the height process and the contour process of a sequence of Galton-Watson trees.

As in Theorem 1.2.1, we consider the height process $(H_n, n\ge 0)$ associated with a sequence of independent μ-Galton-Watson trees. We also let $(C_n, n\ge 0)$ be the contour process associated with this sequence of trees (see Section 1.1). By linear interpolation we can extend the contour process to real values of the time-parameter $t\ge 0$ (cf fig.1). We also set

$$J_n = \mathrm{Card}\{k\in\{1,\dots,n\}, H_k = 0\}$$

and

$$K_n = 2n - H_n - 2J_n.$$

Note that the sequence K_n is strictly increasing and $K_n\ge n$.

Recall that the value at time n of the height process corresponds to the generation of the individual visited at time n, assuming that individuals are visited in lexicographical order one tree after another. It is easily checked by induction on n that $[K_n, K_{n+1}]$ is exactly the time interval during which the contour process goes from the individual n to the individual $n+1$. From this observation, we get

$$\sup_{t\in[K_n,K_{n+1}]} |C_t - H_n| \le |H_{n+1} - H_n| + 1.$$

A more precise argument for this bound follows from the explicit formula for C_t in terms of the height process: For $t \in [K_n, K_{n+1}]$,

$$\begin{aligned} C_t &= H_n - (t - K_n) && \text{if } t \in [K_n, 2n+1-H_{n+1}-J_n-J_{n+1}], \\ C_t &= H_{n+1} - (K_{n+1} - t) && \text{if } t \in [2n+1-H_{n+1}-J_n-J_{n+1}, K_{n+1}]. \end{aligned}$$

These formulas are easily checked by induction on n.

Define a random function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{N}$ by setting $\varphi(t) = n$ iff $t \in [K_n, K_{n+1})$. From the previous bound, we get for every integer $m \geq 1$,

$$\sup_{t \in [0,m]} |C_t - H_{\varphi(t)}| \leq \sup_{t \in [0,K_m]} |C_t - H_{\varphi(t)}| leq 1 + \sup_{n \leq m} |H_{n+1} - H_n|. \tag{1.6}$$

Similarly, it follows from the definition of K_n that

$$\sup_{t \in [0,m]} |\varphi(t) - \frac{t}{2}| \leq \sup_{t \in [0,K_m]} |\varphi(t) - \frac{t}{2}| \leq \frac{1}{2} \sup_{n \leq m} H_n + J_m + 1. \tag{1.7}$$

Theorem 1.3.1 *We have*

$$\left(\frac{1}{\sqrt{p}} C_{pt}\, ,\, t \geq 0 \right) \xrightarrow[p\to\infty]{\text{(d)}} (\frac{\sqrt{2}}{\sigma} \beta_t, t \geq 0). \tag{1.8}$$

where β is a reflected Brownian motion and the convergence holds in the sense of weak convergence in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$.

Proof. For every $p \geq 1$, set $\varphi_p(t) = p^{-1}\varphi(pt)$. By (1.6), we have for every $m \geq 1$,

$$\sup_{t \leq m} \left| \frac{1}{\sqrt{p}} C_{pt} - \frac{1}{\sqrt{p}} H_{p\varphi_p(t)} \right| \leq \frac{1}{\sqrt{p}} + \frac{1}{\sqrt{p}} \sup_{t \leq m} |H_{[pt]+1} - H_{[pt]}| \xrightarrow[p\to\infty]{} 0 \tag{1.9}$$

in probability, by Theorem 1.2.1.

On the other hand, it easily follows from (1.1) that $J_n = -\inf_{k \leq n} S_k$, and so the convergence (1.2) implies that, for every $m \geq 1$,

$$\frac{1}{\sqrt{p}} J_{mp} \xrightarrow[p\to\infty]{\text{(d)}} -\sigma \inf_{t \leq m} B_t. \tag{1.10}$$

Then, we get from (1.7)

$$\sup_{t \leq m} |\varphi_p(t) - \frac{t}{2}| \leq \frac{1}{2p} \sup_{k \leq mp} H_k + \frac{1}{p} J_{mp} + \frac{1}{p} \xrightarrow[p\to\infty]{} 0 \tag{1.11}$$

in probability, by Theorem 1.2.1 and (1.10).

The statement of the theorem now follows from Theorem 1.2.1, (1.9) and (1.11). □

Remark. There is one special case where Theorem 1.3.1 is easy. This is the case where μ is the geometric distribution $\mu(k) = 2^{-k-1}$, which satisfies our assumptions with $\sigma^2 = 2$. In that case, it is not hard to see that our contour process $(C_n, n \geq 0)$ is distributed as a simple random walk reflected at the origin. Thus the statement of Theorem 1.3.1 follows from Donsker's invariance theorem (note that $\sqrt{2}/\sigma = 1$).

Chapter 2

Trees embedded in Brownian excursions

In the previous chapter, we proved that the height process or the contour process of Galton-Watson trees, suitably rescaled, converges in distribution towards reflected Brownian motion. This strongly suggests that Brownian excursions code continuous trees in the same way as the height process or the contour process codes Galton-Watson trees. In this chapter, we will first give an informal description of the tree coded by an excursion, and then provide explicit calculations for the distribution of the tree associated with a Brownian excursion. These calculations play a major role in the approach to superprocesses that will be developed in Chapter 4. When the Brownian excursion is normalized to have length one, the corresponding tree is Aldous' continuum random tree, which has appeared in a number of recent works.

2.1 The tree coded by a continuous function

We denote by $C(\mathbb{R}_+, \mathbb{R}_+)$ the space of all continuous functions from $\mathbb{R}_+$ into $\mathbb{R}_+$, which is equipped with the topology of uniform convergence on the compact subsets of $\mathbb{R}_+$ and the associated Borel σ-field. A special role will be played by the subset $\mathcal{E}$ of excursions : An excursion e is an element of $C(\mathbb{R}_+, \mathbb{R}_+)$ such that $e(0) = 0$ and $e(t) > 0$ if and only if $0 < t < \gamma$ for some $\gamma = \gamma(e) > 0$.

Let us fix $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $f(0) = 0$. We can view this function as coding a continuous tree according to the following informal prescriptions:

(i) Each $s \in \mathbb{R}_+$ corresponds to a vertex of the tree at generation $f(s)$.

(ii) If $s, s' \in \mathbb{R}_+$ the vertex (corresponding to) s is an ancestor of the vertex corresponding to s' iff

$$f(s) = \inf_{r \in [s,s']} f(r) \qquad (f(s) = \inf_{r \in [s',s]} f(r) \text{ if } s' < s).$$

More generally, the quantity $\inf_{r \in [s,s']} f(r)$ is the generation of the last common ancestor to s and s'.

(iii) The distance between vertices s and s' is defined to be

$$d(s,s') = f(s) + f(s') - 2 \inf_{r \in [s,s']} f(r)$$

and we identify s and s' (we write $s \sim s'$) if $d(s,s') = 0$.

With these definitions at hand, the tree coded by f is the quotient set $\mathbb{R}_+/\sim$, equipped with the distance d and the genealogical relation defined in (ii). In this chapter, we will consider the case when f is an excursion with duration γ and then we only need to consider $[0,\gamma]/\sim$, since vertices corresponding to $s \geq \gamma$ are obviously identified with γ.

Note that the line of ancestors of the vertex s is isometric to the line segment $[0, f(s)]$. If $s < s'$, the lines of ancestors of s and s' share a common part isometric to the segment $[0, \inf_{[s,s']} f(r)]$ and then become distinct. More generally, we can define the genealogical tree of ancestors of any p vertices $s_1, \ldots, s_p$, , and our principal aim is to determine the law of the tree when f is randomly distributed according to the law of a Brownian excursion, and $s_1, \ldots, s_p$ are chosen uniformly at random over the duration interval of the excursion.

Aldous' continuum random tree (the CRT) is by definition the random tree coded in the previous sense by the random function equal to twice the normalized Brownian excursion (see [1], [2] for different constructions of the CRT).

We start by recalling basic facts about Brownian excursions.

2.2 The Itô excursion measure

We denote by $(B_t, t \geq 0)$ a linear Brownian motion, which starts at x under the probability measure P_x. We also set $T_0 = \inf\{t \geq 0, B_t = 0\}$. For $x > 0$, the density of the law of T_0 under P_x is

$$q_x(t) = \frac{x}{\sqrt{2\pi t^3}}\, e^{-\frac{x^2}{2t}}.$$

A major role in what follows will be played by the Itô measure $n(de)$ of positive excursions. This is an infinite measure on the set $\mathcal{E}$ of excursions, which can be characterized as follows. Let F be a bounded continuous function on $C(\mathbb{R}_+, \mathbb{R}_+)$ and assume that there exists a number $\alpha > 0$ such that $F(f) = 0$ as soon as $f(t) = 0$ for every $t \geq \alpha$. Then,

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} E_\varepsilon[F(B_{s \wedge T_0}, s \geq 0)] = n(F).$$

The factor 2 in $\frac{1}{2\varepsilon}$ is just a convenient normalization.

For most of our purposes in this chapter, it will be enough to know that $n(de)$ has the following two (characteristic) properties:

(i) For every $t > 0$, and every measurable function $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $g(0) = 0$,

$$\int n(de)\, g(e(t)) = \int_0^\infty dx\, q_x(t)\, g(x). \tag{2.1}$$

(ii) Let $t > 0$ and let Φ and Ψ be two nonnegative measurable functions defined respectively on $C([0,t],\mathbb{R}_+)$ and $C(\mathbb{R}_+,\mathbb{R}_+)$. Then,

$$\int n(de)\,\Phi(e(r),0\le r\le t)\Psi(e(t+r),r\ge 0)$$
$$= \int n(de)\,\Phi(e(r),0\le r\le t)\;E_{e(t)}(\Psi(B_{r\wedge T_0},r\ge 0)).$$

Note that (i) implies $n(\gamma > t) = n(e(t) > 0) = (2\pi t)^{-1/2} < \infty$. Property (ii) means that the process $(e(t), t > 0)$ is Markovian under n with the transition kernels of Brownian motion absorbed at 0.

Let us also recall the useful formula $n(\sup_{s\ge 0} e(s) > \varepsilon) = (2\varepsilon)^{-1}$ for $\varepsilon > 0$.

Lemma 2.2.1 *If g is measurable and nonnegative over $\mathbb{R}_+$ and $g(0) = 0$,*

$$n\Big(\int_0^\infty dt\,g(e(t))\Big) = \int_0^\infty dx\,g(x).$$

Proof. This is a simple consequence of (2.1). □

For every $t \ge 0$, we set $I_t = \inf_{0\le s\le t} B_s$.

Lemma 2.2.2 *If g is measurable and nonnegative over $\mathbb{R}^3$ and $x \ge 0$,*

$$E_x\Big(\int_0^{T_0} dt\,g(t,I_t,B_t)\Big) = 2\int_0^x dy\int_y^\infty dz\int_0^\infty dt\,q_{x+z-2y}(t)\,g(t,y,z) \tag{2.2}$$

In particular, if h is measurable and nonnegative over $\mathbb{R}^2$,

$$E_x\Big(\int_0^{T_0} dt\,h(I_t,B_t)\Big) = 2\int_0^x dy\int_y^\infty dz\,h(y,z). \tag{2.3}$$

Proof. Since

$$E_x\Big(\int_0^{T_0} dt\,g(t,I_t,B_t)\Big) = \int_0^\infty dt\,E_x(g(t,I_t,B_t)\,1_{\{I_t>0\}}),$$

the lemma follows from the explicit formula

$$E_x(g(I_t,B_t)) = \int_{-\infty}^x\int_y^\infty dz\,\frac{2(x+z-2y)}{\sqrt{2\pi t^3}}\,e^{-\frac{(x+z-2y)^2}{2t}}\,g(y,z)$$

which is itself a consequence of the reflection principle for linear Brownian motion. □

2.3 Binary trees

We use the same formalism for trees as in Chapter 1, but we restrict our attention to (ordered rooted) binary trees. Such a tree describes the genealogy of a population starting with one ancestor (the root $\emptyset$), where each individual can have 0 or 2 children, and the population becomes extinct after a finite number of generations (the tree is finite).

Analogously to Chapter 1, we define a tree as a finite subset T of $\cup_{n=0}^{\infty}\{1,2\}^n$ (with $\{1,2\}^0 = \{\emptyset\}$) satisfying the obvious conditions:

(i) $\emptyset \in T$;

(ii) if $(i_1,\ldots,i_n) \in T$ with $n \geq 1$, then $(i_1,\ldots,i_{n-1}) \in T$;

(iii) if $(i_1,\ldots,i_n) \in T$, then either $(i_1,\ldots,i_n,1) \in T$ and $(i_1,\ldots,i_n,2) \in T$, or $(i_1,\ldots,i_n,1) \notin T$ and $(i_1,\ldots,i_n,2) \notin T$.

The elements of T are the vertices (or individuals in the branching process terminology) of the tree. Individuals without children are called leaves. If T and T' are two trees, the concatenation of T and T', denoted by $T * T'$, is defined in the obvious way: For $n \geq 1$, $(i_1,\ldots,i_n)$ belongs to $T * T'$ if and only if $i_1 = 1$ and $(i_2,\ldots,i_n)$ belongs to T, or $i_1 = 2$ and $(i_2,\ldots,i_n)$ belongs to T'. Note that $T * T' \neq T' * T$ in general. For $p \geq 1$, we denote by $\mathbb{T}_p$ the set of all (ordered rooted binary) trees with p leaves. It is easy to compute $a_p = \mathrm{Card}\,\mathbb{T}_p$. Obviously $a_1 = 1$ and if $p \geq 2$, decomposing the tree at the root shows that $a_p = \sum_{j=1}^{p-1} a_j a_{p-j}$. It follows that

$$a_p = \frac{1 \times 3 \times \ldots \times (2p-3)}{p!}\, 2^{p-1}.$$

A *marked tree* is a pair $(T, \{h_v, v \in T\})$, where $h_v \geq 0$ for every $v \in T$. Intuitively, h_v represents the lifetime of individual v.

We denote by $\mathcal{T}_p$ the set of all marked trees with p leaves. Let $\theta = (T, \{h_v, v \in T\}) \in \mathcal{T}_p$, $\theta' = (T', \{h'_v, v \in T')\}) \in \mathcal{T}_{p'}$, and $h \geq 0$. the concatenation

$$\theta \underset{h}{*} \theta'$$

is the element of $\mathcal{T}_{p+p'}$ whose "skeleton" is $T * T'$ and such that the marks of vertices in T, respectively in T', become the marks of the corresponding vertices in $T * T'$, and finally the mark of $\emptyset$ in $T * T'$ is h.

2.4 The genealogy of a finite set of vertices

We will now give a precise definition of the genealogical tree for p given vertices in the tree coded by an excursion e in the sense of Section 1. If the vertices correspond to times $t_1,\ldots,t_p$ in the coding of Section 1, the associated tree denoted by $\theta(e, t_1,\ldots,t_p)$ will be an element of $\mathcal{T}_p$.

For the construction, it is convenient to work in a slightly greater generality. Let $f : [a,b] \longrightarrow \mathbb{R}_+$ be a continuous function defined on a subinterval $[a,b]$ of $\mathbb{R}_+$. For every $a \le u \le v \le b$, we set

$$m(u,v) = \inf_{u \le t \le v} f(t).$$

Let $t_1, \dots, t_p \in \mathbb{R}_+$ be such that $a \le t_1 \le t_2 \le \cdots \le t_p \le b$. We construct the marked tree

$$\theta(f, t_1, \dots, t_p) = (T(f, t_1, \dots, t_p), \{h_v(f, t_1, \dots, t_p), v \in T\}) \in \mathcal{T}_p$$

by induction on p. If $p = 1$, $T(f, t_1)$ is the unique element of $\mathbb{T}_1$, and $h_\emptyset(f, t_1) = f(t_1)$. If $p = 2$, $T(f, t_1, t_2)$ is the unique element of $\mathbb{T}_2$, $h_\emptyset = m(t_1, t_2)$, $h_1 = f(t_1) - m(t_1, t_2)$, $h_2 = f(t_2) - m(t_1, t_2)$.

Then let $p \ge 3$ and suppose that the tree has been constructed up to the order $p - 1$. Let $j = \inf\{i \in \{1, \dots, p-1\}, m(t_i, t_{i+1}) = m(t_1, t_p)\}$. Define f' and f'' by the formulas

$$\begin{aligned} f'(t) &= f(t) - m(t_1, t_p), \qquad & t \in [t_1, t_j], \\ f''(t) &= f(t) - m(t_1, t_p), \qquad & t \in [t_{j+1}, t_p]. \end{aligned}$$

By the induction hypothesis, we can associate with f' and $t_1, \dots, t_j$, respectively with f'' and $t_{j+1}, \dots, t_p$, a tree $\theta(f', t_1, \dots, t_j) \in \mathcal{T}_j$, resp. $\theta(f'', t_{j+1}, \dots, t_p) \in \mathcal{T}_{p-j}$. We set

$$\theta(f, t_1, \dots, t_p) = \theta(f', t_1, \dots, t_j) \underset{m(t_1, t_p)}{*} \theta(f'', t_{j+1}, \dots, t_p).$$

It should be clear that when f is an excursion, the tree $\theta(f, t_1, \dots, t_p)$ describes the genealogy of the vertices corresponding to $t_1, \dots, t_p$, for the continuous tree coded by f in the way explained in Section 1.

The previous construction shows that the tree $\theta(f, t_1, \dots, t_p)$ only depends on the values $f(t_1), \dots, f(t_p)$ of f at times $t_1, \dots, t_p$, and on the minima $m(t_1, t_2), \dots, m(t_{p-1}, t_p)$:

$$\theta(e, t_1, \dots, t_p) = \Gamma_p(m(t_1, t_2), \dots, m(t_{p-1}, t_p), e(t_1), \dots, e(t_p)),$$

where Γ_p is a measurable function from $\mathbb{R}_+^{2p-1}$ into $\mathcal{T}_p$

2.5 The law of the tree coded by an excursion

Our goal is now to determine the law of the tree $\theta(e, t_1, \dots, t_p)$ when e is chosen according to the Itô measure of excursions, and $(t_1, \dots, t_p)$ according to Lebesgue measure on $[0, \gamma(e)]^p$. We keep the notation $m(u,v) = \inf_{u \le t \le v} e(t)$.

Proposition 2.5.1 *Let F be nonnegative and measurable on $\mathbb{R}_+^{2p-1}$. Then*

$$\begin{aligned} & n\Big(\int_{\{0 \le t_1 \le \cdots \le t_p \le \gamma\}} dt_1 \dots dt_p \, f(m(t_1, t_2), \dots, m(t_{p-1}, t_p), e(t_1), \dots, e(t_p)) \Big) \\ &= 2^{p-1} \int_{\mathbb{R}_+^{2p-1}} d\alpha_1 \dots d\alpha_{p-1} d\beta_1 \dots d\beta_p \Big(\prod_{i=1}^{p-1} 1_{[0, \beta_i \wedge \beta_{i+1}]}(\alpha_i) \Big) f(\alpha_1, \dots, \alpha_{p-1}, \beta_1, \dots, \beta_p). \end{aligned}$$

Proof. This is a simple consequence of Lemmas 2.2.1 and 2.2.2. For $p = 1$, the result is exactly Lemma 2.2.1. We proceed by induction on p using property (ii) of the Itô measure and then (2.3):

$$\begin{aligned}
&n\Big(\int_{\{0\le t_1\le\cdots\le t_p\le\gamma\}} dt_1\dots dt_p\, f(m(t_1,t_2),\dots,m(t_{p-1},t_p),e(t_1),\dots,e(t_p))\Big)\\
&= n\Big(\int_{\{0\le t_1\le\cdots\le t_{p-1}\le\gamma\}} dt_1\dots dt_{p-1}\\
&\quad E_{e(t_{p-1})}\Big(\int_0^{T_0} dt\, f(m(t_1,t_2),\dots,m(t_{p-2},t_{p-1}),I_t,e(t_1),\dots,e(t_{p-1}),B_t)\Big)\Big)\\
&= 2\,n\Big(\int_{\{0\le t_1\le\cdots\le t_{p-1}\le\gamma\}} dt_1\dots dt_{p-1}\\
&\quad \int_0^{e(t_{p-1})} d\alpha_{p-1}\int_{\alpha_{p-1}}^{\infty} d\beta_p\, f(m(t_1,t_2),\dots,m(t_{p-2},t_{p-1}),\alpha_{p-1},e(t_1),\dots,e(t_{p-1}),\beta_p)\Big).
\end{aligned}$$

The proof is then completed by using the induction hypothesis. $\square$

The uniform measure Λ_p on $\mathcal{T}_p$ is defined by

$$\int \Lambda_p(d\theta)\,F(\theta) = \sum_{T\in\mathbb{T}_p}\int\prod_{v\in T} dh_v\; F(T,\{h_v, v\in T\}).$$

Theorem 2.5.2 *The law of the tree $\theta(e,t_1,\dots,t_p)$ under the measure*

$$n(de)\,1_{\{0\le t_1\le\cdots\le t_p\le\gamma(e)\}}dt_1\dots dt_p$$

is $2^{p-1}\Lambda_p$.

Proof. Denote by Δ_p the measure on $\mathbb{R}_+^{2p-1}$ defined by

$$\Delta_p(d\alpha_1\dots d\alpha_{p-1}d\beta_1\dots d\beta_p) = \Big(\prod_{i=1}^{p-1} 1_{[0,\beta_i\wedge\beta_{i+1}]}(\alpha_i)\Big)d\alpha_1\dots d\alpha_{p-1}d\beta_1\dots d\beta_p.$$

Recall the notation Γ_p introduced at the end of Section 2.4. In view of Proposition 2.5.1, the proof of Theorem 2.5.2 reduces to checking that $\Gamma_p(\Delta_p) = \Lambda_p$. For $p = 1$, this is obvious.

Let $p \ge 2$ and suppose that the result holds up to order $p-1$. For every $j \in \{1,\dots,p-1\}$, let H_j be the subset of $\mathbb{R}_+^{2p-1}$ defined by

$$H_j = \{(\alpha_1,\dots,\alpha_{p-1},\beta_1,\dots,\beta_p);\ \alpha_j < \alpha_i \text{ for every } i \ne j\}.$$

Then,

$$\Delta_p = \sum_{j=1}^{p-1} 1_{H_j}\cdot\Delta_p.$$

On the other hand, it is immediate to verify that $1_{H_j} \cdot \Delta_p$ is the image of the measure

$$\Delta_j(d\alpha'_1 \dots d\beta'_j) \otimes 1_{(0,\infty)}(h)dh \otimes \Delta_{p-j}(d\alpha''_1 \dots d\beta''_{p-j})$$

under the mapping $\Phi : (\alpha'_1, \dots, \beta'_j, h, \alpha''_1 \dots, \beta''_{p-j}) \longrightarrow (\alpha_1, \dots, \beta_p)$ defined by

$$\begin{array}{ll} \alpha_j = h, & \\ \alpha_i = \alpha'_i + h & \text{for } 1 \le i \le j-1, \\ \beta_i = \beta'_i + h & \text{for } 1 \le i \le j, \\ \alpha_i = \alpha''_{i-j} + h & \text{for } j+1 \le i \le p-1, \\ \beta_i = \beta''_{i-j} + h & \text{for } j+1 \le i \le p. \end{array}$$

The construction by induction of the tree $\theta(e, t_1, \dots, t_p)$ exactly shows that

$$\Gamma_p \circ \Phi(\alpha'_1, \dots, \beta'_j, h, \alpha''_1 \dots, \beta''_{p-j}) = \Gamma_j(\alpha'_1, \dots, \beta'_j) \underset{h}{*} \Gamma_{p-j}(\alpha''_1 \dots, \beta''_{p-j}).$$

Together with the induction hypothesis, the previous observations imply that for any nonnegative measurable function f on $\mathcal{T}_p$,

$$\begin{aligned} \int \Delta_p(du)\, 1_{H_j}(u)\, f(\Gamma_p(u)) &= \int_0^\infty dh \iint \Delta_j(du')\Delta_{p-j}(du'')\, f(\Gamma_p(\Phi(u', h, u''))) \\ &= \int_0^\infty dh \iint \Delta_j(du')\Delta_{p-j}(du'')\, f(\Gamma_j(u') \underset{h}{*} \Gamma_{p-j}(u'')) \\ &= \int_0^\infty dh \int \Lambda_j \underset{h}{*} \Lambda_{p-j}(d\theta)\, f(\theta) \end{aligned}$$

where we write $\Lambda_j \underset{h}{*} \Lambda_{p-j}$ for the image of $\Lambda_j(d\theta)\Lambda_{p-j}(d\theta')$ under the mapping $(\theta, \theta') \longrightarrow \theta \underset{h}{*} \theta'$. To complete the proof, simply note that

$$\Lambda_p = \sum_{j=1}^{p-1} \int_0^\infty dh\, \Lambda_j \underset{h}{*} \Lambda_{p-j}.$$

□

2.6 The normalized excursion and Aldous' continuum random tree

In this section, we propose to calculate the law of the tree $\theta(e, t_1, \dots, t_p)$ when e is chosen according to the law of the Brownian excursion conditioned to have duration 1, and $t_1, \dots, t_p$ are chosen according to the probability measure $p! 1_{\{0 \le t_1 \le \dots \le t_p \le 1\}} dt_1 \dots dt_p$. In contrast with the measure Λ_p of Theorem 4, we get for every p a probability measure on $\mathcal{T}_p$. These probability measures are compatible in a certain sense and they can be identified with the finite-dimensional marginals of Aldous' continuum random tree (this identification is obvious if the CRT is described by the coding explained in Section 1).

We first recall a few basic facts about the normalized Brownian excursion. There exists a unique collection of probability measures $(n_{(s)}, s > 0)$ on $\mathcal{E}$ such that the following properties hold:

(i) For every $s > 0$, $n_{(s)}(\gamma = s) = 1$.

(ii) For every $\lambda > 0$ and $s > 0$, the law under $n_{(s)}(de)$ of $e_\lambda(t) = \sqrt{\lambda} e(t/\lambda)$ is $n_{(\lambda s)}$.

(iii) For every Borel subset A of $\mathcal{E}$,

$$n(A) = \frac{1}{2}(2\pi)^{-1/2} \int_0^\infty s^{-3/2}\, n_{(s)}(A)\, ds.$$

The measure $n_{(1)}$ is called the law of the normalized Brownian excursion.

Our first goal is to get a statement more precise than Theorem 2.5.2 by considering the pair $(\theta(e, t_1, \dots, t_p), \gamma)$ instead of $\theta(e, t_1, \dots, t_p)$. If $\theta = (T, \{h_v, v \in T\})$ is a marked tree, the length of θ is defined in the obvious way by

$$L(\theta) = \sum_{v \in T} h_v.$$

Proposition 2.6.1 *The law of the pair $(\theta(e, t_1, \dots, t_p), \gamma)$ under the measure*

$$n(de)\, 1_{\{0 \le t_1 \le \cdots \le t_p \le \gamma(e)\}} dt_1 \dots dt_p$$

is

$$2^{p-1}\, \Lambda_p(d\theta)\, q_{2L(\theta)}(s) ds.$$

Proof. Recall the notation of the proof of Theorem 2.5.2. We will verify that, for any nonnegative measurable function f on $\mathbb{R}_+^{3p}$,

$$\begin{aligned}
& n\Big(\int_{\{0 \le t_1 \le \cdots \le t_p \le \gamma\}} dt_1 \dots dt_p \\
& \qquad f(m(t_1,t_2), \dots, m(t_{p-1}, t_p), e(t_1), \dots, e(t_p), t_1, t_2 - t_1, \dots, \gamma - t_p)\Big) \\
& = 2^{p-1} \int \Delta_p(d\alpha_1 \dots d\alpha_{p-1} d\beta_1 \dots d\beta_p) \int_{\mathbb{R}_+^{p+1}} ds_1 \dots ds_{p+1}\, q_{\beta_1}(s_1) q_{\beta_1 + \beta_2 - 2\alpha_1}(s_2) \dots \\
& \qquad \dots q_{\beta_{p-1} + \beta_p - 2\alpha_{p-1}}(s_p) q_{\beta_p}(s_{p+1})\, f(\alpha_1, \dots, \alpha_{p-1}, \beta_1, \dots, \beta_p, s_1, \dots, s_{p+1}).
\end{aligned} \tag{2.4}$$

Suppose that (2.4) holds. It is easy to check (for instance by induction on p) that

$$2\, L(\Gamma_p(\alpha_1, \dots, \alpha_{p-1}, \beta_1, \dots, \beta_p)) = \beta_1 + \sum_{i=1}^{p-1} (\beta_i + \beta_{i-1} - 2\alpha_i) + \beta_p.$$

Using the convolution identity $q_x * q_y = q_{x+y}$, we get from (2.4) that

$$
\begin{aligned}
&n\Big(\int_{\{0\le t_1\le\cdots\le t_p\le\gamma\}} dt_1\ldots dt_p\, f(m(t_1,t_2),\ldots,m(t_{p-1},t_p),e(t_1),\ldots,e(t_p),\gamma)\Big)\\
&= 2^{p-1}\int \Delta_p(d\alpha_1\ldots d\alpha_{p-1}d\beta_1\ldots d\beta_p)\int_0^\infty dt\, q_{2L(\Gamma_p(\alpha_1,\ldots,\beta_p))}(t)\, f(\alpha_1,\ldots,\beta_p,t).
\end{aligned}
$$

As in the proof of Theorem 2.5.2, the statement of Proposition 2.6.1 follows from this last identity and the equality $\Gamma_p(\Delta_p) = \Lambda_p$.

It remains to prove (2.4). The case $p = 1$ is easy: By using property (ii) of the Itô measure, then the definition of the function q_x and finally (2.1), we get

$$
\begin{aligned}
\int n(de)\int_0^\gamma dt\, f(e(t),t,\gamma-t) &= \int n(de)\int_0^\gamma dt\, E_{e(t)}(f(e(t),t,T_0))\\
&= \int n(de)\int_0^\gamma dt\int_0^\infty dt'\, q_{e(t)}(t')f(e(t),t,t')\\
&= \int_0^\infty dx\int_0^\infty dt\, q_x(t)\int_0^\infty dt'\, q_x(t')\, f(x,t,t').
\end{aligned}
$$

Let $p \ge 2$. Applying the Markov property under n successively at t_p and at t_{p-1}, and then using (2.2), we obtain

$$
\begin{aligned}
&n\Big(\int_{\{0\le t_1\le\cdots\le t_p\le\gamma\}} dt_1\ldots dt_p\\
&\quad\times f(m(t_1,t_2),\ldots,m(t_{p-1},t_p),e(t_1),\ldots,e(t_p),t_1,t_2-t_1,\ldots,\gamma-t_p)\Big)\\
&=n\Big(\int_{\{0\le t_1\le\cdots\le t_{p-1}\le\gamma\}} dt_1\ldots dt_{p-1}\, E_{e(t_{p-1})}\Big(\int_0^{T_0} dt\int_0^\infty ds\, q_{B_t}(s)\\
&\quad\times f(m(t_1,t_2),\ldots,m(t_{p-2},t_{p-1}),I_t,e(t_1),\ldots,e(t_{p-1}),B_t,t_1,\ldots,t_{p-1}-t_{p-2},t,s)\Big)\Big)\\
&=2n\Big(\int_{\{0\le t_1\le\cdots\le t_{p-1}\le\gamma\}} dt_1\ldots dt_{p-1}\int_0^{e(t_{p-1})} dy\int_y^\infty dz\int_0^\infty dt\int_0^\infty ds\, q_{e(t_{p-1})+z-2y}(t)q_z(s)\\
&\quad\times f(m(t_1,t_2),\ldots,m(t_{p-2},t_{p-1}),y,e(t_1),\ldots,e(t_{p-1}),z,t_1,\ldots,t_{p-1}-t_{p-2},t,s)\Big).
\end{aligned}
$$

It is then straightforward to complete the proof by induction on p. □

We can now state and prove the main result of this section.

Theorem 2.6.2 *The law of the tree $\theta(e,t_1,\ldots,t_p)$ under the probability measure*

$$p!\; 1_{\{0\le t_1\le\cdots\le t_p\le 1\}}dt_1\ldots dt_p\, n_{(1)}(de)$$

is

$$p!\; 2^{p+1}\, L(\theta)\, \exp\big(-2\, L(\theta)^2\big)\, \Lambda_p(d\theta).$$

Proof. We equip $\mathcal{T}_p$ with the obvious product topology. Let $F \in C_{b+}(\mathcal{T}_p)$ and let h be bounded, nonnegative and measurable on $\mathbb{R}_+$. By Proposition 2.6.1,

$$\int n(de)\, h(\gamma) \int_{\{0\le t_1\le\cdots\le t_p\le\gamma\}} dt_1\ldots dt_p\, F(\theta(e,t_1,\ldots,t_p))$$
$$= 2^{p-1}\int_0^\infty ds\, h(s) \int \Lambda_p(d\theta)\, q_{2L(\theta)}(s)\, F(\theta).$$

On the other hand, using the properties of the definition of the measures $n_{(s)}$, we have also

$$\int n(de)\, h(\gamma) \int_{\{0\le t_1\le\cdots\le t_p\le\gamma\}} dt_1\ldots dt_p\, F(\theta(e,t_1,\ldots,t_p))$$
$$= \frac{1}{2}(2\pi)^{-1/2}\int_0^\infty ds\, s^{-3/2}\, h(s) \int n_{(s)}(de) \int_{\{0\le t_1\le\cdots\le t_p\le s\}} dt_1\ldots dt_p\, F(\theta(e,t_1,\ldots,t_p)).$$

By comparing with the previous identity, we get for a.a. $s > 0$,

$$\int n_{(s)}(de) \int_{\{0\le t_1\le\cdots\le t_p\le s\}} dt_1\ldots dt_p\, F(\theta(e,t_1,\ldots,t_p))$$
$$= 2^{p+1}\int \Lambda_p(d\theta)\, L(\theta)\, \exp\big(-\frac{2L(\theta)^2}{s}\big)\, F(\theta).$$

Both sides of the previous equality are continuous functions of s (use the scaling property of $n_{(s)}$ for the left side). Thus the equality holds for every $s > 0$, and in particular for $s = 1$. This completes the proof. $\square$

Concluding remarks. If we pick $t_1,\ldots,t_p$ independently according to Lebesgue measure on $[0,1]$ we can consider the increasing rearrangement $t'_1 \le t'_2 \le \cdots \le t'_p$ of $t_1,\ldots,t_p$ and define $\theta(e,t_1,\ldots,t_p) = \theta(e,t'_1,\ldots,t'_p)$. We can also keep track of the initial ordering and consider the tree $\tilde\theta(e,t_1,\ldots,t_p)$ defined as the tree $\theta(e,t_1,\ldots,t_p)$ where leaves are labelled $1,\ldots,p$, the leaf corresponding to time t_i receiving the label i. (This labelling has nothing to do with the ordering of the tree.) Theorem 2.6.2 implies that the law of the tree $\tilde\theta(e,t_1,\ldots,t_p)$ under the probability measure

$$1_{[0,1]^p}(t_1,\ldots,t_p)dt_1\ldots dt_p\, n_{(1)}(de)$$

has density

$$2^{p+1}L(\theta)\, \exp(-2L(\theta)^2)$$

with respect to $\tilde\Lambda_p(d\theta)$, the uniform measure on the set of labelled marked trees.

We can then "forget" the ordering. Define $\theta^*(e,t_1,\ldots,t_p)$ as the tree $\tilde\theta(e,t_1,\ldots,t_p)$ without the order structure. Since there are 2^{p-1} possible orderings for a given labelled tree, we get that the law (under the same measure) of the tree $\theta^*(e,t_1,\ldots,t_p)$ has density

$$2^{2p}L(\theta)\, \exp(-2L(\theta)^2)$$

with respect to $\Lambda^*_p(d\theta)$, the uniform measure on the set of labelled marked unordered trees. For convenience, replace the excursion e by $2e$ (this simply means that all heights are multiplied by 2). We obtain that the law of the tree $\theta^*(2e, t_1, \ldots, t_p)$ has density

$$L(\theta)\ \exp(-\frac{L(\theta)^2}{2})$$

with respect to $\Lambda_p^*(d\theta)$. It is remarkable that the previous density (apparently) does not depend on p.

In the previous form, we recognize the finite-dimensional marginals of Aldous' continuum random tree [1],[2]. To give a more explicit description, the discrete skeleton $T^*(2e, t_1, \ldots, t_p)$ is distributed uniformly on the set of labelled rooted binary trees with p leaves. (This set has b_p elements, with $b_p = p!\, 2^{-(p-1)} a_p = 1 \times 3 \times \cdots \times (2p-3)$.) Then, conditionally on the discrete skeleton, the heights h_v are distributed with the density

$$b_p \,(\sum h_v)\ \exp\,(-\frac{(\sum h_v)^2}{2})$$

(verify that this is a probability density on $\mathbb{R}_+^{2p-1}$!).

Chapter 3

A brief introduction to superprocesses

3.1 Continuous-state branching processes

In this section we study the limiting behavior of a sequence of rescaled Galton-Watson branching processes, when the initial population tends to infinity whereas the mean lifetime goes to zero. This is the first step needed to understand the construction of superprocesses, which combine the branching phenomenon with a spatial motion.

We start from a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ of the following type:

$$\psi(u) = \alpha u + \beta u^2 + \int \pi(dr)(e^{-ru} - 1 + ru) \ ,$$

where $\alpha \geq 0$, $\beta \geq 0$ and π is a σ-finite measure on $(0, \infty)$ such that $\int \pi(dr)(r \wedge r^2) < \infty$. Note that the function ψ is nonnegative and Lipschitz on compact subsets of $\mathbb{R}_+$. These properties play an important role in what follows.

For every $\varepsilon \in (0,1)$, we then consider a Galton-Watson process in continuous time $X^\varepsilon = (X_t^\varepsilon, t \geq 0)$ where individuals die at rate $\rho_\varepsilon = \alpha_\varepsilon + \beta_\varepsilon + \gamma_\varepsilon$ (the parameters $\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon \geq 0$ will be determined below). When an individual dies, three possibilities may occur:

- with probability $\alpha_\varepsilon/\rho_\varepsilon$, the individual dies without descendants;
- with probability $\beta_\varepsilon/\rho_\varepsilon$, the individual gives rise to 0 or 2 children with probability 1/2;
- with probability $\gamma_\varepsilon/\rho_\varepsilon$, the individual gives rise to a random number of offsprings which is distributed as follows: Let V be a random variable distributed according to

  $$\pi_\varepsilon(dv) = \pi((\varepsilon, \infty))^{-1} 1_{\{v > \varepsilon\}} \pi(dv) \ ;$$

  then, conditionally on V, the number of offsprings is Poisson with parameter $m_\varepsilon V$, where $m_\varepsilon > 0$ is a parameter that will be fixed later.

In other words, the generating function of the branching distribution is:

$$\varphi_\varepsilon(r) = \frac{\alpha_\varepsilon}{\alpha_\varepsilon + \beta_\varepsilon + \gamma_\varepsilon} + \frac{\beta_\varepsilon}{\alpha_\varepsilon + \beta_\varepsilon + \gamma_\varepsilon}\Big(\frac{1 + r^2}{2}\Big) + \frac{\gamma_\varepsilon}{\alpha_\varepsilon + \beta_\varepsilon + \gamma_\varepsilon} \int \ pi_\varepsilon(dv) e^{-m_\varepsilon v(1-r)} \ .$$

We choose the parameters $\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon$ and m_ε so that

(i) $\lim_{\varepsilon\downarrow 0} m_\varepsilon = +\infty$.

(ii) If $\pi \neq 0$, $\lim_{\varepsilon\downarrow 0} m_\varepsilon\gamma_\varepsilon\pi((\varepsilon,\infty))^{-1} = 1$. If $\pi = 0$, $\gamma_\varepsilon = 0$.

(iii) $\lim_{\varepsilon\downarrow 0} \frac{1}{2}m_\varepsilon^{-1}\beta_\varepsilon = \beta$.

(iv) $\lim_{\varepsilon\downarrow 0}(\alpha_\varepsilon - m_\varepsilon\gamma_\varepsilon \int \pi_\varepsilon(dr)r + \gamma_\varepsilon) = \alpha$, and $\alpha_\varepsilon - m_\varepsilon\gamma_\varepsilon \int \pi_\varepsilon(dr)r + \gamma_r epsilon \geq 0$, for every $\varepsilon > 0$.

Obviously it is possible, in many different ways, to choose $\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon$ and m_ε such that these properties hold.

We set $g_\varepsilon(r) = \rho_\varepsilon(\varphi_\varepsilon(r) - r) = \alpha_\varepsilon(1-r) + \frac{\beta_\varepsilon}{2}(1-r)^2 + \gamma_\varepsilon(\int \pi_\varepsilon(dv)e^{-m_\varepsilon v(1-r)} - r)$. Write P_k for the probability measure under which X^ε starts at k. By standard results of the theory of branching processes (see e.g. Athreya-Ney [4]), we have for $r \in [0,1]$,

$$E_1[r^{X_t^\varepsilon}] = v_t^\varepsilon(r)$$

where

$$v_t^\varepsilon(r) = r + \int_0^t g_\varepsilon(v_s^\varepsilon(r))ds \; .$$

We are interested in scaling limits of the processes X^ε: We will start X^ε with $X_0^\varepsilon = [m_\varepsilon x]$, for some fixed $x > 0$, and study the behaviour of $m_\varepsilon^{-1}X_t^\varepsilon$. Thus we consider for $\lambda \geq 0$

$$E_{[m_\varepsilon x]}[e^{-\lambda m_\varepsilon^{-1}X_t^\varepsilon}] = v_t^\varepsilon(e^{-\lambda/m_\varepsilon})^{[m_\varepsilon x]} = \exp\Big([m_\varepsilon x]\log v_t^\varepsilon(e^{-\lambda/m_\varepsilon})\Big) \; . \tag{3.1}$$

This suggests to define $u_t^\varepsilon(\lambda) = m_\varepsilon(1 - v_t^\varepsilon(e^{-\lambda/m_\varepsilon}))$. The function u_t^ε solves the equation

$$u_t^\varepsilon(\lambda) + \int_0^t \psi_\varepsilon(u_s^\varepsilon(\lambda))ds = m_\varepsilon(1 - e^{-\lambda/m_\mathcal{V} ep}) \; , \tag{3.2}$$

where $\psi_\varepsilon(u) = m_\varepsilon\, g_\varepsilon(1 - m_\varepsilon^{-1}u)$.

¿From the previous formulas, we have

$$\begin{aligned}
\psi_\varepsilon(u) &= \alpha_\varepsilon u + m_\varepsilon^{-1}\beta_\varepsilon\frac{u^2}{2} + m_\varepsilon\gamma_\varepsilon\int \pi_\varepsilon(dr)(e^{-ru} - 1 + m_\varepsilon^{-1}u) \\
&= (\alpha_\varepsilon - m_\varepsilon\gamma_\varepsilon\int \pi_\varepsilon(dr)r + \gamma_\varepsilon)u + m_\varepsilon^{-1}\beta_\varepsilon\frac{u^2}{2} \\
&\quad + m_\varepsilon\gamma_\varepsilon\pi((\varepsilon,\infty))^{-1}\int_{(\varepsilon,\infty)} \pi(dr)(e^{-ru} - 1 + ru) \; .
\end{aligned}$$

Proposition 3.1.1 *Suppose that properties* (i) – (iv) *hold. Then, for every $t > 0$, $x \geq 0$, the law of $m_\varepsilon^{-1}X_t^\varepsilon$ under $P_{[m_\varepsilon x]}$ converges as $\varepsilon \to 0$ to a probability measure $P_t(x,dy)$. Furthermore, the kernels $(P_t(x,dy), t > 0, x \in \mathbb{R}_+)$ form a collection of transition kernels on $\mathbb{R}_+$, which satisfy the additivity property*

$$P_t(x + x', \cdot) = P_t(x,\cdot) * P_t(x',\cdot)$$

and are subcritical in the sense that $\int y\, P_t(x,dy) \le x$ for every $x \ge 0$. Finally these kernels are associated with the function ψ in the sense that, for every $\lambda \ge 0$,

$$\int P_t(x,dy) e^{-\lambda y} = e^{-x u_t(\lambda)} ,$$

where the function $(u_t(\lambda), t \ge 0, \lambda \ge 0)$ is the unique nonnegative solution of the integral equation

$$u_t(\lambda) + \int_0^t \psi(u_s(\lambda))\, ds = \lambda . \tag{3.3}$$

Proof. From (i) – (iv) we have

$$\lim_{\varepsilon \downarrow 0} \psi_\varepsilon(u) = \psi(u) \tag{3.4}$$

uniformly over compact subsets of $\mathbb{R}_+$. Let $u_t(\lambda)$ be the unique nonnegative solution of (3.3) ($u_t(\lambda)$ may be defined by: $\int_{u_t(\lambda)}^{\lambda} \psi(v)^{-1} dv = t$ when $\lambda > 0$; this definition makes sense because $\psi(v) \le Cv$ for $v \le 1$, so that $\int_{0+} \psi(v)^{-1} dv = +\infty$).

We then make the difference between (3.2) and (3.3), and use (3.4) and the fact that ψ is Lipschitz over $[0, \lambda]$ to obtain that for $t \in [0,T]$

$$|u_t(\lambda) - u_t^\varepsilon(\lambda)| \le C_\lambda \int_0^t |u_s(\lambda) - u_s^\varepsilon(\lambda)| ds + a_T(\varepsilon, \lambda)$$

where $a_T(\varepsilon,\lambda) \to 0$ as $\varepsilon \to 0$, and the constant C_λ is the Lipschitz constant for ψ on $[0,\lambda]$. We conclude from Gronwall's lemma that for every $\lambda \ge 0$,

$$\lim_{\varepsilon \downarrow 0} u_t^\varepsilon(\lambda) = u_t(\lambda)$$

uniformly on compact sets in t. Coming back to (3.1) we have

$$\lim_{\varepsilon \to 0} E_{[m_\varepsilon x]}[e^{-\lambda m_\varepsilon^{-1} X_t^\varepsilon}] = e^{-x u_t(\lambda)}$$

and the first assertion of the proposition follows from a classical statement about Laplace transforms.

The end of the proof is straightforward. The Chapman-Kolmogorov relation for $P_t(x,dy)$ follows from the identity $u_{t+s} = u_t \circ u_s$, which is easy from (3.3). The additivity property is immediate since

$$\int P_t(x+x', dy) e^{-\lambda y} = e^{-(x+x')u_t(\lambda)} = \Big(\int P_t(x,dy)e^{-\lambda y}\Big)\Big(\int P_t(x',dy)e^{-\lambda y}\Big) .$$

The property $\int P_t(x,dy) y \le x$ follows from the fact that $\limsup_{\lambda \to 0} \lambda^{-1} u_t(\lambda) \le 1$. Finally the kernels $P_t(x,dy)$ are associated with ψ by construction. $\square$

Definition. *The ψ-continuous state branching process (in short, the ψ-CSBP) is the Markov process in $\mathbb{R}_+$ $(X_t, t \ge 0)$ whose transition kernels $P_t(x,dy)$ are associated with the function ψ in the way explained in Proposition 3.1.1. The function ψ is called the branching mechanism of X.*

Examples.

(i) If $\psi(u) = \alpha u$, $X_t = X_0 e^{-\alpha t}$

(ii) If $\psi(u) = \beta u^2$ one can compute explicitely $u_t(\lambda) = \frac{\lambda}{1+\beta\lambda t}$. The corresponding process X is called the Feller diffusion, for reasons that are explained in the exercise below.

(iii) By taking $\alpha = \beta = 0$, $\pi(dr) = c\frac{dr}{r^{1+b}}$ with $1 < b < 2$, one gets $\psi(u) = c' u^b$. This is called the stable branching mechanism.

¿From the form of the Laplace functionals, it is very easy to see that the kernels $P_t(x, dy)$ satisfy the Feller property, as defined in [32] Chapter III (use the fact that linear combinations of functions $e^{-\lambda x}$ are dense in the space of continuous functions on $\mathbb{R}_+$ that tend to 0 at infinity). By standard results, every ψ-CSBP has a modification whose paths are right-continuous with left limits, and which is also strong Markov.

Exercise. Verify that the Feller diffusion can also be obtained as the solution to the stochastic differential equation

$$dX_t = \sqrt{2\beta X_t}\, dB_t$$

where B is a one-dimensional Brownian motion [*Hint*: Apply Itô's formula to see that

$$\exp\Big(-\frac{\lambda X_s}{1+\beta\lambda(t-s)}\Big)\,, \quad 0 \le s \le t$$

is a martingale.]

Exercise. (Almost sure extinction) Let X be a ψ-CSBP started at $x > 0$, and let $T = \inf\{t \ge 0,\ X_t = 0\}$. Verify that $X_t = 0$ for every $t \ge T$, a.s. (use the strong Markov property). Prove that $T < \infty$ a.s. if and only if

$$\int^\infty \frac{du}{\psi(u)} < \infty.$$

(This is true in particular for the Feller diffusion.) If this condition fails, then $T = \infty$ a.s.

3.2 Superprocesses

In this section we will combine the continuous-state branching processes of the previous section with spatial motion, in order to get the so-called superprocesses. The spatial motion will be given by a Markov process $(\xi_s, s \ge 0)$ with values in a Polish space E. We assume that the paths of ξ are càdlàg (right-continuous with left limits) and so ξ may be defined on the canonical Skorokhod space $\mathbb{D}(\mathbb{R}_+, E)$. We write Π_x for the law of ξ started at x. The mapping $x \longrightarrow \Pi_x$ is measurable by assumption. We denote by $\mathcal{B}_{b+}(E)$ (resp. $C_{b+}(E)$) the set of all bounded nonnegative measurable (resp. bounded nonnegative continuous) functions on E.

In the spirit of the previous section, we use an approximation by branching particle systems. Recall the notation ρ_ε, m_ε, φ_ε of the previous section. We suppose that, at time

$t = 0$, we have N_ε particles located respectively at points $x_1^\varepsilon, \ldots, x_{N_\varepsilon}^\varepsilon$ in E. These particles move independently in E according to the law of the spatial motion ξ. Each particle dies at rate ρ_ε and gives rise to a random number of offspring according to the distribution with generating function φ_ε. Let Z_t^ε be the random measure on E defined as the sum of the Dirac masses at the positions of the particles alive at t. Note that the total mass process $\langle Z_t^\varepsilon, 1\rangle$ has the same distribution as the process X_t^ε of the previous section, started at N_ε. (Here and later, we use the notation $\langle \nu, f\rangle = \int f\, d\nu$.) Our goal is to investigate the limiting behavior of $m_\varepsilon^{-1} Z_t^\varepsilon$, for a suitable choice of the initial distribution.

The process $(Z_t^\varepsilon, t \geq 0)$ is a Markov process with values in the set $M_p(E)$ of all point measures on E. We write $\mathbb{P}_\theta^\varepsilon$ for the probability measure under which Z^ε starts at θ.

Fix a Borel function f on E such that $c \leq f \leq 1$ for some $c > 0$. For every $x \in E$, $t \geq 0$ set

$$w_t^\varepsilon(x) = \mathbb{E}_{\delta_x}^\varepsilon(\exp\langle Z_t^\varepsilon, \log\, f\rangle).$$

Note that the quantity $\exp\langle Z_t^\varepsilon, \log\, f\rangle$ is the product of the values of f evaluated at the positions of the particles alive at t.

Proposition 3.2.1 *The function $w_t^\varepsilon(x)$ solves the integral equation*

$$w_t^\varepsilon(x) - \rho_\varepsilon \Pi_x\Big(\int_0^t ds(\varphi_\varepsilon(w_{t-s}^\varepsilon(\xi_s)) - w_{t-s}^\varepsilon(\xi_s))\Big) = \Pi_x(f(\xi_t))\ .$$

Proof. Since the parameter ε is fixed for the moment, we omit it, only in this proof. Note that we have for every positive integer n

$$\mathbb{E}_{n\delta_x}(\exp\langle Z_t, \log\, f\rangle) = w_t(x)^n\ .$$

Under $\mathbb{P}_{\delta_x}$ the system starts with one particle located at x. Denote by T the first branching time and by M the number of offspring of the initial particle. Let also $P_M(dm)$ be the law of M (the generating function of P_M is φ). Then

$$\begin{aligned}
w_t(x) &= \mathbb{E}_{\delta_x}(1_{\{T>t\}}\exp\langle Z_t, \log\, f\rangle) + \mathbb{E}_{\delta_x}(1_{\{T\leq t\}}\exp\langle Z_t, \log\, f\rangle) \\
&= e^{-\rho t}\Pi_x(f(\xi_t)) + \rho\, \Pi_x \otimes P_M\Big(\int_0^t ds\, e^{-\rho s}\mathbb{E}_{m\delta_{\xi_s}}(\exp\langle Z_{t-s}, \log\, f\rangle)\Big) \\
&= e^{-\rho t}\Pi_x(f(\xi_t)) + \rho\, \Pi_x\Big(\int_0^t ds\, e^{-\rho s}\varphi(w_{t-s}(\xi_s))\Big)\ ,
\end{aligned} \tag{3.5}$$

using the fact that $\mathbb{E}_{m\delta_{\xi_s}}(\exp\langle Z_{t-s}, \log\, f\rangle) = (\mathbb{E}_{\delta_{\xi_s}}(\exp\langle Z_{t-s}, \log\, f\rangle))^m$. The integral equation of the proposition is easily derived from this identity: From (3.5) we have

$$\begin{aligned}
&\rho\, \Pi_x\Big(\int_0^t ds\, w_{t-s}(\xi_s)\Big) \\
&= \rho\, \Pi_x\Big(\int_0^t ds\, e^{-\rho s}\Pi_{\xi_s}(f(\xi_{t-s}))\Big) + \rho^2\Pi_x\Big(\int_0^t ds\, \Pi_{\xi_s}\Big(\int_0^{t-s} dr\, e^{-\rho r}\varphi(w_{t-s-r}(\xi_r))\Big)\Big) \\
&= \rho\int_0^t ds\, e^{-\rho s}\Pi_x(f(\xi_t)) + \rho^2\int_0^t ds\int_s^t dr\, e^{-\rho(r-s)}\Pi_x\Big(\Pi_{\xi_s}(\varphi(w_{t-r}(\xi_{r-s})))\Big) \\
&= (1 - e^{-\rho t})\Pi_x(f(\xi_t)) + \rho\, \Pi_x\Big(\int_0^t dr(1 - e^{-\rho r})\varphi(w_{t-r}(\xi_r))\Big)\ .
\end{aligned}$$

By adding this equality to (3.5) we get the desired result. $\square$

We now fix a function $g \in \mathcal{B}_{b+}(E)$, then take $f = e^{-m_\varepsilon^{-1} g}$ in the definition of $w_t^\varepsilon(x)$ and set

$$u_t^\varepsilon(x) = m_\varepsilon(1 - w_t^\varepsilon(x)) = m_\varepsilon\Big(1 - \mathbb{E}^\varepsilon_{\delta_x}(e^{-m_\varepsilon^{-1}\langle Z_t^\varepsilon, g\rangle})\Big) \,.$$

¿From Proposition 3.2.1, it readily follows that

$$u_t^\varepsilon(x) + \Pi_x\Big(\int_0^t ds\, \psi_\varepsilon(u_{t-s}^\varepsilon(\xi_s))\Big) = m_\varepsilon \Pi_x(1 - e^{-m_\varepsilon^{-1} g(\xi_t)}) \tag{3.6}$$

where the function ψ_ε is as in Section 1.

Lemma 3.2.2 *The limit*

$$\lim_{\varepsilon\to 0} u_t^\varepsilon(x) =: u_t(x)$$

exists for every $t \geq 0$ and $x \in E$, and the convergence is uniform on the sets $[0,T]\times E$. Furthermore, $u_t(x)$ is the unique nonnegative solution of the integral equation

$$u_t(x) + \Pi_x\Big(\int_0^t ds\, \psi(u_{t-s}(\xi_s))\Big) = \Pi_x(g(\xi_t)) \,. \tag{3.7}$$

Proof. From our assumptions, $\psi_\varepsilon \geq 0$, and so it follows from (3.6) that $u_t^\varepsilon(x) \leq \lambda := \sup_{x\in E} g(x)$. Also note that

$$\lim_{\varepsilon\to 0} m_\varepsilon \Pi_x(1 - e^{-m_\varepsilon^{-1} g(\xi_t)}) = \Pi_x(g(\xi_t))$$

uniformly in $(t,x) \in \mathbb{R}_+ \times E$ (indeed the rate of convergence only depends on λ). Using the uniform convergence of ψ_ε towards ψ on $[0,\lambda]$ (and the Lipschitz property of ψ as in the proof of Proposition 3.1.1), we get for $\varepsilon > \varepsilon' > 0$ and $t \in [0,T]$,

$$|u_t^\varepsilon(x) - u_t^{\varepsilon'}(x)| \leq C_\lambda \int_0^t ds \sup_{y\in E} |u_s^\varepsilon(y) - u_s^{\varepsilon'}(y)| + b(\varepsilon, T, \lambda)$$

where $b(\varepsilon,T,\lambda) \to 0$ as $\varepsilon \to 0$. From Gronwall's lemma, we obtain that $u_t^\varepsilon(x)$ converges uniformly on the sets $[0,T]\times E$. Passing to the limit in (3.6) shows that the limit satisfies (3.7). Finally the uniqueness of the nonnegative solution of (3.7) is also a consequence of Gronwall's lemma. $\square$

We are now ready to state our basic construction theorem for superprocesses. We denote by $\mathcal{M}_f(E)$ the space of all finite measures on E, which is equipped with the topology of weak convergence.

Theorem 3.2.3 *For every $\mu \in \mathcal{M}_f(E)$ and every $t > 0$, there exists a (unique) probability measure $Q_t(\mu, d\nu)$ on $\mathcal{M}_f(E)$ such that for every $g \in \mathcal{B}_{b+}(E)$,*

$$\int Q_t(\mu, d\nu) e^{-\langle \nu, g\rangle} = e^{-\langle \mu, u_t\rangle} \tag{3.8}$$

where $(u_t(x), x \in E)$ is the unique nonnegative solution of (3.7). The collection $Q_t(\mu, d\nu)$, $t > 0$, $\mu \in \mathcal{M}_f(E)$ is a measurable family of transition kernels on $\mathcal{M}_f(E)$, which satisfies the additivity property

$$Q_t(\mu, \cdot) * Q_t(\mu', \cdot) = Q_t(\mu + \mu', \cdot) \ .$$

The Markov process Z in $\mathcal{M}_f(E)$ corresponding to the transition kernels $Q_t(\mu, d\nu)$ is called the (ξ, ψ)-*superprocess*. By specializing the key formula (3.8) to constant functions, one easily sees that the "total mass process" $\langle Z, 1\rangle$ is a ψ-CSBP. When $\psi(u) = \beta u^2$, we call Z the superprocess with spatial motion ξ and (quadratic) branching rate β. Finally, when ξ is Brownian motion in $\mathbb{R}^d$ and $\psi(u) = \beta u^2$ (quadratic branching mechanism), the process Z is called *super-Brownian motion*.

Proof. Consider the Markov process Z^ε in the case when its initial value Z_0^ε is distributed according to the law of the Poisson point measure on E with intensity $m_\varepsilon \mu$. By the exponential formula for Poisson measures, we have for $g \in \mathcal{B}_{b+}(E)$,

$$\begin{aligned} E[e^{-\langle m_\varepsilon^{-1} Z_t^\varepsilon, g\rangle}] &= E\Big[\exp(\int Z_0^\varepsilon(dx) \log \mathbb{E}^\varepsilon_{\delta_x}(e^{-\langle m_\varepsilon^{-1} Z_t^\varepsilon, g\rangle}))\Big] \\ &= \exp\Big(-m_\varepsilon \int \mu(dx)(1 - \mathbb{E}^\varepsilon_{\delta_x}(e^{-\langle m_\varepsilon^{-1} Z_t^\varepsilon, g\rangle}))\Big) \\ &= \exp(-\langle \mu, u_t^\varepsilon\rangle) \ . \end{aligned}$$

¿From Lemma 3.2.2, we get

$$\lim_{\varepsilon \downarrow 0} E(e^{-\langle m_\varepsilon^{-1} Z_t^\varepsilon, g\rangle}) = \exp(-\langle \mu, u_t\rangle) \ .$$

Furthermore we see from the proof of Lemma 3.2.2 that the convergence is uniform when g varies in the set $\{g \in \mathcal{B}_{b+}(E), 0 \leq g \leq \lambda\} =: \mathcal{H}_\lambda$.

Lemma 3.2.4 *Suppose that $R_n(d\nu)$ is a sequence of probability measures on $\mathcal{M}_f(E)$ such that, for every $g \in \mathcal{B}_{b+}(E)$,*

$$\lim_{n \to \infty} \int R_n(d\nu) e^{-\langle \nu, g\rangle} = L(g)$$

with a convergence uniform on the sets $\mathcal{H}_\lambda$. Then there exists a probability measure $R(d\nu)$ on $\mathcal{M}_f(E)$ such that

$$\int R(d\nu) e^{-\langle \nu, g\rangle} = L(g)$$

for every $g \in \mathcal{B}_{b+}(E)$.

We omit the proof of this lemma (which can be viewed as a generalization of the classical criterion involving Laplace transforms of measures on $\mathbb{R}_+$, see [26] for a detailed argument) and complete the proof of Theorem 3.2.3. The first assertion is a consequence of Lemma 3.2.4 and the beginning of the proof. The uniqueness of $Q_t(\mu, d\nu)$ follows from the fact that a probability measure $R(d\nu)$ on $\mathcal{M}_f(E)$ is determined by the quantities $\int R(d\nu) \exp(-\langle \nu, g\rangle)$

for $g \in \mathcal{B}_{b+}(E)$ (or even $g \in C_{b+}(E)$). To see this, use standard monotone class arguments to verify that the closure under bounded pointwise convergence of the subspace of $\mathcal{B}_b(\mathcal{M}_f(E))$ generated by the functions $\nu \to \exp(-\langle \nu, g \rangle)$, $g \in \mathcal{B}_{b+}(E)$, is $\mathcal{B}_b(\mathcal{M}_f(E))$.

For the sake of clarity, write $u_t^{(g)}$ for the solution of (3.7) corresponding to the function g. The Chapman–Kolmogorov equation $Q_{t+s} = Q_t Q_s$ follows from the identity

$$u_t^{(u_s^{(g)})} = u_{t+s}^{(g)} ,$$

which is easily checked from (3.7). The measurability of the family of kernels $Q_t(\mu, \cdot)$ is a consequence of (3.8) and the measurability of the functions $u_t(x)$. Finally, the additivity property follows from (3.8). □

We will write $Z = (Z_t, t \geq 0)$ for the (ξ, ψ)-superprocess whose existence follows from Theorem 3.2.3. For $\mu \in \mathcal{M}_f(E)$, we denote by $\mathbb{P}_\mu$ the probability measure under which Z starts at μ. From an intuitive point of view, the measure Z_t should be interpreted as uniformly spread over a cloud of infinitesimal particles moving independently in E according to the spatial motion ξ, and subject continuously to a branching phenomenon governed by ψ.

3.3 More Laplace functionals

In this section, we derive certain properties of superprocesses that will be used to make the connection with the different approach presented in the next chapters. Our first result gives the Laplace functional of finite-dimensional distributions of the process. To state this result, it is convenient to denote by $\Pi_{s,x}$ the probability measure under which the spatial motion ξ starts from x *at time* s. Under $\Pi_{s,x}$, ξ_t is only defined for $t \geq s$. We will make the convention that $\Pi_{s,x}(f(\xi_t)) = 0$ if $t < s$.

Proposition 3.3.1 *Let $0 \leq t_1 < t_2 < \cdots < t_p$ and let $f_1, \ldots, f_p \in \mathcal{B}_{b+}(E)$. Then, for any $\theta \in \mathcal{M}_f(E)$,*

$$\mathbb{E}_\theta(\exp - \sum_{i=1}^{p} \langle Z_{t_i}, f_i \rangle) = \exp(-\langle \theta, w_0 \rangle)$$

where the function $(w_t(x), t \geq 0, x \in E)$ is the unique nonnegative solution of the integral equation

$$w_t(x) + \Pi_{t,x}\Big(\int_t^\infty \psi(w_s(\xi_s))\, ds\Big) = \Pi_{t,x}(\sum_{i=1}^{p} f_i(\xi_{t_i})) \ . \tag{3.9}$$

Note that formula (3.9) and the previous convention imply that $w_t(x) = 0$ for $t > t_p$.

Proof. We argue by induction on p. When $p = 1$, (3.9) is merely a rewriting of (3.7): Let $u_t(x)$ be the solution of (3.7) with $\varphi = f_1$, then

$$w_t(x) = 1_{\{t \leq t_1\}} u_{t_1 - t}(x)$$

solves (3.9) and

$$\mathbb{E}_\theta(\exp -\langle Z_{t_1}, f_1\rangle) = \exp -\langle \theta, u_{t_1}\rangle = \exp -\langle \theta, w_0 gle\ .$$

Let $p \geq 2$ and assume that the result holds up to the order $p-1$. By the Markov property at time t_1,

$$\begin{aligned}\mathbb{E}_\theta(\exp -\sum_{i=1}^{p}\langle Z_{t_i}, f_i\rangle) &= \mathbb{E}_\theta\Big(\exp(-\langle Z_{t_1}, f_1\rangle)\mathbb{E}_{Z_{t_1}}(\exp -\sum_{i=2}^{p}\langle Z_{t_i-t_1}, f_i\rangle)\Big) \\ &= \mathbb{E}_\theta(\exp(-\langle Z_{t_1}, f_1\rangle - \langle Z_{t_1}, \tilde{w}_0\rangle)\end{aligned}$$

where $\tilde{w}$ solves

$$\tilde{w}_t(x) + \Pi_{t,x}\Big(\int_t^\infty \psi(\tilde{w}_s(\xi_s))\,ds\Big) = \Pi_{t,x}\Big(\sum_{i=2}^{p} f_i(\xi_{t_i-t_1} r)\Big)\ .$$

By the case $p=1$ we get

$$\mathbb{E}_\theta\Big(\exp -\sum_{i=1}^{p}\langle Z_{t_i}, f_i\rangle\Big) = \exp -\langle \theta, \bar{w}_0\rangle\ ,$$

with

$$\bar{w}_t(x) + \Pi_{t,x}\Big(\int_t^\infty \psi(\bar{w}_s(\xi_s))\,ds\Big) = \Pi_{t,x}(f_1(\xi_{t_1}) + \tilde{w}_0(\xi_{t_1}))\ .$$

We complete the induction by observing that the function $w_t(x)$ defined by

$$w_t(x) = 1_{\{t\leq t_1\}}\bar{w}_t(x) + 1_{\{t>t_1\}}\tilde{w}_{t-t_1}(x)$$

solves (3.9).

Finally the uniqueness of the nonnegative solution of (3.9) easily follows from Gronwall's lemma (note that any nonnegative solution w of (3.9) is automatically bounded and such that $w_t(x) = 0$ for $t > t_p$). □

Remark. The same proof shows that

$$\exp -\langle \theta, w_t\rangle = \mathbb{E}_{t,\theta}\Big(\exp -\sum_{i=1}^{p}\langle Z_{t_i}, f_i\rangle\Big) = \mathbb{E}_\theta\Big(\exp -\sum_{\substack{i=1\\ ptstyle t_i\geq t}}^{p}\langle Z_{t_i-t}, f_i\rangle\Big)\ .$$

¿From now on, we make the mild assumption that the process ξ_t is continuous in probability under Π_x, for every $x \in E$. Then it is a simple exercise to derive from the previous proposition that Z_t is also continuous in probability under $\mathbb{P}_\theta$, for every $\theta \in \mathcal{M}_f(E)$. As a consequence, we can replace Z by a measurable modification, and this allows us to make sense of integrals of the type

$$\int_0^\infty dt \int Z_t(dx)\,F(t,x)$$

for any nonnegative measurable function F on $\mathbb{R}_+ \times E$. Assume furthermore that F is bounded and continuous, and vanishes over $[T, \infty) \times E$ for some $T < \infty$. Then the continuity in probability of Z entails that the previous integral is the limit in probability, as $n \to \infty$, of the quantities

$$\frac{1}{n} \sum_{i=1}^{\infty} \langle Z_{i/n}, F(i/n, \cdot) \rangle.$$

By passing to the limit $n \to \infty$ in the Laplace transform of these quantities (Proposition 3.3.1) we arrive at the following result.

Proposition 3.3.2 *Let $F \in C_{b+}(\mathbb{R}_+ \times E)$ be such that $F(t, x) = 0$ for every $t \geq T$ and $x \in E$, for some $T < \infty$. Then,*

$$\mathbb{E}_\theta(\exp - \int_0^\infty dt \int Z_t(dx)\, F(t, x)) = \exp(-\langle \theta, w_0 \rangle)$$

where the function $(w_t(x), t \geq 0, x \in E)$ is the unique nonnegative solution of the integral equation

$$w_t(x) + \Pi_{t,x}\Big(\int_t^\infty \psi(w_s(\xi_s))\, ds \Big) = \Pi_{t,x}(\int_t^\infty F(s, \xi_s)\, ds) \; . \tag{3.10}$$

3.4 Moments in the quadratic branching case

In this section and the next one we concentrate on the quadratic case $\psi(u) = \beta u^2$. ¿From the expression of the Laplace functional of Z_t, one easily derives formulas for the moments of this random measure. For simplicity, we consider only first and second moments. We denote by Q_t the semigroup of the Markov process ξ: $Q_t f(x) = \Pi_x(f(\xi_t))$.

Proposition 3.4.1 *For any nonnegative measurable function f on E*

$$\mathbb{E}_\theta(\langle Z_t, f \rangle) = < \theta, Q_t f > = \int \theta(dx) E_x(f(\xi_t)), \tag{3.11}$$

and, for any nonnegative measurable function φ on $E \times E$,

$$\begin{aligned} \mathbb{E}_\theta\Big(\int\!\!\int Z_t(dy) Z_t(dy')\, \varphi(y, y') \Big) &= \int\!\!\int \theta Q_t(dy)\, \theta Q_t(dy')\, \varphi(y, y') \\ +2\beta \int_0^t ds \int \theta Q_s(dz) \int Q_{t-s}(z, dy) &\int Q_{t-s}(z, dy')\, \varphi(y, y'). \end{aligned} \tag{3.12}$$

Remark. The first moment formula provided the motivation for the name superprocess. The two terms in the second moment formula can be interpreted as follows in terms of the approximating branching particle system. The first term corresponds to the case when the particles located at y, y' at time t have different ancestors at time 0. The second term takes account of the case when the particles have the same ancestors up to time $s \in [0, t]$.

Proof. By (3.8) and (3.7), we have for $\lambda > 0$,

$$\mathbb{E}_\theta(\exp -\lambda\langle Z_t, f\rangle) = \exp -\langle \theta, v_t^\lambda\rangle,$$

where

$$v_t^\lambda(x) = \lambda Q_t f(x) - \beta \int_0^t Q_{t-s}((v_s^\lambda)^2)(x)\, ds.$$

In particular, $0 \le v_t^\lambda(x) \le \lambda Q_t f(x) \le \lambda \|f\|_\infty$, so that $v_t^\lambda(x) = \lambda Q_t f(x) + O(\lambda^2)$, and then

$$v_t^\lambda(x) = \lambda Q_t f(x) - \beta\lambda^2 \int_0^t Q_{t-s}((Q_s f)^2)(x)\, ds + O(\lambda^3). \tag{3.13}$$

It follows that

$$\begin{aligned}
&\mathbb{E}_\theta(\exp -\lambda\langle Z_t, f\rangle)\\
&= 1 - \lambda\langle\theta, Q_t f\rangle + \frac{\lambda^2}{2}\Big(\langle\theta, Q_t f\rangle^2 + 2\beta\int_0^t \langle\theta Q_{t-s}, (Q_s f)^2\rangle ds\Big) + o(\lambda^2).
\end{aligned}$$

The formula for $\mathbb{E}_\theta(\langle Z_t, f\rangle)$ follows immediately, as well as the formula (3.12) in the special case when $\varphi(y, y') = f(y)f(y')$. The general case follows by standard arguments. □

3.5 Sample path properties

In this last section, we consider super-Brownian motion, that is we asssume both that $\psi(u) = \beta u^2$ and that the underlying spatial motion ξ is Brownian motion in $\mathbb{R}^d$. Thus, $Q_t(x, dy) = q_t(x, y)dy$, where

$$q_t(x, y) = (2\pi t)^{-d/2} \exp(-\frac{|y - x|^2}{2t})$$

is the Brownian transition density. We fix $t > 0$ and our goal is to derive some information about the random measure Z_t. An important role will be played by the function

$$H_x(y, y') = \int_0^t ds \int dz\, q_{t-s}(z - x)\, q_s(y - z)\, q_s(y' - z), \qquad x, y, y' \in \mathbb{R}^d.$$

We start with the case $d = 1$.

Theorem 3.5.1 *If $d = 1$, there exists a process $(Y_t(y), y \in \mathbb{R})$ continuous in the L^2-norm, such that $Z_t(dy) = Y_t(y)\, dy$ a.s.*

Proof. For $a \in \mathbb{R}$ and $\varepsilon > 0$, set

$$Y^\varepsilon(a) = \frac{1}{2\varepsilon} Z_t((a - \varepsilon, a + \varepsilon))$$

By (3.12), we have

$$\begin{aligned}
\mathbb{E}_\theta(Y^\varepsilon(a)Y^{\varepsilon'}(a')) &= \frac{1}{4\varepsilon\varepsilon'}\mathbb{E}_\theta\Big(Z_t((a-\varepsilon,a+\varepsilon))Z_t((a'-\varepsilon',a'+\varepsilon'))\Big)\\
&= \frac{1}{4\varepsilon\varepsilon'}\Big(\theta Q_t((a-\varepsilon,a+\varepsilon))\theta Q_t((a'-\varepsilon',a'+\varepsilon'))\\
&\quad +2\beta\int_{a-\varepsilon}^{a+\varepsilon}dy\int_{a'-\varepsilon'}^{a'+\varepsilon'}dy'\int\theta(dx)\,H_x(y,y')\Big).
\end{aligned}$$

The measure θQ_t has density $q_t*\theta$, which is bounded and uniformly continuous over $\mathbb{R}$. Hence,

$$\frac{1}{4\varepsilon\varepsilon'}\theta Q_t((a-\varepsilon,a+\varepsilon))\theta Q_t((a'-\varepsilon',a'+\varepsilon'))\underset{\varepsilon,\varepsilon'\to 0}{\longrightarrow} q_t*\theta(a)q_t*\theta(a')$$

uniformly in $a,a'\in\mathbb{R}$. On the other hand, it is easily checked that if $d=1$, the function $(x,y,y')\to H_x(y,y')$ is uniformly continuous on $\mathbb{R}^3$. In consequence,

$$\frac{1}{4\varepsilon\varepsilon'}\int\theta(dx)\int_{a-\varepsilon}^{a+\varepsilon}dy\int_{a'-\varepsilon'}^{a'+\varepsilon'}dy'\,H_x(y,y')\underset{\varepsilon,\varepsilon'\to 0}{\longrightarrow}\int\theta(dx)\,H_x(a,a'),$$

again uniformly in a, a'. We conclude that $E_\theta(Y^\varepsilon(a)Y^{\varepsilon'}(a'))$ converges as ε,ε' go to 0. In particular, the collection $(Y^\varepsilon(a),\varepsilon>0)$ is Cauchy in L^2, and we can set

$$\bar{Y}_t(a)=\lim_{\varepsilon\to 0}Y^\varepsilon(a) \tag{3.14}$$

where the convergence holds in L^2, uniformly in a. We can then choose a sequence $\varepsilon_p\downarrow 0$ such that convergence (3.14) holds a.s. along this sequence, for every $a\in\mathbb{R}$. We then set, for every $a\in\mathbb{R}$,

$$Y_t(a)=\liminf_{p\to\infty}Y^{\varepsilon_p}(a),\qquad \forall a\in\mathbb{R},$$

in such a way that $Y_t(a)=\bar{Y}_t(a)$, p.s., and the process $(Y_t(a),a\in\mathbb{R})$ is measurable.

Then, if φ is continuous with compact support on $\mathbb{R}$,

$$\begin{aligned}
<Z_t,\varphi> &= \lim_{\varepsilon\downarrow 0}\int Z_t(dy)\,\frac{1}{2\varepsilon}\int_{y-\varepsilon}^{y+\varepsilon}da\,\varphi(a)\\
&= \lim_{\varepsilon\downarrow 0}\int da\,\varphi(a)\,Y^\varepsilon(a)\\
&= \int da\,\varphi(a)\,Z_t(a),
\end{aligned}$$

where the last convergence holds in $L^2(\mathbb{P}_\theta)$.

Finally,

$$E_\theta(Y_t(a)Y_t(a'))=\lim_{\varepsilon\to 0}\mathbb{E}_\theta(Y^\varepsilon(a)Y^\varepsilon(a'))=q_t*\theta(a)q_t*\theta(a')+2\beta\int\theta(dx)\,H_x(a,a'),$$

which implies

$$\begin{aligned}
E_\theta((Y_t(a) - Y_t(a'))^2) &= (q_t * \theta(a) - q_t * \theta(a'))^2 \\
&+ 2\beta \int \theta(dx)\,(H_x(a,a) - 2H_x(a,a') + H_x(a',a')),
\end{aligned}$$

and gives the desired continuity of $a \to Y_t(a)$. □

In higher dimensions, we can use similar techniques to get a lower bound on the Hausdorff dimension of the support of super-Brownian motion. We denote by $\operatorname{supp} Z_t$ the topological support of the measure Z_t.

Proposition 3.5.2 *If $d \geq 2$,*

$$\dim \operatorname{supp} Z_t \geq 2 \qquad a.s.\ on\ \{Z_t \neq 0\}.$$

Remark. As will be proved in the next chapter, the lower bound of the proposition is in fact an equality, and this implies that the measure Z_t is singular with respect to Lebesgue measure in dimension $d \geq 3$. The latter property also holds in dimension $d = 2$ but a finer argument is required.

Proof. Let $B_K = B(0,K)$ be the ball of radius K centered at 0 in $\mathbb{R}^d$. From Frostman's lemma (see e.g. [21]), it is enough to check that, for every $K > 0$, $\varepsilon > 0$,

$$\iint_{B_K^2} Z_t(dy)\, Z_t(dy')\, |y - y'|^{\varepsilon - 2} < \infty, \qquad \text{a.s.} \tag{3.15}$$

However, by (3.12),

$$\begin{aligned}
\mathbb{E}_\theta\left(\iint_{B_K^2} Z_t(dy)\, Z_t(dy')\, |y - y'|^{\varepsilon-2}\right) &= \iint_{B_K^2} dy\, dy'\, \theta * q_t(y)\, \theta * q_t(y')\, |y - y'|^{\varepsilon - 2} \\
&+ 2\beta\int \theta(dx) \iint_{B_K^2} dy\, dy'\, H_x(y,y')\, |y - y'|^{\varepsilon-2}.
\end{aligned}$$

Then, on the one hand, the function $\theta * q_t$ is bounded on $\mathbb{R}^d$, on the other hand, some simple estimates give

$$H_x(y,y') \leq \begin{cases} C\Big(1 + \log_+(1/|y-y'|)\Big) & \text{if } d = 2, \\ C(1 + |y - y'|^{2-d}) & \text{if } d \geq 3. \end{cases}$$

It follows that the expected value of the left side of (3.15) is finite, which completes the proof.

□

Chapter 4

The Brownian snake approach

4.1 Combining spatial motion with a branching structure

Throughout this chapter, we consider a Markov process (ξ_t, Π_x) with values in a Polish space E. We will make stronger continuity assumptions than in the previous chapter. Namely, we assume that ξ has continuous sample paths and that there exist an integer $p > 2$ and positive constants C and ε such that for every $x \in E$ and $t > 0$,

$$\Pi_x\Big(\sup_{0\le r\le t} \delta(x,\xi_r)^p\Big) \le C t^{2+\varepsilon}, \tag{4.1}$$

where δ denotes the distance on E. This assumption is not really necessary, but it will simplify our treatment. It holds for Brownian motion or for solutions of stochastic differential equations with smooth coefficients in $\mathbb{R}^d$ or on a manifold.

We denote by $\mathcal{W}$ the set of finite E-valued paths. An element of $\mathcal{W}$ is a continuous mapping $\mathrm{w} : [0,\zeta] \to E$, where $\zeta = \zeta_{(\mathrm{w})} \ge 0$ depends on w and is called the lifetime of w. The final point of w will be denoted by $\hat{\mathrm{w}} = \mathrm{w}(\zeta)$. If $x \in E$, the trivial path w such that $\zeta_{(\mathrm{w})} = 0$ and $\mathrm{w}(0) = x$ is identified with the point x, so that E is embedded in $\mathcal{W}$. The set $\mathcal{W}$ is a Polish space for the distance

$$d(\mathrm{w},\mathrm{w}') = |\zeta - \zeta'| + \sup_{t\ge 0} \delta(\mathrm{w}(t\wedge\zeta), \mathrm{w}'(t\wedge\zeta')).$$

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous mapping with $f(0) = 0$. We saw in Chapter 2 that f codes a continuous genealogical structure, where vertices are labelled by reals $s \ge 0$, and by construction the generation of the last common ancestor to s and s' is $\inf_{[s,s']} f(r)$. We can now combine this genealogical structure with spatial motions distributed according to the law of the process ξ.

Notation. Let $\mathrm{w} : [0,\zeta] \to E$ be an element of $\mathcal{W}$, let $a \in [0,\zeta]$ and $b \ge a$. We denote by $R_{a,b}(\mathrm{w}, d\mathrm{w}')$ the unique probability measure on $\mathcal{W}$ such that

(i) $\zeta' = b$, $R_{a,b}(\mathrm{w}, d\mathrm{w}')$ a.s.

(ii) $\mathrm{w}'(t) = \mathrm{w}(t)$, $\forall t \leq a$, $R_{a,b}(\mathrm{w}, d\mathrm{w}')$ a.s.

(iii) The law under $R_{a,b}(\mathrm{w}, d\mathrm{w}')$ of $(\mathrm{w}'(a+t), 0 \leq t \leq b-a)$ coincides with the law of $(\xi_t, 0 \leq t \leq b-a)$ under $\Pi_{\mathrm{w}(a)}$.

Under $R_{a,b}(\mathrm{w}, d\mathrm{w}')$, the path w' is the same as w up to time a and then behaves according to the spatial motion ξ up to time b.

Proposition 4.1.1 *Assume that f is Hölder continuous with exponent $\frac{1}{2}-\alpha$ for every $\alpha > 0$, and let $x \in E$. There exists a unique probability measure Θ_x^f on $C(\mathbb{R}_+, \mathcal{W})$ such that $W_0 = x$, Θ_x^f a.s., and, under Θ_x^f, the canonical process $(W_s, s \geq 0)$ is (time-inhomogeneous) Markov with transition kernels*

$$R_{m(s,s'),f(s')}(\mathrm{w}, d\mathrm{w}')$$

where $m(s,s') = \inf_{[s,s']} f(r)$.

The intuitive meaning of this construction should be clear if we think of s and s' as labelling two vertices in a continuous tree, which have the same ancestors up to level $m(s,s')$. The respective spatial motions W_s and $W_{s'}$ must be the same up to level $m(s,s')$ and then behave independently. Thus, the path $W_{s'}$ is obtained from the path W_s through the kernel $R_{m(s,s'),f(s')}(W_s, d\mathrm{w}')$.

Proof. For each choice of $0 \leq t_1 \leq t_2 \leq \cdots \leq t_p$, we can consider the probability measure $\pi^{x,f}_{t_1,\ldots,t_p}$ on $\mathcal{W}^p$ defined by

$$\pi^{x,f}_{t_1,\ldots,t_p}(d\mathrm{w}_1 \ldots d\mathrm{w}_p) = R_{0,f(t_1)}(x, d\mathrm{w}_1) R_{m(t_1,t_2),f(t_2)}(\mathrm{w}_1, d\mathrm{w}_2) \ldots R_{m(t_{p-1},t_p),f(t_p)}(\mathrm{w}_{p-1}, d\mathrm{w}_p).$$

It is easy to verify that this collection is consistent when p and $t_1, \ldots, t_p$ vary. Hence the Kolmogorov extension theorem yields the existence of process $(W_s, s \geq 0)$ with values in $\mathcal{W}$ (in fact in $\mathcal{W}_x := \{\mathrm{w} \in \mathcal{W} : \mathrm{w}(0) = x\}$) whose finite-dimensional marginals are the measures $\pi^{x,f}_{t_1,\ldots,t_p}$.

To complete the proof, we have to verify that $(W_s, s \geq 0)$ has a continuous modification. Thanks to the classical Kolmogorov lemma, it is enough to show that, for every $T > 0$ there are constants $\beta > 0$ and C such that

$$E[d(W_s, W_{s'})^p] \leq C|s-s'|^{1+\beta}, \tag{4.2}$$

for every $s \leq s' \leq T$.

By our construction, the joint distribution of $(W_s, W_{s'})$ is

$$R_{0,f(s)}(x, d\mathrm{w}) R_{m(s,s'),f(s')}(\mathrm{w}, d\mathrm{w}').$$

This means that W_s and $W_{s'}$ are two random paths that coincide up to time $m(s,s')$ and then behave independently according to the law of the process ξ. Using the definition of the distance d, we get for every $s, s' \in [0,T]$, $s \leq s'$,

$$\begin{aligned}
E(d(W_s, W_{s'})^p) &\leq c_p\Big(|f(s)-f(s')|^p + 2\,\Pi_x\Big(\Pi_{\xi_{m(s,s')}}\big(\sup_{0 \leq t \leq (f(s)\vee f(s'))-m(s,s')} \delta(\xi_0, \xi_t)^p\big)\Big)\Big) \\
&\leq c_p\Big(|f(s)-f(s')|^p + 2C|(f(s)\vee f(s')) - m(s,s')|^{2+\varepsilon}\Big) \\
&\leq c_p\Big(C^p_{\eta,T}|s-s'|^{p(\frac{1}{2}-\alpha)} + 2C\,C^{2+\varepsilon}_{\eta,T}|s-s'|^{(2+\varepsilon)(\frac{1}{2}-\alpha)}\Big)\,,
\end{aligned}$$

where we used assumption (4.1) in the second inequality. We can choose $\alpha > 0$ small enough so that $p(\frac{1}{2} - \alpha) > 1$ and $(2+\varepsilon)(\frac{1}{2} - \alpha) > 1$. The bound (4.2) then follows. $\square$

Remark. If we no longer assume that $f(0) = 0$, but consider instead an element $\mathrm{w}_0 \in \mathcal{W}$ with $\zeta_{(\mathrm{w}_0)} = f(0)$, a similar argument gives the existence of a probability measure $\Theta^f_{\mathrm{w}_0}$ on $C(\mathbb{R}_+, \mathcal{W})$ such that $W_0 = \mathrm{w}_0$, $\Theta^f_{\mathrm{w}_0}$ a.s. and the process W_s is (time-inhomogeneous) Markovian under $\Theta^f_{\mathrm{w}_0}$ with transition kernels $R_{m(s,s'),f(s')}(\mathrm{w}, d\mathrm{w}')$.

4.2 The Brownian snake

We now randomize f in the construction of the previous section. For every $r \geq 0$, we denote by $P_r(df)$ the law of reflected Brownian motion started at r (the law of $(|B_t|, t \geq 0)$ if B is a linear Brownian motion started at r). For every $s > 0$, we denote by $\rho^r_s(da\, db)$ the law under P_r of the pair

$$(\inf_{0 \leq u \leq s} f(u), f(s)).$$

The reflection principle easily gives the explicit form of $\rho^r_s(da\, db)$:

$$\begin{aligned}\rho^r_s(da\, db) &= \frac{2(x+b-2a)}{(2\pi s^3)^{1/2}} \exp -\frac{(r+b-2a)^2}{2s}\, 1_{(0<a<b\wedge r)}\, da\, db \\ &\quad + 2\,(2\pi s)^{-1/2} \exp -\frac{(r+b)^2}{2s}\, 1_{(0<b)} \delta_0(da)\, db.\end{aligned}$$

Theorem 4.2.1 *Let $\mathbb{P}_x$ be the probability measure on $C(\mathbb{R}_+, \mathbb{R}_+) \times C(\mathbb{R}_+, \mathcal{W})$ defined by*

$$\mathbb{P}_x(df\, d\omega) = P_0(df)\Theta^f_x(d\omega).$$

The canonical process $W_s(f, \omega) = \omega(s)$ is under $\mathbb{P}_x$ a continuous Markov process with values in $\mathcal{W}$, with initial value x and transition kernels

$$\mathbb{Q}_s(\mathrm{w}, d\mathrm{w}') = \int \rho^{\zeta_{(\mathrm{w})}}_s(dadb)\, R_{a,b}(\mathrm{w}, d\mathrm{w}').$$

This process is called the Brownian snake with spatial motion ξ.

Proof. The continuity is obvious and it is also clear that $W_0 = x$, $\mathbb{P}_x$ a.s. As for the Markov property, we write

$$\begin{aligned}&\mathbb{E}_x[F(W_{s_1}, \ldots, W_{s_p})G(W_{s_{p+1}})] \\ &= \int P_0(df) \int_{\mathcal{W}^{p+1}} R_{0,f(s_1)}(x, d\mathrm{w}_1) \ldots R_{m(s_p,s_{p+1}),f(s_{p+1})}(\mathrm{w}_p, d\mathrm{w}_{p+1})\, F(\mathrm{w}_1, \ldots, \mathrm{w}_p)G(\mathrm{w}_{p+1}) \\ &= \int_{\mathbb{R}_+^{2(p+1)}} \rho^0_{s_1}(da_1 db_1)\rho^{b_1}_{s_2-s_1}(da_2 db_2) \ldots \rho^{b_p}_{s_{p+1}-s_p}(da_{p+1} db_{p+1}) \\ &\qquad \int_{\mathcal{W}^{p+1}} R_{0,b_1}(x, d\mathrm{w}_1) \ldots R_{a_{p+1},b_{p+1}}(\mathrm{w}_p, d\mathrm{w}_{p+1})\, F(\mathrm{w}_1, \ldots, \mathrm{w}_p)G(\mathrm{w}_{p+1})\end{aligned}$$

$$= \int_{\mathbb{R}_+^{2(p+1)}} \rho^0_{s_1}(da_1db_1)\rho^{b_1}_{s_2-s_1}(da_2db_2)\dots\rho^{b_{p-1}}_{s_p-s_{p-1}}(da_pdb_p)$$

$$\int_{\mathcal{W}^p} R_{0,b_1}(x,d\mathrm{w}_1)\dots R_{a_p,b_p}(\mathrm{w}_{p-1},d\mathrm{w}_p)\,F(\mathrm{w}_1,\dots,\mathrm{w}_p)\mathbb{Q}_{s_{p+1}-s_p}G(\mathrm{w}_p)$$

$$= \mathbb{E}_x[F(W_{s_1},\dots,W_{s_p})\mathbb{Q}_{s_{p+1}-s_p}G(W_{s_p})]$$

□

Remark. For every $\mathrm{w}\in\mathcal{W}$, we can similarly define the law of the Brownian snake started at w by setting

$$\mathbb{P}_\mathrm{w}(df\,d\omega) = P_{\zeta_{(\mathrm{w})}}(df)\Theta^f_\mathrm{w}(d\omega).$$

We denote by $\zeta_s = \zeta_{(W_s)}$ the lifetime of W_s. Under $\mathbb{P}_x$ (resp. under $\mathbb{P}_\mathrm{w}$), the process $(\zeta_s, s\geq 0)$ is a reflected started at 0 (resp. at $\zeta_{(\mathrm{w})}$). This property is obvious since by construction $\zeta_s(f,\omega) = f(s)$, $\mathbb{P}_x$ a.s.

Excursion measures. For every $x\in E$, the excursion measure $\mathbb{N}_x$ is the σ-finite measure on $C(\mathbb{R}_+,\mathbb{R}_+)\times C(\mathbb{R}_+,\mathcal{W})$ defined by

$$\mathbb{N}_x(df\,d\omega) = n(df)\Theta^f_x(d\omega),$$

where $n(df)$ denotes Itô's excursion measure as in Chapter 2. As in Chapter 2, we will use the notation $\gamma=\gamma(f)$ for the duration of the excursion under $\mathbb{N}_x(df\,d\omega)$. Alternatively, the law of $(W_s, s\geq 0)$ under $\mathbb{N}_x$ is easily identified as the excursion measure from x of the Brownian snake.

We will see later that excursion measures play a major role in connections between the Brownian snake and superprocesses.

4.3 Finite-dimensional distributions under the excursion measure

In this section, we briefly derive a description of the finite-dimensional marginals of the Brownian snake, in terms of the marked trees that were introduced in the Chapter 2.

Let $\theta\in\mathcal{T}_p$ be a marked tree with p branches. We associate with θ a probability measure on $(\mathcal{W}_x)^p$ denoted by Π^θ_x, which is defined inductively as follows.

If $p=1$, then $\theta = (\{\emptyset\}, h)$ for some $h\geq 0$ and we let $\Pi^\theta_x = \Pi^h_x$ be the law of $(\xi_t, 0\leq t\leq h)$ under Π_x.

If $p\geq 2$, then we can write in a unique way

$$\theta = \theta' \underset{h}{*} \theta''\ ,$$

where $\theta'\in\mathcal{T}_j$, $\theta''\in\mathcal{T}_{p-j}$, and $j\in\{1,\dots,p-1\}$. We then define Π^θ_x by

$$\int \Pi^\theta_x(d\mathrm{w}_1,\dots,d\mathrm{w}_p)F(\mathrm{w}_1,\dots,\mathrm{w}_p) \;=\; \Pi_x\Big(\iint \Pi^{\theta'}_{\xi_h}(d\mathrm{w}'_1,\dots,d\mathrm{w}'_j)\Pi^{\theta''}_{\xi_h}(d\mathrm{w}''_1,\dots,d\mathrm{w}''_{p-j})$$
$$F(\xi_{[0,h]}\odot\mathrm{w}'_1,\dots,\xi_{[0,h]}\odot\mathrm{w}'_j,\xi_{[0,h]}\odot\mathrm{w}''_1,\dots,\xi_{[0,h]}\odot\mathrm{w}''_{p-j})\Big)$$

where $\xi_{[0,h]} \odot \mathrm{w}$ denotes the concatenation (defined in an obvious way) of the paths $(\xi_t, 0 \leq t \leq h)$ and $(\mathrm{w}(t), 0 \leq t \leq \zeta_{\mathrm{w}})$.

Informally, Π_x^θ is obtained by running independent copies of ξ along the branches of the tree θ.

Proposition 4.3.1 (i) *Let $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that $f(0) = 0$, and let $0 \leq t_1 \leq t_2 \leq \cdots \leq t_p$. Then the law under Θ_x^f of $(\omega(t_1), \ldots, \omega(t_p))$ is $\Pi_x^{\theta(f,t_1,\ldots,t_p)}$.*

(ii) *For any $F \in \mathcal{B}_+(\mathrm{w}_x^p)$,*

$$\mathbb{N}_x\Big(\int_{\{0\leq s_1\leq\cdots\leq s_p\leq\gamma\}} ds_1 \ldots ds_p\, F(W_{t_1}, \ldots, W_{t_p})\Big) = 2^{p-1} \int \Lambda_p(d\theta)\, \Pi_x^\theta(F)\,.$$

Proof. Assertion (i) follows easily from the definition of Θ_x^f and the construction of the trees $\theta(f, t_1, \ldots, t_p)$. A precise argument can be given using induction on p, but we leave details to the reader. To get (ii), we write

$$\begin{aligned}
&\mathbb{N}_x\Big(\int_{\{0\leq t_1\leq\cdots\leq t_p\leq\gamma\}} dt_1 \ldots dt_p\, F(W_{t_1}, \ldots, W_{t_p})\Big) \\
&= \int n(df) \int_{\{0\leq t_1\leq\cdots\leq t_p\leq\gamma\}} dt_1 \ldots dt_p\, \Theta_x^f\Big(F(W_{t_1}, \ldots, W_{t_p})\Big) \\
&= \int n(df) \int_{\{0\leq t_1\leq\cdots\leq t_p\leq\gamma\}} dt_1 \ldots dt_p\, \Pi_x^{\theta(f,t_1,\ldots,t_p)}(F) \\
&= 2^{p-1} \int \Lambda_p(d\theta)\, \Pi_x^\theta(F).
\end{aligned}$$

The first equality is the definition of $\mathbb{N}_x$, the second one is part (i) of the proposition, and the last one is Theorem 2.5.2. $\square$

The cases $p = 1$ and $p = 2$ of Proposition 4.3.1 (ii) are of special interest. Let us rewrite the corresponding formulas in a special case. Recall that we denote by $\hat{\mathrm{w}}$ the terminal point of w. For any $g \in \mathcal{B}_+(E)$, we have

$$\mathbb{N}_x\Big(\int_0^\gamma ds\, g(\widehat{W}_s)\Big) = \Pi_x\Big(\int_0^\infty dt\, g(\xi_t)\Big),$$

and

$$\mathbb{N}_x\Big(\Big(\int_0^\gamma ds\, g(\widehat{W}_s)\Big)^2\Big) = 4\, \Pi_x\Big(\int_0^\infty dt\, \Big(\Pi_{\xi_t}\Big(\int_0^\infty dr\, g(\xi_r)\Big)\Big)^2\Big).$$

These formulas are reminiscent of the moment formulas for superprocesses obtained in Chapter 3. We will see in the next section that this analogy is not a coincidence.

4.4 The connection with superprocesses

We start with a key technical result.

Proposition 4.4.1 *Let $g \in \mathcal{B}_{b+}(\mathbb{R}_+ \times E)$ such that $g(t,y) = 0$ for $t \geq A > 0$. Then the function*

$$u_t(x) = \mathbb{N}_x\Big(1 - \exp - \int_0^\gamma ds\, g(t + \zeta_s, \widehat{W}_s)\Big)$$

solves the integral equation

$$u_t(x) + 2\,\Pi_{t,x}\Big(\int_t^\infty dr (u_r(\xi_r))^2\Big) = \Pi_{t,x}\Big(\int_t^\infty dr\, g(r, \xi_r)\Big) \tag{4.3}$$

(recall that the process ξ starts from x at time t under the probability measure $\Pi_{t,x}$).

Proof. For every integer $p \geq 1$, set

$$T^p g(t,x) = \frac{1}{p!}\mathbb{N}_x\Big((\int_0^\gamma ds\, g(t + \zeta_s, \widehat{W}_s))^p\Big)\ .$$

By the case $p = 1$ of Proposition 4.3.1 (ii), we have

$$T^1 g(t,x) = \Pi_x\Big(\int_0^\infty dr\, g(t + r, \xi_r)\Big). \tag{4.4}$$

Then let $p \geq 2$. Using Proposition 4.3.1 (ii) again we have

$$\begin{aligned} T^p g(t,x) &= \mathbb{N}_x\Big(\int_{\{0 \leq s_1 \leq \cdots \leq s_p \leq \gamma\}} ds_1 \ldots ds_p \prod_{i=1}^p g(t + \zeta_{s_i}, \widehat{W}_{s_i})\Big) \\ &= 2^{p-1} \int \Lambda_p(d\theta) \int \Pi_x^\theta(dw_1 \ldots dw_p) \prod_{i=1}^p g(t + \zeta_{w_i}, \widehat{w}_i). \end{aligned}$$

At this stage, we decompose the tree θ at its first node, using the relation

$$\Lambda_p = \sum_{j=1}^{p-1} \int_0^\infty dh\, \Lambda_j \underset{h}{*} \Lambda_{p-j}$$

and the fact that if $\theta = \theta' \underset{h}{*} \theta''$, we can construct Π_x^θ by first considering a path of ξ started at x on the time interval $[0,h]$ and then concatenating paths $w'_1, \ldots, w'_j$ with distribution $\Pi_{\xi_h}^{\theta'}$ and paths $w''_1, \ldots, w''_{p-j}$ with distribution $\Pi_{\xi_h}^{\theta''}$. It follows that

$$\begin{aligned} T^p g(t,x) &= 2^{p-1} \sum_{j=1}^{p-1} \int_0^\infty dh \iint \Lambda_j(d\theta')\Lambda_{p-j}(d\theta'') \\ &\qquad \Pi_x\Big((\int \Pi_{\xi_h}^{\theta'}(dw'_1 \cdots dw'_j) \prod_{i=1}^j g(t + h + \zeta_{w'_i}, \hat{w}'_i)) \\ &\qquad \times (\int \Pi_{\xi_h}^{\theta''}(dw''_1 \cdots dw''_{p-j}) \prod_{i=1}^{p-j} g(t + h + \zeta_{w''_i}, \hat{w}''_i))\Big) \\ &= 2 \sum_{j=1}^{p-1} \Pi_x\Big(\int_0^\infty dh\, T^j g(t+h, \xi_h) T^{p-j} g(t+h, \xi_h)\Big). \end{aligned} \tag{4.5}$$

For $p = 1$, (4.4) gives the bound

$$T^1 g(t,x) \leq C 1_{[0,A]}(t) \; .$$

Recall from Chapter 2 the definition of the numbers a_p satisfying $a_p = \sum_{j=1}^{p-1} a_j a_{p-j}$. ¿From the bound for $p = 1$ and (4.5), we easily get $T^p g(t,x) \leq (2A)^{p-1} C^p a_p \, 1_{[0,A]}(t)$ by induction on p. Since $a_p \leq 4^p$, we obtain

$$T^p g(t,x) \leq (C')^p 1_{[0,A]}(t) \; .$$

It follows that, for $0 < \lambda < \lambda_0 := (C')^{-1}$,

$$\sum_{p=1}^{\infty} \lambda^p T^p g(t,x) \leq K \, 1_{[0,A]}(t), \tag{4.6}$$

for some constant $K < \infty$.

By expanding the exponential we get for $\lambda \in (0, \lambda_0)$

$$u_t^\lambda(x) := \mathbb{N}_x\Big(1 - \exp(-\lambda \int_0^\gamma ds \, g(t + \zeta_s, dehatW_s))\Big) = \sum_{p=1}^{\infty} (-1)^{p+1} \lambda^p T^p g(t,x) \; .$$

By (4.5), we have also

$$\begin{aligned} 2\Pi_x\Big(\int_0^\infty dr (u_{t+r}^\lambda(\xi_r))^2\Big) &= 2\Pi_x\Big(\int_0^\infty dr (\sum_{p=1}^{\infty} (-1)^{p+1} \lambda^p T^p g(t+r, \xi_r))^2\Big) \\ &= 2 \sum_{p=2}^{\infty} (-1)^p \lambda^p \sum_{j=1}^{p-1} \Pi_x\Big(\int_0^\infty dr T^j g(t+r, \xi_r) T^{p-j} g(t+r, \xi_r)\Big) \\ &= \sum_{p=2}^{\infty} (-1)^p \lambda^p T^p g(t,x) \; . \end{aligned}$$

(The use of Fubini's theorem in the second equality is justified thanks to (4.6).) ¿From the last equality and the previous formula for $u_t^\lambda(x)$, we get, for $\lambda \in (0, \lambda_0)$,

$$u_t^\lambda(x) + 2\Pi_x\Big(\int_0^\infty dr (u_{t+r}^\lambda(\xi_r))^2\Big) = \lambda T^1 g(t,x) = \lambda \Pi_x\Big(\int_0^\infty dr \, g(t+r, \xi_r)\Big) \; .$$

This is the desired integral equation, except that we want it for $\lambda = 1$. Note however that the function $\lambda \longrightarrow u_t^\lambda(x)$ is holomorphic on the domain $\{\mathrm{Re}\,\lambda > 0\}$. Thus, an easy argument of analytic continuation shows that if the previous equation holds for $\lambda \in (0, \lambda_0)$, it must hold for every $\lambda > 0$. This completes the proof. □

Theorem 4.4.2 *Let $\mu \in \mathcal{M}_f(E)$ and let*

$$\sum_{i \in I} \delta_{(x_i, f_i, \omega_i)}$$

be a Poisson point measure with intensity $\mu(dx)\mathbb{N}_x(df\,d\omega)$. Write $W^i_s = W_s(f_i,\omega_i)$, $\zeta^i_s = \zeta_s(f_i,\omega_i)$ and $\gamma_i = \gamma(f_i)$ for every $i \in I$, $s \geq 0$. Then there exists a superprocess $(Z_t, t \geq 0)$ with spatial motion ξ and quadratic branching rate 4, started at $Z_0 = \mu$, such that for every $\varphi \in \mathcal{B}_{b+}(\mathbb{R}_+ \times E)$,

$$\int_0^\infty dt \int Z_t(dx)\,\varphi(t,x) = \sum_{i\in I} \int_0^{\gamma_i} \varphi(\zeta^i_s, \widehat{W}^i_s)\,ds\;. \tag{4.7}$$

More precisely, Z_t can be defined for $t > 0$ by

$$\langle Z_t, g\rangle = \sum_{i\in I} \int_0^{\gamma_i} d\ell^t_s(\zeta^i) g(\widehat{W}^i_s)\;, \tag{4.8}$$

where $\ell^t_s(\zeta^i)$ denotes the local time at level t and at time s of $(\zeta^i_r, r \geq 0)$.

Remarks. (i) The local time $\ell^t_s(\zeta^i)$ can be defined via the usual approximation

$$\ell^t_s(\zeta^i) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^s dr\, 1_{(t,t+\varepsilon)}(\zeta^i_r).$$

The function $(\ell^t_s(\zeta^i), s \geq 0)$ is continuous and increasing, for every $i \in I$, a.s., and the notation $d\ell^t_s(\zeta^i)$ refers to integration with respect to this function.

(ii) The superprocess Z has branching rate 4, but a trivial modification will give a superprocess with an arbitrary branching rate. Simply observe that, for every $\lambda > 0$, the process $(\lambda Z_t, t \geq 0)$ is a superprocess with spatial motion ξ and branching rate 4λ.

Proof. Let $\mathcal{L}$ denote the random measure on $\mathbb{R}_+ \times E$ defined by

$$\int \mathcal{L}(dt\,dy)\varphi(t,y) = \sum_{i\in I}\int_0^{\gamma_i} \varphi(\zeta^i_s, \widehat{W}^i_s)ds\;.$$

Suppose that $\varphi(t,y) = 0$ for $t \geq A$, for some $A < \infty$. By the exponential formula for Poisson measures and then Proposition 4.4.1, we get

$$\begin{aligned} E(\exp - \int \mathcal{L}(dt\,dy)\varphi(t,y)) &= \exp\Big(-\int \mu(dx)\mathbb{N}_x(1 - \exp - \int_0^\gamma ds\,\varphi(\zeta_s, hatW_s))\Big) \\ &= \exp(-\langle \mu, u_0\rangle) \end{aligned}$$

where $(u_t(x), t \geq 0, x \in E)$ is the unique nonnegative solution of

$$u_t(x) + 2\Pi_{t,x}\Big(\int_t^\infty dr (u_r(\xi_r))^2\Big) = \Pi_{t,x}\Big(\int_t^\infty dr\,\varphi(r,\xi_r)\Big)\;.$$

By comparing with Proposition 3.3.2, we see that the random measure $\mathcal{L}$ has the same distribution as

$$dt\,Z'_t(dy)$$

where Z' is a superprocess with spatial motion ξ and branching rate 4, started at $Z'_0 = \mu$.

Since Z' is continuous in probability we easily obtain that, for every $t \geq 0$,

$$Z'_t = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} Z'_r dr \ ,$$

in probability. It follows that for every $t \geq 0$ the limit

$$Z_t(dy) := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathcal{L}(dr\, dy)$$

exists in probability. Clearly the process Z has the same distribution as Z' and is thus also a superprocess with spatial motion ξ and branching rate 4 started at μ.

Then, if $t > 0$ and $g \in C_{b+}(E)$,

$$\begin{aligned} langleZ_t, g\rangle &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int \mathcal{L}(dr\, dy) 1_{[t,t+\varepsilon]}(r) g(y) \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \sum_{i \in I} \int_0^{\gamma_i} ds\, 1_{[t,t+\varepsilon]}(\zeta^i_s) g(\widehat{W}^i_s) \ . \end{aligned}$$

Note that there is only a finite number of nonzero terms in the sum over $i \in I$ (for $t > 0$, $\mathbb{N}_x(\sup \zeta_s \geq t) = n(\sup e(s) \geq t) < \infty$). Furthermore, the usual approximation of Brownian local time, and the continuity of the mapping $s \to \widehat{W}_s$ give

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^{\gamma} ds\, 1_{[t,t+\varepsilon]}(\zeta_s) g(\widehat{W}_s) = \int_0^{\gamma} d\ell^t_s(\zeta) g(\widehat{W}_s)$$

$\mathbb{N}_x$ a.s., for every $x \in E$. Formula (4.8) now follows, and (4.7) is then a consequence of the occupation time formula for Brownian local time. □

Let us comment on the representation provided by Theorem 4.4.2. Define under $\mathbb{N}_x$ a measure-valued process $(\mathcal{Z}_t, t > 0)$ by the formula

$$\langle \mathcal{Z}_t, g\rangle = \int_0^{\gamma} d\ell^t_s(\zeta) g(\widehat{W}_s) \ . \tag{4.9}$$

The "law" of $(\mathcal{Z}_t, t > 0)$ under $\mathbb{N}_x$ is sometimes called the canonical measure (of the superprocess with spatial motion ξ and branching rate 4) with initial point x. Intuitively the canonical measure represents the contributions to the superprocess of the descendants of one single "individual" alive at time 0 at the point x. (This intuitive explanation could be made rigorous by an approximation by discrete branching particle systems in the spirit of Chapter 3.) The representation of the theorem can be written as

$$Z_t = \sum_{i \in I} \mathcal{Z}^i_t$$

and means (informally) that the population at time t is obtained by superimposing the contributions of the different individuals alive at time 0.

The canonical representation can be derived independently of the Brownian snake approach: Up to some point, it is a special case of the Lévy-Khintchine decomposition for infinitely divisible random measures. One advantage of the Brownian snake approach is that it gives the explicit formula (4.9) for the canonical measure.

4.5 Some applications of the Brownian snake representation

The representation derived in Theorem 4.4.2 has some interesting consequences for path properties of superprocesses.

Theorem 4.5.1 *Let Z be a superprocess with spatial motion ξ. Then $(Z_t, t \geq 0)$ has a continuous modification. For this modification,* $\operatorname{supp} Z_t$ *is compact for every $t > 0$ a.s.*

Proof. We may assume that the branching rate is 4 (cf remarks after Theorem 4.4.2) and that Z is given by the representation formula (4.8):

$$\langle Z_t, g\rangle = \sum_{i\in I} \int_0^{\gamma_i} d\ell_s^t(\zeta^i) g(\widehat{W}_s^i).$$

Note that the set $I_\delta = \{i \in I : \sup_{s\geq 0} \zeta_s^i \geq \delta\}$ is finite a.s., and that if $t \in [\delta, \infty)$ only the terms corresponding to $i \in I_\delta$ can be non-zero in the right-hand side of (4.8). Furthermore, the joint continuity of Brownian local times implies that the mapping $t \to d\ell_s^t(\zeta)$ is continuous from $\mathbb{R}_+$ into $\mathcal{M}_f(\mathbb{R}_+)$, $\mathbb{N}_x$ a.e. Since we also know that the mapping $s \to \widehat{W}_s^i$ is continuous, we immediately see that the process Z_t , as defined by (4.8), is continuous over $(0, \infty)$.

Before dealing with the continuity at $t = 0$, let us observe that the last assertion of the theorem is easy from the previous arguments. Indeed, for $t \in [\delta, \infty)$, we have

$$\operatorname{supp} Z_t \subset \bigcup_{i\in I_\delta} \{\widehat{W}_s^i : s \geq 0\}$$

where I_δ is finite a.s., and for every i the set $\{\widehat{W}_s^i, s \geq 0\}$ is compact as the image of the compact interval $[0, \gamma_i]$ under the continuous mapping $s \to \widehat{W}_s^i$.

To deal with the continuity at $t = 0$, let g be a bounded nonnegative Lipschitz function on E, and let $v_t(x)$ be the (nonnegative) solution of the integral equation

$$v_t(x) + 2\Pi_x\Big(\int_0^t v_{t-s}(\xi_s)^2 ds\Big) = \Pi_x(g(\xi_t)) \ . \tag{4.10}$$

Then for every fixed $t > 0$,

$$\exp -\langle Z_r, v_{t-r}\rangle = E(\exp -\langle Z_t, g\rangle | Z_r)$$

is a martingale indexed by $r \in [0, t]$. By standard results on martingales,

$$\lim_{r\downarrow 0} \langle Z_r, v_{t-r}\rangle$$

exists a.s., at least along rationals. On the other hand, it easily follows from (4.10) and assumption (4.1) that $v_t(x)$ converges to $g(x)$ as $t \downarrow 0$, uniformly in $x \in E$. Hence,

$$\limsup_{r\downarrow 0,\, r\in\mathbb{Q}} \langle Z_r, g\rangle \leq \limsup_{r\downarrow 0,\, r\in\mathbb{Q}} \langle Z_r, v_{t-r}\rangle + \varepsilon(t)$$

with $\varepsilon(t) \to 0$ as $t \downarrow 0$, and similarly for the lim inf. We conclude that $\lim_{r\downarrow 0}\langle Z_r, g\rangle$ exists a.s. and the limit must be $\langle \mu, g\rangle$ by the continuity in probability. □

Theorem 4.5.2 *Let Z be super-Brownian motion in $\mathbb{R}^d$. Then:*

(i) *For every $\delta > 0$,*

$$\dim\Big(\overline{\bigcup_{t\geq\delta} \mathrm{supp}\, Z_t}\Big) = 4 \wedge d, \qquad a.s.\ on\ \{Z_\delta \neq 0\}$$

(ii) *For every $t > 0$,*

$$\dim(\mathrm{supp}\, Z_t) = 2 \wedge d, \qquad a.s.\ on\ \{Z_t \neq 0\}$$

Proof. (i) We only prove the upper bound

$$\dim\Big(\overline{\bigcup_{t\geq\delta} \mathrm{supp}\, Z_t}\Big) \leq 4, \qquad \text{a.s.}$$

The lower bound is easily proved by an application of Frostman's lemma similar to the proof of Proposition 3.5.2.

We may again assume that Z is given by formula (4.8). Then, as in the previous proof, we have

$$\overline{\bigcup_{t\geq\delta} \mathrm{supp}\, Z_t} \subset \bigcup_{i\in I_\delta} \{\widehat{W}^i_s : s \geq 0\}.$$

The proof thus reduces to showing that

$$\dim \bigcup_{i\in I_\delta} \{\widehat{W}^i_s : s \geq 0\} \leq 4$$

or equivalently,

$$\dim\{\widehat{W}_s : s \geq 0\} \leq 4 \qquad \mathbb{N}_x \text{ a.e.} \tag{4.11}$$

To this end we use more detailed information about the Hölder continuity of the mapping $s \to \widehat{W}_s$. Note that when ξ is Brownian motion in $\mathbb{R}^d$, assumption (4.1) holds with any $p > 4$ and $2 + \varepsilon = p/2$. The argument of the proof of Proposition 4.1.1 shows that for any continuous $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $f(0) = 0$, for $s \leq s'$,

$$\Theta^f_x(|\widehat{W}_s - \widehat{W}_{s'}|^p) \leq c_p\Big(|f(s) - f(s')|^p + (f(s) + f(s') - \inf_{r\in[s,s']} f(r))^{p/2}\Big).$$

If f is Hölder continuous with exponent $1/2 - \varepsilon$ (which holds $n(df)$ a.e.), we get

$$\Theta^f_x(|\widehat{W}_s - \widehat{W}_{s'}|^p) \leq C_{p,f,\varepsilon}|s - s'|^{\frac{p}{2}(\frac{1}{2}-\varepsilon)}.$$

Then the Kolmogorov lemma implies that the mapping $s \to \widehat{W}_s$ is Hölder continuous with exponent $\frac{1}{4} - \varepsilon$, for every $\varepsilon > 0$, $\mathbb{N}_x$ a.e. The bound (4.11) is an immediate consequence.

(ii) The lower bound $\dim(\mathrm{supp}\, Z_t) \geq 2 \wedge d$ on $\{Z_t \neq 0\}$ was already obtained in Chapter 3 (Proposition 3.5.2). For the upper bound, we note that by formula (4.8), we have for $t \geq \delta$,

$$\mathrm{supp}\, Z_t \subset \bigcup_{i\in I_\delta} \{\widehat{W}^i_s : s \geq 0, \zeta^i_s = t\}$$

(recall that the local time measure $d\ell^t_s(\zeta^i)$ is supported on $\{s \geq 0 : \zeta^i_s = t\}$). Hence we need to verify that

$$\dim \{\widehat{W}_s : s \geq 0, \zeta_s = t\} \leq 2 \qquad \mathbb{N}_x \text{ a.e.} \tag{4.12}$$

By the first part of the proof, the mapping $s \to \widehat{W}_s$ is Hölder continuous with exponent $\frac{1}{4} - \varepsilon$, and on the other hand, it is well-known that the level sets of Brownian motion have dimension 1/2:

$$\dim \{s \geq 0 : \zeta_s = t\} \leq \frac{1}{2}, \qquad n(d\zeta) \text{ a.e.}$$

The bound (4.12) follows, which completes the proof. □

The Brownian snake has turned out to be a powerful tool for investigating path properties of super-Brownian motion. See e.g. [29] for a typical example of such applications.

4.6 Integrated super-Brownian excursion

In this last section, we discuss the random measure known as integrated super-Brownian excursion (ISE). The motivation for studying this random measure comes from limit theorems showing that ISE arises in the asymptotic behavior of certain models of statistical mechanics.

We suppose that the spatial motion ξ is Brownian motion in $\mathbb{R}^d$. We use the notation $\mathcal{J}$ for the total occupation measure of the Brownian snake under $\mathbb{N}_x$:

$$\langle \mathcal{J}, g \rangle = \int_0^\gamma ds\, g(\widehat{W}_s)\ , \qquad g \in \mathcal{B}_+(\mathbb{R}^d).$$

Informally, ISE is $\mathcal{J}$ under $\mathbb{N}_0(\cdot \mid \gamma = 1)$.

To give a cleaner definition, recall the notation $n_{(1)}$ for the law of the normalized Brownian excursion (cf Chapter 2). With the notation of Proposition 4.1.1, define a probability measure $\mathbb{N}^{(1)}_x$ on $C(\mathbb{R}_+, \mathbb{R}_+) \times C(\mathbb{R}_+, \mathcal{W}_x)$ by setting

$$\mathbb{N}^{(1)}_x(df\, d\omega) = n_{(1)}(df)\, \Theta^f_x(d\omega).$$

Definition. *ISE is the random measure $\mathcal{J}$ on $\mathbb{R}^d$ defined under $\mathbb{N}^{(1)}_0$ by*

$$\langle \mathcal{J}, g \rangle = \int_0^1 ds\, g(\widehat{W}_s)\ , \qquad g \in \mathcal{B}_+(\mathbb{R}^d).$$

¿From Theorem 4.5.2 (i) and a scaling argument, it is straightforward to verify that $\dim \operatorname{supp} \mathcal{J} = 4 \wedge d$ a.s.

One can use Theorem 2.5.2 to get an explicit formula for the moments of ISE. These moment formulas are important in the proof of the limit theorems involving ISE: See Derbez and Slade [11].

Before stating the result, recall the notation Π^θ_x introduced in Section 3 above. We use the tree formalism described in Section 2.3. In particular, a tree T is defined as the set of its vertices, which are elements of $\cup_{n=0}^\infty \{1,2\}^n$. We denote by L_T the set of all leaves of T and if $v \in T$, $v \neq \emptyset$, we denote by $\tilde{v}$ the father of v.

Proposition 4.6.1 *Let $p \geq 1$ be an integer and let $F \in \mathcal{B}_+(\mathcal{W}^p)$. Then,*

$$\begin{aligned}
&\mathbb{N}_0^{(1)}\Big(\int_{0\leq s_1\leq s_2\leq\cdots\leq s_p\leq 1\}} ds_1 \dots ds_p\, F(W_{s_1},\dots,W_{s_p})\Big)\\
&= 2^{p+1}\int \Lambda_p(d\theta)\, L(\theta)\, \exp(-2L(\theta)^2)\, \Pi_0^\theta(F(w_1,\dots,w_p)). \qquad (4.13)
\end{aligned}$$

Let $g \in \mathcal{B}_+(\mathbb{R}^d)$. Then,

$$\begin{aligned}
\mathbb{N}_0^{(1)}(\langle \mathcal{J}, g\rangle^p) &= p!\, 2^{p+1}\sum_{T\in\mathbb{T}_p}\int_{(\mathbb{R}_+)^T}\prod_{v\in T} dh_v\, (\sum_{v\in T} h_v)\exp\big(-2(\sum_{v\in T} h_v)^2\big)\\
&\quad\times\int_{(\mathbb{R}^d)^T}\prod_{v\in T} dy_v\Big(\prod_{v\in T} q_{h_v}(y_{\tilde v}, y_v)\Big)\prod_{v\in L_T} g(y_v), \qquad (4.14)
\end{aligned}$$

where $y_{\tilde v} = 0$ if $v = \emptyset$ by convention, and $q_t(y,y')$ denotes the Brownian transition density.

Proof. Formula (4.13) is an immediate consequence of Theorem 2.5.2 and Proposition 4.3.1 (i), along the lines of the proof of Proposition 4.3.1 (ii). Formula (4.14) follows as a special case (taking $F(w_1,\dots,w_p) = g(\hat w_1)\dots g(\hat w_p))$ using the construction of Π_0^θ and the definition of $\Lambda_p(d\theta)$. □

Formula (4.13) obviously contains more information than (4.14). For instance, it yields as easily the moment functionals for space-time ISE, which is the random measure $\mathcal{J}^*$ on $\mathbb{R}_+\times\mathbb{R}^d$ defined under $\mathbb{N}_0^{(1)}$ as

$$\langle \mathcal{J}^*, g\rangle = \int_0^1 ds\, g(\zeta_s, \widehat{W}_s)\ , \qquad g \in \mathcal{B}_+(\mathbb{R}_+\times\mathbb{R}^d).$$

The formula analogous to (4.14) when $\mathcal{J}$ is replaced by $\mathcal{J}^*$ is left as an exercise for the reader.

Remark. The second formula of Proposition 4.6.1 can be rewritten in several equivalent ways. Arguing as in the concluding remarks of Chapter 2, we may replace the sum over ordered binary trees with p leaves by a sum over (unordered) binary trees with p labelled leaves. The formula is unchanged, except that the factor $p!\,2^{p+1}$ is replaced by 2^{2p}. In this way, we (almost) get the usual form of the moment functionals of ISE: See Aldous [3] or Derbez and Slade [11]. There are still some extra factors 2 due to the fact that in the usual definition of ISE, $n_{(1)}(df)$ is replaced by its image under the mapping $f \to 2f$. To recover exactly the usual formula, simply replace $p_{h_v}(y_{\tilde v}, y_v)$ by $p_{2h_v}(y_{\tilde v}, y_v)$.

Chapter 5

Lévy processes and branching processes

Our goal in this chapter is to extend the Brownian snake approach of Chapter 4 to superprocesses with a general branching mechanism. This extension will rely on properties of spectrally positive Lévy processes. Most of the properties of Lévy processes that we will need can be found in Bertoin's monograph [5], especially in Chapter VII.

5.1 Lévy processes

In this section we introduce the class of Lévy processes that will be relevant to our purposes and we record some of their properties.

We start from a function ψ of the type considered in Chapter 3:

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} \pi(dr)(e^{-\lambda r} - 1 + \lambda r)$$

where $\alpha \geq 0$, $\beta \geq 0$ and π is a σ-finite measure on $(0, \infty)$ such that $\int \pi(dr)(r \wedge r^2) < \infty$.

Then there exists a Lévy process (real-valued process with stationary independent increments) $Y = (Y_t, t \geq 0)$ started at $Y_0 = 0$, whose Laplace exponent is ψ, in the sense that for every $t \geq 0$, $\lambda \geq 0$:

$$E[e^{-\lambda Y_t}] = e^{t\psi(\lambda)} .$$

The measure π is the Lévy measure of Y, β corresponds to its Brownian part, and $-\alpha$ to a drift coefficient (after compensation of the jumps). Since π is supported on $(0, \infty)$, Y has no negative jumps. In fact, under our assumptions, Y can be the most general Lévy process without negative jumps that does not drift to $+\infty$ (i.e. we cannot have $Y_t \to \infty$ as $t \to \infty$, a.s.). This corresponds to the fact that we consider only critical or subcritical branching. The point 0 is always regular for $(-\infty, 0)$, with respect to Y, meaning that

$$P(\inf\{t > 0, Y_t < 0\} = 0) = 1 .$$

It is not always true that 0 is regular for $(0,\infty)$, but this holds if

$$\beta > 0\ ,\ \text{or}\ \beta = 0 \text{ and } \int_0^1 r\pi(dr) = \infty\ . \tag{5.1}$$

¿From now on we will assume that (5.1) holds. This is equivalent to the property that the paths of Y are a.s. of infinite variation. A parallel theory can be developed in the finite variation case, but the cases of interest in relation with superprocesses (in particular the stable case where $\pi(dr) = cr^{-2-\alpha}dr$, $0 < \alpha < 1$) do satisfy (5.1).

Consider the maximum and minimum processes of Y:

$$S_t = \sup_{s\leq t} Y_s\ ,\quad I_t = \inf_{s\leq t} Y_s\ .$$

Both $S-Y$ and $Y-I$ are Markov processes in $\mathbb{R}_+$ (this is true indeed for any Lévy process). ¿From the previous remarks on the regularity of 0, it immediately follows that 0 is a regular point (for itself) with respect to both $S-Y$ and $Y-I$. We can therefore consider the (Markov) local time of both $S-Y$ and $Y-I$ at level 0.

It is easy to see that the process $-I$ provides a local time at 0 for $Y-I$. We will denote by N the associated excursion measure. By abuse of notation, we still denote by Y the canonical process under N. Under N, Y takes nonnegative values and $Y_t > 0$ if and only if $0 < t < \gamma$, where γ denotes the duration of the excursion.

We denote by $L = (L_t, t \geq 0)$ the local time at 0 of $S-Y$. Here we need to specify the normalization of L. This can be done by the following approximation:

$$L_t = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^t 1_{\{S_s - Y_s < \varepsilon\}} ds\ , \tag{5.2}$$

in probability. If $L^{-1}(t) = \inf\{s, L_s > t\}$ denotes the right-continuous inverse of L, formula (5.2) follows from the slightly more precise result

$$\lim_{\varepsilon\to 0} E\Big[\Big(\frac{1}{\varepsilon}\int_0^{L^{-1}(t)} 1_{\{S_s - Y_s<\varepsilon\}} ds - (t\wedge L_\infty)\Big)^2\Big] = 0 \tag{5.3}$$

which can be derived from excursion theory for $S-Y$ (after choosing the proper normalization for L, see [19]).

The process $(S_{L^{-1}(t)}, t\geq 0)$ is a subordinator (that is, a Lévy process with nondecreasing paths) and a famous formula of fluctuation theory (cf Theorem VII-4 (ii) in [5]) gives its Laplace transform

$$E(\exp -\lambda S_{L^{-1}(t)}) = \exp\big(-t\frac{\psi(\lambda)}{\lambda}\big)\ . \tag{5.4}$$

Note that there should be a normalization constant in the exponential of the right side, but this constant is equal to 1 under our normalization of local time. We have

$$\frac{\psi(\lambda)}{\lambda} = \alpha + \beta\lambda + \int_0^\infty dr\,\pi((r,\infty))(1-e^{-\lambda r})$$

so that the subordinator $(S_{L^{-1}(t)}, t \geq 0)$ has Lévy measure $\pi((r,\infty))dr$, drift β and is killed at rate α. In particular for every $s \geq 0$, if m denotes Lebesgue measure on $\mathbb{R}_+$, we have a.s.

$$m(\{S_{L^{-1}(r)}; 0 \leq r \leq s, L^{-1}(r) < \infty\}) = \beta(s \wedge L_\infty)$$

from which it easily follows that

$$m(\{S_r, 0 \leq r \leq t\}) = \beta L_t \ . \tag{5.5}$$

Note that when $\beta > 0$ this formula yields an explicit expression for L_t.

5.2 The height process

Recall formula (1.1) in Lemma 1.1.1, which gives an expression for the height process of a sequence of independent Galton-Watson trees with an arbitrary offspring distribution (this formula does not depend on the particular assumptions made on the offspring distribution in Chapter 1). If we formally try to extend this formula to our continuous setting, replacing the random walk S by the Lévy process Y, we are lead to define H_t as the Lebesgue measure of the set $\{s \leq t, Y_s = I_t^s\}$, where

$$I_t^s = \inf_{s \leq r \leq t} Y_r \ .$$

Under our assumptions however, this Lebesgue measure is always zero (if $s < t$, we have $P(Y_s = I_t^s) = 0$ because 0 is regular for $(-\infty, 0)$) and so we need to use some kind of local time that will measure the size of the set in consideration. More precisely, for a fixed $t > 0$, we introduce the time-reversed process

$$\hat{Y}_r^{(t)} = Y_t - Y_{(t-r)-} \ , \quad 0 \leq r \leq t \qquad (Y_{0-} = 0 \text{ by convention})$$

and its supremum process

$$\hat{S}_r^{(t)} = \sup_{0 \leq s \leq r} \hat{Y}_s^{(t)} \ , \quad 0 \leq r \leq t \ .$$

Note that $(\hat{Y}_r^{(t)}, \hat{S}_r^{(t)}; 0 \leq r \leq t)$ has the same distribution as $(Y_r, S_r; 0 \leq r \leq t)$. Via time-reversal, the set $\{s \leq t, Y_s = I_t^s\}$ corresponds to the set $\{s \leq t, \hat{S}_s^{(t)} = \hat{Y}_s^{(t)}\}$. This leads us to the following definition.

Definition. *For every $t \geq 0$, we let H_t be the local time at 0, at time t, of the process $\hat{S}^{(t)} - \hat{Y}^{(t)}$. The process $(H_t, t \geq 0)$ is called the height process.*

Obviously, the normalization of local time is the one that was described in the previous section. From this definition it is not clear that the sample paths of $(H_t, t \geq 0)$ have any regularity property. In this work, in order to avoid technical difficulties, we will reinforce assumption (5.1) by imposing that

$$\beta > 0 \ . \tag{5.6}$$

We emphasize that (5.6) is only for technical convenience and that all theorems and propositions that follow hold under (5.1) (for a suitable choice of a modification of $(H_t, t \geq 0)$).

Under (5.6) we can get a simpler expression for H_t. Indeed from (5.5) we have

$$H_t = \frac{1}{\beta} m(\{\hat{S}_r^{(t)}, 0 \le r \le t\}) \,,$$

or equivalently,

$$H_t = \frac{1}{\beta} m(\{I_t^r, 0 \le r \le t\}) \,. \tag{5.7}$$

The right side of the previous formula obviously gives a continuous modification of H (recall that Y has no negative jumps). From now on we deal only with this modification.

If $\psi(u) = \beta u^2$, Y is a (scaled) linear Brownian motion and has continuous paths. The previous formula then implies that $H_t = \frac{1}{\beta}(Y_t - I_t)$ is a (scaled) reflected Brownian motion, by a famous theorem of Lévy.

We can now state our main results. The key underlying idea is that H codes the genealogy of a ψ-CSBP in the same way as reflected Brownian motion codes the genealogy of the Feller diffusion. Our first theorem shows that the local time process of H (evaluated at a suitable stopping time), as a function of the space variable, is a ψ-CSBP.

Theorem 5.2.1 *For every $r > 0$, set $\tau_r = \inf\{t, I_t = -r\}$. There exists a ψ-CSBP $X = (X_a, a \ge 0)$ started at r, such that for every $h \in \mathcal{B}_{b+}(\mathbb{R}_+)$,*

$$\int_0^\infty da\, h(a) X_a = \int_0^{\tau_r} ds\, h(H_s) \,.$$

Obviously X can be defined by

$$X_a = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^{\tau_r} ds\, 1_{\{a < H_s < a+\varepsilon\}}, \text{ a.s.}$$

Remark. It is easy to verify that a.s. for every $t \ge 0$, $H_t = 0$ iff $Y_t = I_t$. (The implication $Y_t = I_t \Longrightarrow H_t = 0$ is trivial.) Since τ_r is the inverse local time at 0 of $Y - I$, we can also interpret τ_r as the inverse local time at 0 of H. Indeed, Theorem 5.2.1 implies

$$r = X_0 = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^{\tau_r} ds\, 1_{\{0 < H_s < \varepsilon\}} \,, \text{ a.s.}$$

from which it easily follows that for every $t \ge 0$

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t ds\, 1_{\{0 < H_s < \varepsilon\}} = -I_t \,, \text{ a.s.}$$

Using this remark, we see that the case $\psi(u) = \beta\, u^2$ of the previous theorem reduces to a classical Ray-Knight theorem on the Markovian properties of Brownian local times.

We now turn to a snake-like construction of (ξ, ψ)-superprocesses, that generalizes the construction of Chapter 4. As in Chapter 4 we consider a Markov process ξ with values in a Polish space E, satisfying the assumptions in Section 4.1. Also recall that N stands for the excursion measure of $Y - I$ away from 0. The definition of the process $(H_s, s \geq 0)$ (via formula (5.7)) makes sense under N, and H has continuous sample paths under N.

For every fixed $x \in E$, we can construct a σ-finite measure $\mathbb{N}_x$ and a process $(W_s, s \geq 0)$ with values in $\mathcal{W}_x$, defined under the measure $\mathbb{N}_x$, whose law is characterized by the following two properties (we use the notation of Section 4.1):

(i) $\zeta_s = \zeta_{(W_s)}$, $s \geq 0$ is distributed under $\mathbb{N}_x$ as the process H_s, $s \geq 0$ under N.

(ii) Conditionally on $\zeta_s = f(s)$, $s \geq 0$, the process W has distribution Θ_x^f.

Note that this is exactly similar to the construction of Chapter 4, but the role of the Itô excursion measure is played by the law of the process H under N. There is another significant difference: The process W is not Markovian, because H itself is not. This explains why we constructed only the excursion measures $\mathbb{N}_x$ and not the law of the process W started at an arbitrary starting point (this would not make sense, see however Section 4 below).

Our assumption $\beta > 0$ implies that H has Hölder continuous sample paths with exponent η for any $\eta < 1/2$ (see [19]). By arguing as in the proof of Proposition 4.1.1, it follows that $(W_s, s \geq 0)$ has a continuous modification under $\mathbb{N}_x$. Hence, without loss of generality, we can assume in the next theorem (but not necessarily in the remainder of this chapter) that $\mathbb{N}_x$ is defined on the canonical space $C(\mathbb{R}_+, \mathbb{R}_+) \times C(\mathbb{R}_+, \mathcal{W})$ in such a way that $\zeta_s(f, \omega) = f(s)$ and $W_s = \omega(s)$, $\mathbb{N}_x(df\,d\omega)$ a.e.

Theorem 5.2.2 *Let $\mu \in \mathcal{M}_f(E)$ and let*

$$\sum_{i \in I} \delta_{(x_i, f_i, \omega_i)}$$

be a Poisson point measure with intensity $\mu(dx)\mathbb{N}_x(df\,d\omega)$. Write $W_s^i = W_s(f_i, \omega_i)$, $\zeta_s^i = \zeta_s(f_i, \omega_i)$ and $\gamma_i = \gamma(f_i)$ for every $i \in I$, $s \geq 0$. Then there exists a (ξ, ψ)-superprocess $(Z_t, t \geq 0)$ with $Z_0 = \mu$, such that for every $\varphi \in \mathcal{B}_{b+}(\mathbb{R}_+ \times E)$,

$$\int_0^\infty dt \int Z_t(dx)\, \varphi(t, x) = \sum_{i \in I} \int_0^{\gamma_i} \varphi(\zeta_s^i, \hat{W}_s^i)\, ds \,. \tag{5.8}$$

Remark. As in Theorem 4.4.2 we could have given a somewhat more explicit formula for Z by using the local times of the height process at every level: See [19].

5.3 The exploration process

Before we proceed to the proofs, we need to introduce a crucial tool. We noticed that H is in general not a Markov process. For the calculations that will follow it is important to consider another process which contains more information than H and is Markovian.

Definition. *The exploration process $(\rho_t, t \geq 0)$ is the process with values in $\mathcal{M}_f(\mathbb{R}_+)$ defined by*

$$\langle \rho_t, g \rangle = \int_{[0,t]} d_s I_t^s \, g(H_s) \,,$$

for $g \in \mathcal{B}_{b+}(\mathbb{R}_+)$*. The integral in the right side is with respect to the increasing function* $s \longrightarrow I_t^s$*.*

We can easily obtain a more explicit formula for ρ_t: A change of variables using (5.7) shows that

$$\begin{aligned}
\langle \rho_t, g \rangle &= \int_0^t d_s\, I_t^s\, g(\beta^{-1} m(\{I_s^r, r \le s\})) \\
&= \int_0^t d_s\, I_t^s\, g(\beta^{-1} m(\{I_t^r, r \le s\})) \\
&= \beta \int_0^{H_t} da\, g(a) + \sum_{s \le t : Y_{s-} < I_t^s} (I_t^s - Y_{s-}) g(H_s)
\end{aligned}$$

so that

$$\rho_t(da) = \beta 1_{[0,H_t]}(a) da + \sum_{s \le t : Y_{s-} < I_t^s} (I_t^s - Y_{s-}) \delta_{H_s}(da) \ . \tag{5.9}$$

¿From this formula, it is clear that

$$\operatorname{supp} \rho_t = [0, H_t] \ , \ \text{for every } t \ge 0 \ , \ \text{a.s.}$$

The definition of ρ_t also shows that

$$\langle \rho_t, 1 \rangle = Y_t - I_t \ .$$

If $\mu \in \mathcal{M}_f(\mathbb{R}_+)$ and $a \in \mathbb{R}$ we define $k_a \mu \in \mathcal{M}_f(\mathbb{R}_+)$ by the formula

$$k_a \mu([0, r]) = \mu([0, r]) \wedge a^+ \ .$$

When $a \le 0$, $k_a \mu = 0$, and when $a > 0$, $k_a \mu$ can be interpreted as the measure μ "truncated at mass a".

If $\mu \in \mathcal{M}_f(\mathbb{R}_+)$ has compact support and $\nu \in \mathcal{M}_f(\mathbb{R}_+)$, the concatenation $[\mu, \nu]$ is defined by

$$\int [\mu, \nu](dr) h(r) = \int \mu(dr) h(r) + \int \nu(dr) h(H(\mu) + r)$$

where $H(\mu) = \sup(\operatorname{supp} \mu)$.

Proposition 5.3.1 *The process* $(\rho_t, t \ge 0)$ *is a càdlàg strong Markov process with values in the space* $\mathcal{M}_f(\mathbb{R}_+)$ *of all finite measures on* $\mathbb{R}_+$*. If* $\theta \in \mathcal{M}_f(\mathbb{R}_+)$*, the process started at* θ *can be defined by the explicit formula*

$$\rho_t^\theta = [k_{<\theta,1>+I_t} \theta, \rho_t].$$

Proof. The càdlàg property of paths follows from the explicit formula (5.9). This formula shows more precisely that t is a discontinuity time for ρ iff it is so for Y, and $\rho_t = \rho_{t-} + \Delta Y_t\, \delta_{H_t}$.

Then, let T be a stopping time of the canonical filtration $(\mathcal{F}_t)_{t\geq 0}$ of Y. Consider the shifted process

$$Y_t^{(T)} = Y_{T+t} - Y_T, \quad t \geq 0,$$

which has the same distribution as Y and is independent of $\mathcal{F}_T$. Then, from the explicit formulas for ρ and H, one easily verifies that, a.s. for every $t > 0$,

$$\rho_{T+t} = [k_{<\rho_T,1>+I_t^{(T)}}\rho_T, \rho_t^{(T)}],$$

with an obvious notation for $\rho_t^{(T)}$ and $I_t^{(T)}$. The statement of the proposition now follows from the fact that $(I_t^{(T)}, \rho_t^{(T)})$ has the same distribution as (I_t, ρ_t) and is independent of $\mathcal{F}_T$. □

Remark. Let ρ^θ be defined as in Proposition 5.3.1 and set $T_0 = \inf\{t \geq 0 : \rho_t^\theta = 0\}$. Then the formula of Proposition 5.3.1 implies that $T_0 = \inf\{t \geq 0 : Y_t = -\langle\theta, 1\rangle\}$. For $t \leq T_0$, the total mass of ρ_t^θ is equal to $\langle\theta, 1\rangle + I_t + \langle\rho_t, 1\rangle = \langle\theta, 1\rangle + Y_t$. Furthermore, if (a, b) is an excursion interval of $Y - I$ before time T_0, we have $\rho_t^\theta = [k_{<\theta,1>+I_a}\theta, \rho_t]$ for every $t \in [a, b]$, and in particular $\rho_a = \rho_b = k_{<\theta,1>+I_a}\theta$. These observations will be useful in Section 4 below.

The following proposition gives an explicit formula for the invariant measure of ρ. Before stating this result, we give some important observations. Recall that N denotes the excursion measure of $Y - I$ away from 0. Formulas (5.7) and (5.9) providing explicit expressions for the processes ρ and H still make sense under the excursion measure N. Furthermore, these formulas show that both ρ_t and H_t depend only on the values taken by $Y - I$ on the excursion e_t of $Y - I$ that straddles t, and

$$\rho_t = \rho_{t-a_t}(e_t)\ , \qquad H_t = H_{t-a_t}(e_t),$$

if a_t denotes the starting time of this excursion. Since $\langle\rho_t, 1\rangle = Y_t - I_t$ the excursion intervals of ρ away from 0 are the same as those of $Y - I$, and the "law" of $(\rho_t, t \geq 0)$ under N is easily identified with the excursion measure of the Markov process ρ away from 0.

We set $\widetilde{\psi}(u) = u^{-1}\psi(u)$ and denote by $U = (U_t, t \geq 0)$ a subordinator with Laplace exponent $\widetilde{\psi}$, i.e. with killing rate α, drift β and Lévy measure $\pi([r, \infty))dr$.

Proposition 5.3.2 *For every nonnegative measurable function Φ on $\mathcal{M}_f(\mathbb{R}_+)$,*

$$N\Big(\int_0^\gamma dt\, \Phi(\rho_t)\Big) = \int_0^\infty da\, E(\Phi(J_a)),$$

where $J_a(dr) = 1_{[0,a]}(r)\, dU_r$ if U is not killed before time a, and we make the convention that $\Phi(J_a) = 0$ if U is killed before time a .

Proof. We may assume that Φ is bounded and continuous. Recall the notation τ_r in Theorem 5.2.1. From excursion theory for $Y - I$ and the remarks preceding the proposition, we have for every $\varepsilon > 0$, $C > 0$,

$$\begin{aligned} N\Big(\int_0^\gamma dt\, \Phi(\rho_t)\, 1_{\{H_t\leq C\}}\Big) &= \frac{1}{\varepsilon}E\Big(\int_0^{\tau_\varepsilon} dt\, \Phi(\rho_t)\, 1_{\{H_t\leq C\}}\Big) \\ &= \frac{1}{\varepsilon}\int_0^\infty dt\, E(1_{\{t<\tau_\varepsilon, H_t\leq C\}}\Phi(\rho_t)). \end{aligned}$$

Then, for every fixed $t > 0$, we use time-reversal at time t. Recalling the definition of H and ρ, we see that

$$\rho_t = \hat{\eta}_t^{(t)}$$

where $\hat{\eta}_t^{(t)}$ is defined by

$$\langle \hat{\eta}_t^{(t)}, f \rangle = \int_0^t d\hat{S}_r^{(t)} \, f(\hat{L}_t^{(t)} - \hat{L}_r^{(t)})$$

and $\hat{L}_r^{(t)} = \beta^{-1} m(\{\hat{S}_s^{(t)}, 0 \le s \le t\})$ as in (5.5). Similarly,

$$\{t < \tau_\varepsilon, H_t \le C\} = \{\hat{S}_t^{(t)} - \hat{Y}_t^{(t)} < \varepsilon, \hat{L}_t^{(t)} \le C\}$$

and so we can write

$$E(1_{\{t<\tau_\varepsilon, H_t \le C\}} \Phi(\rho_t)) = E(1_{\{\hat{S}_t^{(t)} - \hat{Y}_t^{(t)} < \varepsilon, \hat{L}_t^{(t)} \le C\}} \Phi(\hat{\eta}_t^{(t)})) = E(1_{\{S_t - Y_t < \varepsilon, L_t \le C\}} \Phi(\eta_t))$$

where

$$\langle \eta_t, f \rangle = \int_0^t dS_r \, f(L_t - L_r).$$

Summarizing, we have for every $\varepsilon > 0$

$$N\Big(\int_0^\gamma dt \, \Phi(\rho_t) \, 1_{\{H_t \le C\}}\Big) = E\Big(\frac{1}{\varepsilon} \int_0^\infty dt \, 1_{\{S_t - Y_t < \varepsilon, L_t \le C\}} \Phi(\eta_t)\Big).$$

Note from (5.2) that the random measures $\varepsilon^{-1} 1_{\{S_t - Y_t < \varepsilon\}} dt$ converge in probability to the measure dL_t. Furthermore, (5.3) allows us to pass to the limit under the expectation sign and we arrive at

$$\begin{aligned}
\lim_{\varepsilon \to 0} E\Big(\frac{1}{\varepsilon} \int_0^\infty dt \, 1_{\{S_t - Y_t < \varepsilon, L_t \le C\}} \Phi(\eta_t)\Big) &= E\Big(\int_0^\infty dL_t \, 1_{\{L_t \le C\}} \, \Phi(\eta_t)\Big) \\
&= E\Big(\int_0^{L_\infty \wedge C} da \, \Phi(\eta_{L^{-1}(a)})\Big).
\end{aligned}$$

We finally let C tend to ∞ to get

$$N\Big(\int_0^\gamma dt \, \Phi(\rho_t)\Big) = E\Big(\int_0^{L_\infty} da \, \Phi(\eta_{L^{-1}(a)})\Big).$$

Then note that, on the event $\{a < L_\infty\}$,

$$\langle \eta_{L^{-1}(a)}, f \rangle = \int_0^{L^{-1}(a)} dS_r \, f(a - L_r) = \int_0^a dV_s \, f(a - s),$$

where $V_s = S_{L^{-1}(s)}$ is a subordinator with exponent $\frac{\psi(\lambda)}{\lambda}$ (cf (5.4)). The desired result now follows. $\square$

5.4 The Lévy snake

If $\mu \in \mathcal{M}_f(\mathbb{R}_+)$, we set

$$H(\mu) = \sup(\operatorname{supp} \mu) \in [0, \infty]$$

where $H(\mu) = 0$ if $\mu = 0$ by convention. It is important to observe that $H(\rho_s) = H_s$ for every $s \geq 0$, P a.s. or N a.e. (this was pointed out in the previous section).

We denote by $\mathcal{M}_f^0$ the set of all measures $\mu \in \mathcal{M}_f^0$ such that $H(\mu) < \infty$ and the support of μ is equal to the interval $[0, H(\mu)]$ (or to $\emptyset$ in the case $\mu = 0$).

¿From Proposition 5.3.1 it is easy to see that the process ρ started at $\mu \in \mathcal{M}_f^0$ will remain forever in $\mathcal{M}_f^0$. Therefore we may and will consider the exploration process as a Markov process in $\mathcal{M}_f^0$. It also follows from the representation in Proposition 5.3.1, that $H(\rho_s)$ has continuous sample paths whenever $\rho_0 \in \mathcal{M}_f^0$ (this is the reason for the condition $\operatorname{supp} \mu = [0, H(\mu)]$ in the definition of $\mathcal{M}_f^0$).

Finally, for every $x \in E$, we let Δ_x be the set of all pairs $(\mu, \mathrm{w}) \in \mathcal{M}_f^0 \times \mathcal{W}_x$ such that $\zeta_{(\mathrm{w})} = H(\mu)$. Recall from the end of Section 4.1 the definition of the distributions Θ_{w}^f.

Definition. *The ψ-Lévy snake with initial point $x \in E$ is the Markov process (ρ_s, W_s) with values in Δ_x whose law is characterized by the following properties. If $(\mu, \mathrm{w}) \in \Delta_x$, the Lévy snake started at (μ, w) is such that:*

(i) The first coordinate $(\rho_s, s \geq 0)$ is the exploration process started at μ.

(ii) Conditionally on $(\rho_s, s \geq 0)$, the process $(W_s, s \geq 0)$ has distribution $\Theta_{\mathrm{w}}^{H(\rho)}$.

The fact that the process (ρ_s, W_s) satisfying properties (i) and (ii) above is Markovian follows from the Markov property of ρ by using arguments similar to the proof of Theorem 4.2.1. We will denote by $\mathbb{P}_{\mu,\mathrm{w}}$ the probability measure under which the Lévy snake starts at (μ, w). We also write $\mathbb{P}^*_{\mu,\mathrm{w}}$ for the law of the same process stopped when it hits 0. As previously, we will use the notation $\zeta_s = \zeta_{(W_s)}$ for the lifetime of W_s. By property (ii), $\zeta_s = H(\rho_s)$ for every $s \geq 0$, $\mathbb{P}_{\mu,\mathrm{w}}$ a.s.

Again an important role will be played by excursion measures. The excursion measure $\mathbb{N}_x$ of the Lévy snake away from $(0, x)$ is the σ-finite measure such that

(i') The distribution of $(\rho_s, s \geq 0)$ under $\mathbb{N}_x$ coincides with its distribution under N.

(ii') Under $\mathbb{N}_x$, conditionally on $(\rho_s, s \geq 0)$, the process $(W_s, s \geq 0)$ has distribution $\Theta_x^{H(\rho)}$.

This is consistent with our previous use of the notation $\mathbb{N}_x$ in Section 2 above, up to a slight abuse of notation (to be specific, we were considering in Theorem 5.2.2 the law of the pair $(H(\rho_s), W_s; s \geq 0)$ under $\mathbb{N}_x$).

¿From the Markov property of the exploration process under N and property (ii') above, it easily follows that the process (ρ_s, W_s) is Markovian under $\mathbb{N}_x$ with the transition kernels of the Lévy snake stopped when it hits 0.

The next result is an immediate consequence of Proposition 5.3.2 and property (ii') above of the excursion measure. We keep the notation introduced before Proposition 5.3.2.

Proposition 5.4.1 *For every nonnegative measurable function Φ on Δ_x,*

$$\mathbb{N}_x\Big(\int_0^\gamma dt\,\Phi(\rho_t,W_t)\Big)=\int_0^\infty da\,E\otimes\Pi_x(\Phi(J_a,(\xi_r,0\le r\le a))).$$

We need still another result giving information about the law of the Lévy snake started at $(\mu,\mathrm{w})\in\mathcal{W}_x$. First observe that $Y_t=\langle\rho_t,1\rangle$ is distributed under $\mathbb{P}^*_{\mu,\mathrm{w}}$ as the underlying Lévy process started at $\langle\mu,1\rangle$ and stopped when it first hits 0. We write $I_t=\inf_{r\le t}Y_r$, and we denote by (α_i,β_i), $i\in J$ the excursion intervals of $Y-I$ away from 0, before time $T_0=\inf\{t\ge 0: Y_t=0\}$. Consider one such excursion interval (α_i,β_i). From Proposition 5.3.1 and the remark following this proposition, it is easy to see that for every $s\in[\alpha_i,\beta_i]$, the minimum of $H(\rho)$ over $[0,s]$ is equal to $h_i=H(\rho_{\alpha_i})=H(\rho_{\beta_i})$. Hence, by the snake property (property (ii) above), it follows that $W_s(t)=\mathrm{w}(t)$ for every $t\in[0,h_i]$ and $s\in[\alpha_i,\beta_i]$, $\mathbb{P}^*_{\mu,\mathrm{w}}$ a.s. We then define the pair (ρ^i,W^i) by the formulas

$$\begin{array}{ll}\langle\rho^i_s,\varphi\rangle=\int_{(h_i,\infty)}\rho_{\alpha_i+s}(dr)\,\varphi(r-h_i) & \text{if } 0\le s\le\beta_i-\alpha_i\\ \rho^i_s=0 & \text{if } s>\beta_i-\alpha_i,\end{array}$$

and

$$\begin{array}{ll}W^i_s(t)=W_{\alpha_i+s}(h_i+t),\quad \zeta^i_s=H(\rho_{\alpha_i+s})-h_i & \text{if } 0<s<\beta_i-\alpha_i\\ W^i_s=\mathrm{w}(h_i) & \text{if } s=0 \text{ or } s\ge\beta_i-\alpha_i.\end{array}$$

Lemma 5.4.2 *Let $(\mu,\mathrm{w})\in\Theta_x$. The point measure*

$$\sum_{i\in J}\delta_{(h_i,\rho^i,W^i)}$$

is under $\mathbb{P}^*_{\mu,\mathrm{w}}$ a Poisson point measure with intensity

$$\mu(dh)\,\mathbb{N}_{\mathrm{w}(h)}(d\rho\,dW).$$

Proof. Consider first the point measure

$$\sum_{i\in J}\delta_{(h_i,\rho^i)}.$$

If $J_s=I_s-\langle\mu,1\rangle$, we have $h_i=H(\rho_{\alpha_i})=H(k_{-J_{\alpha_i}}\mu)$. Excursion theory for $Y-I$ ensures that

$$\sum_{i\in I}\delta_{(-J_{\alpha_i},\rho^i)}$$

is under $\mathbb{P}^*_{\mu,\mathrm{w}}$ a Poisson point measure with intensity $1_{[0,<\mu,1>]}(u)\,du\,N(d\rho)$. (By abuse of notation, we write $N(d\rho)$ for the distribution of ρ under N.) Since the image measure of $1_{[0,<\mu,1>]}(u)\,du$ under the mapping $u\longrightarrow H(k_u\mu)$ is precisely the measure μ, it follows that

$$\sum_{i\in J}\delta_{(h_i,\rho^i)}$$

is a Poisson point measure with intensity $\mu(dh)\,N(d\rho)$. To complete the proof, it remains to obtain the conditional distribution of $(W^i, i\in J)$ knowing $(\rho_s, s\geq 0)$. However, the form of the conditional law $\Theta^{H(\rho)}_{\mathrm{w}}$ easily implies that under $\Theta^{H(\rho)}_{\mathrm{w}}$, the processes W^i, $i\in J$ are independent, and furthermore the conditional distribution of W^i is $\Theta^{H^i}_{\mathrm{w}(h_i)}$, where $H^i_s = H(\rho^i_s)$. It follows that

$$\sum_{i\in J}\delta_{(h_i,\rho^i,W^i)}$$

is a Poisson measure with intensity

$$\mu(dh)\,N(d\rho)\,\Theta^{H(\rho)}_{\mathrm{w}(h)}(dW) = \mu(dh)\,\mathbb{N}_{\mathrm{w}(h)}(d\rho dW).$$

This completes the proof. □

We will use the following consequence of the lemma. Let φ be a nonnegative measurable function on $\mathbb{R}_+\times E$. We can then compute

$$\mathbb{E}_{\mu,\mathrm{w}}\Big(\exp-\int_0^{T_0} ds\,\varphi(\zeta_s,\widehat{W}_s)\Big)$$

in the following way. Note that the set $\{s\leq T_0 : Y_s = I_s\}$ has Lebesgue measure 0 (because 0 is regular for $(-\infty,0)$ for the underlying Lévy process). Hence, with the notation of the lemma, we have

$$\int_0^{T_0} ds\,\varphi(\zeta_s,\widehat{W}_s) = \sum_{i\in J}\int_{\alpha_i}^{\beta_i} ds\,\varphi(\zeta_s,\widehat{W}_s) = \sum_{i\in J}\int_0^{\beta_i-\alpha_i} ds\,\varphi(h_i+\zeta^i_s,\widehat{W}^i_s).$$

¿From Lemma 5.4.2, it now follows that

$$\mathbb{E}_{\mu,\mathrm{w}}\Big(\exp-\int_0^{T_0} ds\,\varphi(\zeta_s,\widehat{W}_s)\Big) = \exp-\int\mu(dh)\,\mathbb{N}_{\mathrm{w}(h)}\Big(1-\exp-\int_0^{\gamma} ds\,\varphi(h+\zeta_s,\widehat{W}_s)\Big). \tag{5.10}$$

5.5 Proof of the main theorem

We prove only Theorem 5.2.2, since the “Ray-Knight theorem” (Theorem 5.2.1) then follows immediately by taking functions $\varphi(t,x)$ that do not depend on x (use excursion theory for $Y-I$ and recall that the total mass process of the (ξ,ψ)-superprocess is a ψ-CSBP). Let $\varphi\in C_{b+}(\mathbb{R}_+\times E)$ be such that $\varphi(t,x)=0$ for every $t\geq T$ and $x\in E$. In view of Proposition 3.3.2, it is enough to verify that

$$\mathbb{E}_\theta\Big(\exp-\sum_{i\in I}\int_0^{\gamma_i}\varphi(\zeta^i_s,\hat{W}^i_s)\,ds\Big) = \exp(-\langle\mu,u_0\rangle) \tag{5.11}$$

where the function $(u_t(x), t\geq 0, x\in E)$ is the unique nonnegative solution of the integral equation

$$u_t(x)+\Pi_x\Big(\int_0^\infty\psi(u_{t+s}(\xi_s))\,ds\Big) = \Pi_x\Big(\int_0^\infty\varphi(t+s,\xi_s)\,ds\Big)\,. \tag{5.12}$$

Indeed, by comparing with Proposition 3.3.2, this will imply that the measure

$$\varphi \longrightarrow \sum_{i\in I} \int_0^{\gamma_i} \varphi(\zeta^i_s, \hat{W}^i_s)\, ds$$

has the same Laplace functional as the measure

$$\varphi \longrightarrow \int_0^\infty dt \int Z_t(dx)\, \varphi(t,x)$$

when Z is a (ξ,ψ)-superprocess started at μ, and the desired result follows.

By the exponential formula for Poisson measures, the left-hand side of (5.11) is equal to

$$\exp\Big(-\int \mu(dx)\, \mathbb{N}_x\Big(1-\exp-\int_0^\gamma ds\, \varphi(\zeta_s,\widehat{W}_s)\Big)\Big).$$

So, if we set

$$u_t(x) = \mathbb{N}_x\Big(1-\exp-\int_0^\gamma ds\, \varphi(t+\zeta_s,\widehat{W}_s)\Big)$$

it only remains to show that $u_t(x)$ solves the integral equation (5.12).

Now note that

$$1-\exp-\int_0^\gamma ds\, \varphi(t+\zeta_s,\widehat{W}_s) = \int_0^\gamma ds\, \varphi(t+\zeta_s,\widehat{W}_s)\, \exp\Big(-\int_s^\gamma dr\, \varphi(t+\zeta_r,\widehat{W}_r)\Big).$$

By using the Markov property of the Lévy snake under $\mathbb{N}_x$, it follows that

$$u_t(x) = \mathbb{N}_x\Big(\int_0^\gamma ds\, \varphi(t+\zeta_s,\widehat{W}_s)\, \mathbb{E}^*_{\rho_s,W_s}\Big(\exp-\int_0^{T_0} dr\, \varphi(t+\zeta_r,\widehat{W}_r)\Big)\Big).$$

However, by (5.10), we have

$$\begin{aligned}
&\mathbb{E}^*_{\rho_s,W_s}\Big(\exp-\int_0^{T_0} dr\, \varphi(t+\zeta_r,\widehat{W}_r)\Big)\\
&\quad = \exp\Big(-\int \rho_s(dh)\mathbb{N}_{W_s(h)}\Big(1-\exp-\int_0^\gamma dr\, \varphi(t+h+\zeta_r,\widehat{W}_r)\Big)\Big)\\
&\quad = \exp\Big(-\int \rho_s(dh)\, u_{t+h}(W_s(h))\Big).
\end{aligned}$$

Hence,

$$u_t(x) = \mathbb{N}_x\Big(\int_0^\gamma ds\, \varphi(t+\zeta_s,\widehat{W}_s)\, \exp\Big(-\int \rho_s(dh)\, u_{t+h}(W_s(h))\Big)\Big).$$

We can use Proposition 5.4.1 to evaluate the latter quantity. From the definition of J_a (and with the same convention as in Proposition 5.3.2), we have for any nonnegative function f,

$$E\Big(\exp-\int J_a(dh)\, f(h)\Big) = \exp\Big(-\int_0^a dh\, \widetilde{\psi}(f(h))\Big).$$

By using this together with Proposition 5.4.1, we arrive at

$$\begin{aligned} u_t(x) &= \int_0^\infty da\,\Pi_x\Big(g(t+a,\xi_a)\,E\Big(\exp\Big(-\int J_a(dh)\,u_{t+h}(\xi_h)\Big)\Big) \\ &= \int_0^\infty da\,\Pi_x\Big(g(t+a,\xi_a)\,\exp\Big(-\int_0^a dh\,\widetilde{\psi}(u_{t+h}(\xi_h))\Big)\Big). \end{aligned}$$

¿From the last formula the proof is now easily completed by means of the standard Feynman-Kac argument:

$$\begin{aligned} &\Pi_x\Big(\int_0^\infty da\,g(t+a,\xi_a)\Big) - u_t(x) \\ &= \Pi_x\Big(\int_0^\infty da\,g(t+a,\xi_a)\Big(1-\exp\Big(-\int_0^a dh\,\widetilde{\psi}(u_{t+h}(\xi_h))\Big)\Big) \\ &= \Pi_x\Big(\int_0^\infty da\,g(t+a,\xi_a)\int_0^a db\,\widetilde{\psi}(u_{t+b}(\xi_b))\exp\Big(-\int_b^a dh\,\widetilde{\psi}(u_{t+h}(\xi_h))\Big)\Big) \\ &= \Pi_x\Big(\int_0^\infty db\,\widetilde{\psi}(u_{t+b}(\xi_b))\int_b^\infty da\,g(t+a,\xi_a)\exp\Big(-\int_b^a dh\,\widetilde{\psi}(u_{t+h}(\xi_h))\Big)\Big) \\ &= \Pi_x\Big(\int_0^\infty db\,\widetilde{\psi}(u_{t+b}(\xi_b))\,\Pi_{\xi_b}\Big(\int_0^\infty da\,g(t+b+a,\xi_a)\exp\Big(-\int_0^a dh\,\widetilde{\psi}(u_{t+b+h}(\xi_h))\Big)\Big)\Big) \\ &= \Pi_x\Big(\int_0^\infty db\,\widetilde{\psi}(u_{t+b}(\xi_b))\,u_{t+b}(\xi_b)\Big) \\ &= \Pi_x\Big(\int_0^\infty db\,\psi(u_{t+b}(\xi_b))\Big). \end{aligned}$$

We have thus obtained the desired integral equation (5.12), and this completes the proof.

Remark. In the case $\psi(u) = \beta u^2$, the underlying Lévy process is Brownian motion, the law of H under N is the Itô excursion measure and Theorem 5.2.2 yields the Brownian snake representation of Chapter 4. The approach presented here (taken from [19]) is different and in a sense more probabilistic than the proof of Theorem 4.4.2 in Chapter 4. An approach of Theorem 5.2.2 in the spirit of Chapter 4 can be found in [28].

Chapter 6

Some connections

In this chapter, we briefly discuss without proofs some connections of the preceding results with other topics. We restrict our attention to the quadratic branching case, and so we do not use the results of Chapter 5. Sections 1,2,3 below can be read independently.

Throughout this chapter, we use the notation of Chapter 4. The spatial motion ξ is always Brownian motion in $\mathbb{R}^d$.

6.1 Partial differential equations

When $\psi(u) = \beta u^2$ and ξ is Brownian motion in $\mathbb{R}^d$, the integral equation (3.7) is the integral form of the parabolic partial differential equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\Delta u - \beta u^2 \\ \\ u_0 = \varphi \end{cases}$$

This suggests that super-Brownian motion could be used to give a probabilistic approach to elliptic problems involving the nonlinear operator $\frac{1}{2}\Delta u - \beta u^2$. This program was carried over by Dynkin in a series of important papers [13],[14],[15]. We give here a brief presentation in terms of the Brownian snake (see e.g. [26] for more details).

In the classical relations between Brownian motion and the Laplace equation, a key role is played by the exit point of Brownian motion from a domain. Similarly, the probabilistic approach to the Dirichlet problem for the operator $\frac{1}{2}\Delta u - \beta u^2$ will involve the exit points of the Brownian snake paths $W_s, s \geq 0$ under the excursion measures $\mathbb{N}_x$. Precisely, let D be a domain in $\mathbb{R}^d$, and for any path $\mathrm{w} \in \mathcal{W}$, let

$$\tau(\mathrm{w}) = \inf\{t \geq 0 : \mathrm{w}(t) \notin D\} \qquad (\inf \emptyset = \infty).$$

We set

$$\mathcal{E}^D := \{W_s(\tau(W_s)) : s \geq 0, \tau(W_s) < \infty\},$$

which represents the set of exit points of the paths W_s, for those paths that do exit the domain D.

Our first task is to construct a random measure that is in some sense uniformly distributed over $\mathcal{E}^D$.

Proposition 6.1.1 *Let $x \in D$. The limit*

$$\langle \mathcal{Z}^D, f\rangle = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^{\gamma} f(W_s(\tau(W_s))) 1_{\{\tau(W_s)<\zeta_s<\tau(W_s)+\varepsilon\}}\, ds$$

exists for every $f \in C(\mathbb{R}^d, \mathbb{R}_+)$, $\mathbb{N}_x$ a.e. and defines a random measure supported on $\mathcal{E}^D$ called the exit measure from D.

The approximation used to define the exit measure is reminiscent of the classical approximation for Brownian local time, and in fact the existence of the limit is easily obtained by considering the local time at 0 of the process $(\zeta_s - \tau(W_s))^+$ (see [26], Chapter V for details).

The exit measure leads to the probabilistic solution of the Dirichlet problem for the operator $\Delta u - u^2$ due to Dynkin. For simplicity, we assume that D is bounded and has a smooth boundary.

Theorem 6.1.2 [13] *Let g be continuous and nonnegative over ∂D. Then the function*

$$u(x) = \mathbb{N}_x(1 - \exp(-\langle \mathcal{Z}^D, g\rangle), \qquad x \in D$$

is the unique nonnegative solution of the problem

$$\begin{cases} \Delta u = 4u^2 & \text{in } D \\ u_{|\partial D} = g\,. \end{cases}$$

A very interesting feature of the probabilistic representation of the previous theorem is that it is robust in the sense that various limiting procedures will lead from this representation to analogous formulas for several classes of singular solutions. In particular, one gets a representation for the maximal nonnegative solution in a domain. We denote by $\mathcal{R}$ the range of the Brownian snake, defined by

$$\mathcal{R} = \{\widehat{W}_s : s \geq 0\}.$$

Corollary 6.1.3 [13] *Let D be any open set in $\mathbb{R}^d$. The function*

$$u(x) = \mathbb{N}_x(\mathcal{R} \cap D^c \neq \emptyset), \qquad x \in D$$

is the maximal nonnegative solution of $\Delta u = 4u^2$ in D. In particular, if K is a compact subset of $\mathbb{R}^d$, K is polar (in the sense that $\mathbb{N}_x(\mathcal{R} \cap K \neq \emptyset) = 0$ for every $x \in \mathbb{R}^d \backslash K$) if and only if there is no nontrivial nonnegative solution of $\Delta u = 4u^2$ in $\mathbb{R}^d \backslash K$.

This corollary has allowed Dynkin to give an analytic characterization of polar sets for super-Brownian motion [13]. Until now there exists no probabilistic proof of this characterization.

It is natural to ask whether a probabilistic representation of the type of Theorem 6.1.2 holds for any nonnegative solution of $\Delta u = u^2$. It turns out that the answer is yes in two dimensions.

Theorem 6.1.4 [25] *Assume that $d = 2$ and ∂D is of class C^2. There is a $1-1$ correspondence between*

- *nonnegative solutions u of $\Delta u = 4u^2$ in D*
- *pairs (K, ν), where K is a compact subset of ∂D and ν is a Radon measure on $\partial D \backslash K$.*

This correspondence is given by

$$u(x) = \mathbb{N}_x\Big(1 - 1_{\{\mathcal{E}^D \cap K = \emptyset\}} \exp -\langle \nu, z^D \rangle\Big), \qquad x \in D$$

where $(z_D(y), y \in \partial D)$ is the continuous density of the exit measure $\mathcal{Z}^D$ with respect to Lebesgue measure on the boundary.

In the correspondence of Theorem 6.1.4, both the compact set K and the measure ν can be recovered from the boundary behavior of the solution u. The pair (K, ν) is called the trace of ν.

Marcus and Véron [31] have used analytic methods to rederive the previous theorem (except for the probabilistic representation) and to extend it to the equation $\Delta u = u^p$, when $p > 1$ and $d < \frac{p+1}{p-1}$. In the supercritical case (i.e. $d \geq \frac{p+1}{p-1}$, in particular when $d \geq 3$ for equation $\Delta u = u^2$), things become more complicated. One can still define the trace (K, ν) of a solution but there is no longer a one-to-one correspondence between a solution and its trace. Interesting results in this connection have been obtained by Dynkin and Kuznetsov [17]. Dynkin and Kuznetsov obtain a satisfactory probabilistic representation and classification for all σ-moderate solutions (a nonnegative solution is moderate if it is bounded above by a harmonic function, and a solution is σ-moderate if it is the increasing limit of a sequence of moderate solutions). The key remaining question [18] is:

Is every positive solution σ-moderate ?

6.2 Interacting particle systems

Several recent papers [12],[8],[6] show that the asymptotic behavior of certain interacting particle systems (contact process, voter model) can be analysed in terms of super-Brownian motion. The rough idea is that in the scaling limit the interaction reduces to a branching phenomenon. Here we will concentrate on the voter model and present some results of [6].

The voter model is one of the most classical interacting particle systems. At each site $x \in \mathbb{Z}^d$ sits an individual who can have two possible opinions, say 0 or 1. At rate 1, each individual forgets his opinion and gets a new one by choosing one of his nearest neighbors uniformly at random, and taking his opinion. Our goal is to understand the way opinions propagate in space. We restrict our attention to dimensions $d \geq 2$ and we denote by $\xi_t(x)$ the opinion of x at time t.

We consider the simple situation where all individuals have type (opinion) 0 at the initial time, except for the individual at the origin who has type 1 (in other words $\xi_0(x) = 1_{\{x=0\}}$).

Then with a high probability, type 1 will disappear. More precisely, if $\mathcal{U}_t = \{x \in \mathbb{Z}^d : \xi_t(x) = 1\}$ denotes the set of individuals who have type 1 at time t, Bramson and Griffeath [7] proved that for t large,

$$p_t := P[\mathcal{U}_t \neq \emptyset] \sim \begin{cases} \dfrac{\gamma_2 \log t}{t} & \text{if } d = 2 \\ \dfrac{\gamma_d}{t} & \text{if } d \geq 3 \end{cases}$$

where γ_d is a positive constant. One may then ask about the shape of the set $\mathcal{U}_t$ conditional on the event $\{\mathcal{U}_t \neq \emptyset\}$.

Theorem 6.2.1 [6] *The law of the random set $\frac{1}{\sqrt{t}}\mathcal{U}_t$ conditional on $\{\mathcal{U}_t \neq \emptyset\}$ converges as $t \to \infty$ to the law under $\mathbb{N}_0(\cdot \mid \sup \zeta_s \geq 1)$ of the set*

$$\{\frac{1}{\sqrt{d}} W_s(1) : s \geq 0, \zeta_s \geq 1\}.$$

Here the convergence of sets is in the sense of the Hausdorff metric on compact subsets of $\mathbb{R}^d$. One can prove functional versions of the previous theorem, which show that conditionally on non-extinction on a large time interval, the set of 1's evolves like super-Brownian motion under its excursion measure (more precisely, like the process $\mathcal{Z}_t$ of (4.9) under $\mathbb{N}_0(\cdot \mid \sup \zeta_s \geq 1)$).

Theorem 6.2.1 yields interesting complements to the Bramson-Griffeath result recalled above. Let A be an open subset of $\mathbb{R}^d$ and assume for simplicity that A has a smooth boundary. As a simple consequence of Theorem 6.2.1 and connections between super-Brownian motion and partial differential equations, we get the limiting behavior of the probability that $\mathcal{U}_t$ intersects $\sqrt{t}A$.

Corollary 6.2.2 [6] *We have*

$$\lim_{t\to\infty} P(\mathcal{U}_t \cap \sqrt{t}A \neq \emptyset \mid \mathcal{U}_t \neq \emptyset) = \mathbb{N}_0(\exists s \geq 0 : \zeta_s \geq 1 \text{ and } \widehat{W}_s \in A \mid \sup \zeta_s \geq 1) = u_1(0)$$

where $(u_t(x), t > 0, x \in \mathbb{R}^d)$ is the unique nonnegative solution of the problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{1}{2d}\Delta u - u^2 \qquad on\ (0,\infty) \times \mathbb{R}^d\,, \\ &u_0(x) = +\infty, \qquad x \in A\,, \\ &u_0(x) = 0, \qquad x \in \mathbb{R}^d \backslash \bar{A}\,. \end{aligned}$$

6.3 Lattice trees

A d-dimensional lattice tree with n bonds is a connected subgraph of $\mathbb{Z}^d$ with n bonds and $n+1$ vertices in which there are no loops. One is interested in describing the typical shape of a lattice tree when n is large. To this end, let $Q_n(d\omega)$ be the uniform probability measure on the set of all lattice trees with n bonds that contain the origin. For every tree ω, let $X_n(\omega)$ be the probability measure on $\mathbb{R}^d$ obtained by putting mass $\frac{1}{n+1}$ to each vertex of the rescaled tree $cn^{-1/4}\omega$. Here $c = c(d)$ is a positive constant that must be fixed properly for the following to hold.

The following theorem was conjectured by Aldous [3] and then proved by Derbez and Slade [11]. Recall from Section 4.6 the definition of ISE.

Theorem 6.3.1 [11] *If d is large enough, the law of X_n under Q_n converges weakly as $n \to \infty$ to the law of ISE.*

It is believed that $d > 8$ should be the right condition for the previous theorem to hold (under this condition, the topological support of ISE is indeed a tree). Derbez and Slade have proved a version of the previous result in dimension $d > 8$, but only for sufficiently spread-out trees.

The proof of Theorem 6.3.1 uses the lace expansion method developed by Brydges and Spencer. Another recent work of Hara and Slade [23] shows that ISE also arises as a scaling limit of the incipient infinite percolation cluster at the critical probability, again in high dimensions. Furthermore, a work in preparation of van der Hofstad, Hara and Slade indicates that super-Brownian motion under its excursion measure appears in scaling limits for oriented percolation.

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