Subordinators, Lévy processes with no negative jumps, and branching processes

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These notes are based on a concentrated advanced course on Lévy Processes and Branching Processes given at MaPhySto in late August 2000, and are related to my monograph [1] on Lévy processes and to lecture notes [3] for the École d'Été de Probabilités de Saint-Flour. I should like to thank the participants for their comments, and express my gratitude to Ole E. Barndorff-Nielsen for his kind invitation.

Summary. The purpose of this course is to present some simple relations connecting subordinators, Lévy processes with no negative jumps, and continuous state branching processes. To start with, we develop the main ingredients on subordinators (the Lévy-Khintchine formula, the Lévy-Itô decomposition, the law of the iterated logarithm, the renewal theory for the range, and the link with local times of Markov processes). Then we consider Lévy processes with no negative jumps, first in the simple case given by a subordinator with negative drift, and then in the case with unbounded variation. The main formulas of fluctuation theory are presented in this setting, including those related to the so-called two-sided exit problem. Last, we turn our attention to continuous state branching processes. We first discuss the construction by Lamperti based on a simple time-substitution of a Lévy process with no negative jumps. Then we dwell on the connection with Bochner's subordination for subordinators and its application to the genealogy of continuous state branching processes.

Key words. Subordinator, Lévy process, continuous state branching process, fluctuation theory.

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Chapter 1

Subordinators

The purpose of this chapter is to introduce standard material on subordinators. We start by recalling basic results on Poisson point processes, which are a key to the probabilistic structure of subordinators (or more generally, of Lévy processes).

1.1 Preliminaries on Poisson point processes

The results stated in this section are well-known and their proofs can be found e.g. in section XII.1 in Revuz and Yor [30].

A Poisson process with parameter (or intensity) $c > 0$, $N = (N_t, t \geq 0)$, is an increasing integer valued process with independent and stationary increments, such that for every $t > 0$, N_t has the Poisson distribution with parameter ct, i.e.

$$\mathbb{P}(N_t = k) \;=\; \mathrm{e}^{-ct}(ct)^k/k!\,, \qquad k \in \mathbb{N}\,.$$

When $(\mathcal{G}_t)$ is a filtration which satisfies the usual conditions, we say that N is a $(\mathcal{G}_t)$-Poisson process if N is a Poisson process which is adapted to $(\mathcal{G}_t)$ and for every $s, t \geq 0$, the increment $N_{t+s} - N_t$ is independent of $\mathcal{G}_t$. In particular, N is a $(\mathcal{G}_t)$-Poisson process if $(\mathcal{G}_t)$ is the natural filtration of N. In this direction, recall the following useful criterion for the independence of Poisson processes. If $N^{(i)}, i = 1, \cdots, d$ are $(\mathcal{G}_t)$-Poisson processes, then they are independent if and only if they never jump simultaneously, that is for every i, j with $i \neq j$

$$N_t^{(i)} - N_{t-}^{(i)} = 0 \text{ or } N_t^{(j)} - N_{t-}^{(j)} = 0 \quad \text{for all } t > 0, \text{ a.s.},$$

where $N_{t-}^{(k)}$ stands for the left limit of $N^{(k)}$ at time t.

Next, let ν be a sigma-finite measure on $\mathbb{R}^\star = \mathbb{R}\backslash\{0\}$. We call a random measure φ on $\mathbb{R}^\star$ *a Poisson measure with intensity* ν if it satisfies the following. For every Borel subset B of $\mathbb{R}^\star$ with $\nu(B) < \infty$, $\varphi(B)$ has a Poisson distribution with parameter $\nu(B)$, and if $B_1, \cdots, B_n$ are disjoint Borel sets, the variables $\varphi(B_1), \cdots, \varphi(B_n)$ are independent. Plainly, φ is then a sum of Dirac point masses.

We then consider the product space $[0,\infty[\times\mathbb{R}^\star$ endowed with the product measure $\mu = m\otimes\nu$ where m stands for the Lebesgue measure on $[0,\infty[$. Let φ be Poisson measure on $[0,\infty[\times\mathbb{R}^\star$ with intensity μ. It is easy to check that a.s., $\varphi(\{t\}\times\mathbb{R}^\star) = 0$ or 1 for all $t \geq 0$. This enables us to represent φ in terms of a stochastic process taking values in $\mathbb{R}$ where 0 serves as an isolated point added to $\mathbb{R}^\star$. Specifically, if $\varphi(\{t\}\times\mathbb{R}^\star) = 0$, then put $e(t) = 0$. If $\varphi(\{t\}\times\mathbb{R}^\star) = 1$, then the restriction of φ to the section $\{t\}\times\mathbb{R}^\star$ is a Dirac point mass, say at (t,ϵ), and we put $e(t) = \epsilon$. We can now express the Poisson measure as

$$\varphi \;=\; \sum_{t\geq 0} \delta_{(t,e(t))}.$$

The process $e = (e(t), t\geq 0)$ is called a *Poisson point process* with characteristic measure ν. We denote its natural filtration after the usual completion by $(\mathcal{G}_t)$.

For every Borel subset B of $\mathbb{R}^\star$, we call

$$N_t^B \;=\; \mathrm{Card}\{s\leq t : e(s)\in B\} \;=\; \varphi(B\times[0,t]) \qquad (t\geq 0)$$

the *counting process* of B. It is a $(\mathcal{G}_t)$-Poisson process with parameter $\nu(B)$. Conversely, suppose that $e = (e(t), t\geq 0)$ is a stochastic process taking values in $\mathbb{R}$ such that, for every Borel subset B of $\mathbb{R}^\star$, the counting process $N_t^B = \mathrm{Card}\{s\leq t : e(s)\in B\}$ is a Poisson process with intensity $\nu(B)$ in a given filtration $(\mathcal{G}_t)$. Observe that counting processes associated to disjoint Borel sets never jump simultaneously and thus are independent. One then deduces that the associated random measure $\varphi = \sum_{t\geq 0}\delta_{(t,e(t))}$ is a Poisson measure with intensity ν.

Throughout the rest of this section, we assume that $(e(t), t\geq 0)$ is a Poisson point process on $\mathbb{R}^\star$ with characteristic measure ν. In practice, it is important to calculate certain expressions in terms of the characteristic measure. The following two formulas are the most useful:

Compensation Formula. *Let $H = (H_t, t\geq 0)$ be a $(\mathcal{G}_t)$-predictable process taking values in the space of nonnegative measurable functions on $\mathbb{R}$, such that $H_t(0) = 0$ for all $t\geq 0$. We have*

$$\mathbb{E}\Big(\sum_{0\leq t<\infty} H_t(e(t))\Big) \;=\; \mathbb{E}\Big(\int_0^\infty dt \int_{\mathbb{R}^\star} d\nu(\epsilon) H_t(\epsilon)\Big)\,.$$

Exponential Formula. *Let f be a complex-valued Borel function on $\mathbb{R}$ with $f(0) = 0$ and*

$$\int_{\mathbb{R}^\star} |1-\mathrm{e}^{f(\epsilon)}|\nu(d\epsilon) \;<\; \infty\,.$$

Then $\sum_{0\leq s\leq t}|f(e(s))| < \infty$ a.s. for every $t\geq 0$ and one has

$$\mathbb{E}\Big(\exp\Big\{\sum_{0\leq s\leq t} f(e(s))\Big\}\Big) \;=\; \exp\left\{-t\int_{\mathbb{R}^\star}(1-\mathrm{e}^{f(\epsilon)})\nu(d\epsilon)\right\}\,.$$

1.2 Subordinators as Markov processes

Let $(\Omega, \mathbb{P})$ denote a probability space endowed with a right-continuous and complete filtration $(\mathcal{F}_t)_{t\geq 0}$. We consider right-continuous increasing adapted processes started from 0 and with values in the extended half-line $[0, \infty]$, where ∞ serves as a cemetery point (i.e. ∞ is an absorbing state). If $\sigma = (\sigma_t, t \geq 0)$ is such a process, we denote its lifetime by

$$\zeta \;=\; \inf\{t \geq 0 : \sigma_t = \infty\}$$

and call σ a *subordinator* if it has independent and homogeneous increments on $[0, \zeta)$. That is to say that for every $s, t \geq 0$, conditionally on $\{t < \zeta\}$, the increment $\sigma_{t+s} - \sigma_t$ is independent of $\mathcal{F}_t$ and has the same distribution as σ_s (under $\mathbb{P}$). When the lifetime is infinite a.s., we say that σ is a subordinator in the strict sense. The terminology has been introduced by Bochner [11]; see the forthcoming Section 3.2.

Here is a standard example that will be generalized in Section 2.1. Consider a linear Brownian motion $B = (B_t : t \geq 0)$ started at 0, and the first passage times

$$\tau_t \;=\; \inf\{s \geq 0 : B_s > t\}\,, \qquad t \geq 0$$

(it is well-known that $\tau_t < \infty$ for all $t \geq 0$, a.s.). We write $\mathcal{F}_t$ for the complete sigma-field generated by the Brownian motion stopped at time τ_t, viz. $(B_{s\wedge\tau_t} : s \geq 0)$. According to the strong Markov property, $B'_s = B_{s+\tau_t} - t$, $s \geq 0$ is independent of $\mathcal{F}_t$ and is again a Brownian motion. Moreover, it is clear that for every $s \geq 0$

$$\tau_{t+s} - \tau_t \;=\; \inf\{u \geq 0 : B'_u > s\}\,.$$

This shows that $\tau = (\tau_t : t \geq 0)$ is an increasing $(\mathcal{F}_t)$-adapted process with independent and homogeneous increments. Its paths are right-continuous and have an infinite lifetime a.s.; and hence τ is a strict subordinator.

We assume henceforth that σ is a subordinator. The independence and homogeneity of the increments immediately yield the (simple) Markov property: For every fixed $t \geq 0$, conditionally on $\{t < \zeta\}$, the process $\sigma' = (\sigma'_s = \sigma_{s+t} - \sigma_t, s \geq 0)$ is independent of $\mathcal{F}_t$ and has the same law as σ. The simple Markov property can easily be reinforced, i.e. extended to stopping times:

Proposition 1.1 *If T is a stopping time, then, conditionally on $\{T < \zeta\}$, the process $\sigma' = (\sigma'_t = \sigma_{T+t} - \sigma_T, t \geq 0)$ is independent of $\mathcal{F}_T$ and has the same law as σ (under $\mathbb{P}$).*

Proof: For an elementary stopping time, the statement merely rephrases the simple Markov property. If T is a general stopping time, then there exists a sequence of elementary stopping times $(T_n)_{n\in\mathbb{N}}$ that decrease towards T, a.s. For each integer n, conditionally on $\{T_n < \zeta\}$, the shifted process $(\sigma_{T_n+t} - \sigma_{T_n}, t \geq 0)$ is independent of $\mathcal{F}_{T_n}$ (and thus of $\mathcal{F}_T$), and has the same law as σ. Letting $n \to \infty$ and using the right-continuity of the paths, this entails our assertion. $\blacksquare$

The one-dimensional distributions of σ

$$p_t(dy) = \mathbb{P}(\sigma_t \in dy, t < \zeta), \qquad t \geq 0, y \in [0, \infty[$$

thus give rise to a convolution semigroup $(P_t, t \geq 0)$ by

$$P_t f(x) = \int_{[0,\infty[} f(x+y) p_t(dy) = \mathbb{E}\left(f(\sigma_t + x), t < \zeta\right)$$

where f stands for a generic nonnegative Borel function. It can be checked that this semigroup has the Feller property, cf. Proposition I.5 in [1] for details.

A subordinator is a transient Markov process; its potential measure $U(dx)$ is called the *renewal measure*. It is given by

$$\int_{[0,\infty[} f(x) U(dx) = \mathbb{E}\left(\int_0^\infty f(\sigma_t) dt\right).$$

The distribution function of the renewal measure

$$U(x) = \mathbb{E}\left(\int_0^\infty \mathbf{1}_{\{\sigma_t \leq x\}} dt\right), \qquad x \geq 0$$

is known as the *renewal function*. It is immediate to deduce from the Markov property that the renewal function is subadditive, that is

$$U(x+y) \leq U(x) + U(y) \qquad \text{for all } x, y \geq 0.$$

The law of a subordinator is specified by the Laplace transforms of its one-dimensional distributions. To this end, it is convenient to use the convention that $\mathrm{e}^{-\lambda \times \infty} = 0$ for any $\lambda \geq 0$, so that

$$\mathbb{E}\left(\exp\{-\lambda \sigma_t\}, t < \zeta\right) = \mathbb{E}\left(\exp\{-\lambda \sigma_t\}\right), \qquad t, \lambda \geq 0.$$

The independence and homogeneity of the increments then yield the multiplicative property

$$\mathbb{E}\left(\exp\{-\lambda \sigma_{t+s}\}\right) = \mathbb{E}\left(\exp\{-\lambda \sigma_t\}\right) \mathbb{E}\left(\exp\{-\lambda \sigma_s\}\right)$$

for every $s, t \geq 0$. We can therefore express these Laplace transforms in the form

$$\mathbb{E}\left(\exp\{-\lambda \sigma_t\}\right) = \exp\{-t\Phi(\lambda)\}, \qquad t, \lambda \geq 0 \tag{1.1}$$

where the function $\Phi : [0, \infty[\to [0, \infty[$ is called the *Laplace exponent* of σ. Note that the Laplace transform of the renewal measure is

$$\mathcal{L}U(\lambda) = \int_{[0,\infty[} \mathrm{e}^{-\lambda x} U(dx) = \frac{1}{\Phi(\lambda)}, \qquad \lambda > 0,$$

in particular the renewal measure characterizes the law of the subordinator.

For instance, in the case when $\sigma = \tau$ is the first passage process of Brownian motion, the one-dimensional distributions are given by

$$p_t(dy) = \frac{t}{\sqrt{2\pi y^3}} \exp\left(-\frac{t^2}{2y}\right) dy,$$

and the characteristic exponent by $\Phi(q) = \sqrt{2q}$.

1.3 The Lévy-Khintchine formula

The next theorem gives a necessary and sufficient analytic condition for a function to be the Laplace exponent of a subordinator.

Theorem 1.2 (de Finetti, Lévy, Khintchine)(i) *If* Φ *is the Laplace exponent of a subordinator, then there exist a unique pair* $(\mathtt{k}, \mathtt{d})$ *of nonnegative real numbers and a unique measure* Π *on* $]0,\infty[$ *with* $\int (1 \wedge x)\, \Pi(dx) < \infty$*, such that for every* $\lambda \geq 0$

$$\Phi(\lambda) \;=\; \mathtt{k} + \mathtt{d}\lambda + \int_{]0,\infty[} \left(1 - \mathrm{e}^{-\lambda x}\right) \Pi(dx)\,. \tag{1.2}$$

(ii) *Conversely, any function* Φ *that can be expressed in the form* (1.2) *is the Laplace exponent of a subordinator.*

Equation (1.2) will be referred to as the *Lévy-Khintchine formula*; one calls $\mathtt{k}$ the *killing rate*, $\mathtt{d}$ the *drift coefficient* and Π the *Lévy measure* of σ. It is sometimes convenient to perform an integration by parts and rewrite the Lévy-Khintchine formula as

$$\Phi(\lambda)/\lambda \;=\; \mathtt{d} + \int_0^\infty \mathrm{e}^{-\lambda x}\overline{\Pi}(x)dx\,, \qquad \text{with } \overline{\Pi}(x) \;=\; \mathtt{k} + \Pi\left(]x,\infty[\right)\,.$$

We call $\overline{\Pi}$ the *tail of the Lévy measure.* Note that the killing rate and the drift coefficient are given by

$$\mathtt{k} \;=\; \Phi(0) \quad , \quad \mathtt{d} \;=\; \lim_{\lambda\to\infty} \frac{\Phi(\lambda)}{\lambda}\,.$$

In particular, the lifetime ζ has an exponential distribution with parameter $\mathtt{k} \geq 0$ ($\zeta \equiv \infty$ for $\mathtt{k} = 0$).

Before we proceed to the proof of Theorem 1.2, we present some well-known examples of subordinators. The simplest is the Poisson process with intensity $c > 0$, which corresponds to the Laplace exponent

$$\Phi(\lambda) \;=\; c(1 - \mathrm{e}^{-\lambda})\,,$$

that is the killing rate $\mathtt{k}$ and the drift coefficient $\mathtt{d}$ are zero and the Lévy measure $c\delta_1$, where δ_1 stands for the Dirac point mass at 1. Then the so-called standard stable subordinator with index $\alpha \in]0,1[$ has a Laplace exponent given by

$$\Phi(\lambda) \;=\; \lambda^\alpha \;=\; \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - \mathrm{e}^{-\lambda x}) x^{-1-\alpha} dx\,.$$

The restriction on the range of the index is due to the requirement $\int (1 \wedge x)\, \Pi(dx) < \infty$. The boundary case $\alpha = 1$ is degenerate since it corresponds to the deterministic process $\sigma_t \equiv t$, and is usually implicitly excluded. A third family of examples is provided by the Gamma processes with parameters $a, b > 0$, for which the Laplace exponent is

$$\Phi(\lambda) = a \log(1 + \lambda/b) = \int_0^\infty (1 - \mathrm{e}^{-\lambda x}) a x^{-1} \mathrm{e}^{-bx} dx\,.$$

We see that the Lévy measure is $\Pi^{(a,b)}(dx) = ax^{-1}\mathrm{e}^{-bx}dx$ and the killing rate and the drift coefficient are zero.

Proof of Theorem 1.2: (i) Making use of the independence and homogeneity of the increments in the second equality below, we get from (1.1) that for every $\lambda \geq 0$

$$\begin{aligned}\Phi(\lambda) \;=\; \lim_{n\to\infty} n\left(1-\exp\{-\Phi(\lambda)/n\}\right) \;&=\; \lim_{n\to\infty} n\mathbb{E}\left(1-\exp\{-\lambda\sigma_{1/n}\}\right)\\ &=\; \lambda\lim_{n\to\infty}\int_0^\infty \mathrm{e}^{-\lambda x} n\mathbb{P}\left(\sigma_{1/n}\geq x\right)dx\,.\end{aligned}$$

Write $\overline{\Pi}_n(x) = n\mathbb{P}\left(\sigma_{1/n}\geq x\right)$, so that

$$\frac{\Phi(\lambda)}{\lambda} \;=\; \lim_{n\to\infty}\int_0^\infty \mathrm{e}^{-\lambda x}\overline{\Pi}_n(x)dx\,.$$

This shows that the sequence of absolutely continuous measures $\overline{\Pi}_n(x)dx$ converges vaguely as $n\to\infty$. As each function $\overline{\Pi}_n(\cdot)$ decreases, the limit has necessarily the form $\mathtt{d}\delta_0(dx)+\overline{\Pi}(x)dx$, where $\mathtt{d}\geq 0$, $\overline{\Pi}:]0,\infty[\to[0,\infty[$ is a non-increasing function, and δ_0 stands for the Dirac point mass at 0. Thus

$$\frac{\Phi(\lambda)}{\lambda} \;=\; \mathtt{d} + \int_0^\infty \mathrm{e}^{-\lambda x}\overline{\Pi}(x)dx$$

and this yields (1.2) with $\mathtt{k} = \overline{\Pi}(\infty)$ and $\Pi(dx) = -d\overline{\Pi}(x)$ on $]0,\infty[$. It is plain that we must have $\int_{]0,1[} x\Pi(dx) < \infty$ since otherwise $\Phi(\lambda)$ would be infinite. Uniqueness is obvious.

(ii) Consider a Poisson point process $\Delta = (\Delta_t, t\geq 0)$ valued in $]0,\infty[$ with characteristic measure Π, and introduce an independent time ζ that is exponentially distributed with parameter $\mathtt{k}$ (with the convention that $\zeta\equiv\infty$ when $\mathtt{k}=0$). Then define $\Sigma = (\Sigma_t, t\geq 0)$ by

$$\Sigma_t \;=\; \begin{cases} \mathtt{d}t + \sum_{0\leq s\leq t}\Delta_s & \text{if } t<\zeta\\ \infty & \text{otherwise.}\end{cases}$$

The condition $\int(1\wedge x)\,\Pi(dx) < \infty$ ensures that $\Sigma_t < \infty$ whenever $t<\zeta$, a.s. It is plain that Σ is a right-continuous increasing process started at 0, with lifetime ζ, and that its increments are stationary and independent on $[0,\zeta[$. In other words, Σ is a subordinator. Finally, the exponential formula for a Poisson point process yields for every $t,\lambda\geq 0$

$$\mathbb{E}\left(\exp\{-\lambda\Sigma_t\}\right) \;=\; \exp\left\{-t\left(\mathtt{k}+\mathtt{d}\lambda+\int_{]0,\infty[}(1-\mathrm{e}^{-\lambda x})\Pi(dx)\right)\right\}\,,$$

which shows that the Laplace exponent of Σ is given by (1.2). ■

More precisely, the proof of (ii) contains relevant information on the canonical decomposition of a subordinator as the sum of its continuous part and its jumps. The following important result is known as the Lévy-Itô decomposition.

Proposition 1.3 (Itô [19]) *One has a.s., for every $t \in [0, \zeta[$:*

$$\sigma_t \;=\; \mathtt{d}t + \sum_{0 \le s \le t} \Delta_s \,,$$

where $\Delta = (\Delta_s, s \ge 0)$ is a Poisson point process with values in $]0, \infty[$ and characteristic measure Π, which is independent of the lifetime ζ.

Sometimes it may be convenient to formulate slightly differently Proposition 1.3 by saying that the jump process $\Delta\sigma$ of σ is a Poisson point process on $]0, \infty]$ with characteristic measure $\Pi + \mathtt{k}\delta_\infty$, stopped when it takes the value ∞.

As a consequence, we see that a subordinator is a step process if its drift coefficient is $\mathtt{d} = 0$ and its Lévy measure has a finite mass, $\Pi(]0, \infty[) < \infty$ (this is also equivalent to the boundedness of the Laplace exponent). Otherwise σ is a strictly increasing process. In the first case, we say that σ is a *compound Poisson process*. A compound Poisson process can be identified as a random walk time-changed by an independent Poisson process; and in many aspects, it can be thought of as a process in discrete time. Because we are mostly concerned with 'truly' continuous time problems, it will be more convenient to concentrate on strictly increasing subordinators in the sequel.

Henceforth, the case when σ is a compound Poisson process is implicitly excluded.

As a consequence, if we introduce the inverse L of the strictly increasing process σ,

$$L_x \;=\; \sup\{t \ge 0 : \sigma_t \le x\} \;=\; \inf\{t > 0 : \sigma_t > x\} \,, \qquad x \ge 0 \,, \tag{1.3}$$

one gets a process with continuous sample paths which will play an important role in this text. We shall refer to L as the local time. Note in particular that the renewal function gives the first moments of the local time,

$$U(x) \;=\; \mathbb{E}\left(\int_0^\infty \mathbf{1}_{\{\sigma_t \le x\}} dt\right) \;=\; \mathbb{E}(L_x) \,.$$

1.4 Law of the iterated logarithm

We continue the study of a subordinator σ and its inverse L by presenting a remarkable law of the iterated logarithm.

Theorem 1.4 (Fristedt and Pruitt [17]) *There exists a positive and finite constant c_Φ such that*

$$\limsup_{t \to 0+} \frac{L_t \Phi\left(t^{-1} \log\log \Phi(t^{-1})\right)}{\log\log \Phi(t^{-1})} \;=\; c_\Phi \qquad a.s.$$

There is also a version of Theorem 1.4 for large times, which follows from a simple variation of the arguments for small times. Specifically, suppose that the killing rate is $\mathtt{k} = 0$. Then there exists $c'_\Phi \in]0, \infty[$ such that

$$\limsup_{t \to \infty} \frac{L_t \Phi\left(t^{-1} \log|\log \Phi(t^{-1})|\right)}{\log|\log \Phi(t^{-1})|} \;=\; c'_\Phi \qquad \text{a.s.} \tag{1.4}$$

The proof of Theorem 1.4 relies on two technical lemmas. We write

$$f(t) \;=\; \frac{\log\log\Phi(t^{-1})}{\Phi\left(t^{-1}\log\log\Phi(t^{-1})\right)}\,, \qquad t \text{ small enough,}$$

and denote the inverse function of Φ by φ.

Lemma 1.5 *For every integer $n \geq 2$, put*

$$t_n \;=\; \frac{\log n}{\varphi(\mathrm{e}^n \log n)} \quad , \quad a_n = f(t_n)\,.$$

(i) *The sequence $(t_n : n \geq 2)$ decreases, and we have $a_n \sim \mathrm{e}^{-n}$.*

(ii) *The series $\Sigma\mathbb{P}\left(L_{t_n} > 3a_n\right)$ converges*

Proof: (i) The first assertion follows readily from the fact that φ is convex and increasing. On the one hand, since Φ increases, we have for $n \geq 3$

$$\Phi(t_n^{-1}) \;=\; \Phi(\varphi(\mathrm{e}^n \log n)/\log n) \;\leq\; \Phi(\varphi(\mathrm{e}^n \log n)) \;=\; \mathrm{e}^n \log n\,.$$

On the other hand, since Φ is concave, we have for $n \geq 3$

$$\Phi(t_n^{-1}) \;=\; \Phi(\varphi(\mathrm{e}^n \log n)/\log n) \;\geq\; \Phi(\varphi(\mathrm{e}^n \log n))/\log n \;=\; \mathrm{e}^n\,.$$

This entails

$$\log\log\Phi(t_n^{-1}) \;\sim\; \log n \tag{1.5}$$

and then

$$t_n^{-1}\log\log\Phi(t_n^{-1}) \;\sim\; \varphi(\mathrm{e}^n \log n)\,.$$

Note that if $\alpha_n \sim \beta_n$, then $\Phi(\alpha_n) \sim \Phi(\beta_n)$ (because Φ is concave and increasing). We deduce that

$$\Phi\left(t_n^{-1}\log\log\Phi(t_n^{-1})\right) \;\sim\; \mathrm{e}^n \log n\,, \tag{1.6}$$

and our assertion follows from (1.5).

(ii) The probability of the event $\{L_{t_n} > 3a_n\} = \{\sigma_{3a_n} < t_n\}$ is bounded from above by

$$\exp\{\lambda t_n\}\mathbb{E}\left(\exp\{-\lambda\sigma_{3a_n}\}\right) \;=\; \exp\{\lambda t_n - 3a_n\Phi(\lambda)\}$$

for every $\lambda \geq 0$. We choose $\lambda = \varphi(\mathrm{e}^n \log n)$; so $\Phi(\lambda) = \mathrm{e}^n \log n$ and $\lambda t_n = \log n$. Our statement follows now from (i). ■

Lemma 1.6 *For every integer $n \geq 2$, put*

$$s_n \;=\; \frac{2\log n}{\varphi(2\exp\{n^2\}\log n)} \quad , \quad b_n = f(s_n)\,.$$

(i) *We have $b_n \sim \exp\{-n^2\}$.*

(ii) *The series $\Sigma\mathbb{P}\left(\sigma(b_n/3) < 2s_n/3\right)$ diverges*

Proof: (i) Just note that $s_n = t_{n^2}$ and apply Lemma 1.5(i).
(ii) For every b, s and $\lambda \geq 0$, we have

$$\mathbb{P}(\sigma_b \geq s) \leq \left(1 - \mathrm{e}^{-\lambda s}\right)^{-1} \mathbb{E}\left(1 - \exp\{-\lambda \sigma_b\}\right) ,$$

which entails

$$\mathbb{P}(\sigma_b < s) \geq \frac{\mathrm{e}^{-b\Phi(\lambda)} - \mathrm{e}^{-\lambda s}}{1 - \mathrm{e}^{-\lambda s}} . \tag{1.7}$$

Apply this to $b = b_n/3$, $s = 2s_n/3$ and $\lambda = \varphi(2\exp\{n^2\}\log n)$, and observe that then $\Phi(\lambda) = 2\exp\{n^2\}\log n$, $\lambda s = \frac{4}{3}\log n$ and $b\Phi(\lambda) \sim \frac{2}{3}\log n$ (by (i)). In particular $\mathrm{e}^{-b\Phi(\lambda)} \geq n^{-3/4}$ for every sufficiently large n; we thus obtain

$$2\mathbb{P}\left(\sigma(b_n/3) < 2s_n/3\right) \geq \frac{n^{-3/4} - n^{-4/3}}{1 - n^{-4/3}} ,$$

and our claim follows. ■

We are now able to establish the law of the iterated logarithm, using a standard method based on the Borel-Cantelli lemma.

Proof of Theorem 1.4: 1. To prove the upper-bound, we use the notation of Lemma 1.5. Take any $t \in [t_{n+1}, t_n]$, so, provided that n is large enough

$$f(t) \geq \frac{\log\log\Phi(t_n^{-1})}{\Phi(t_{n+1}^{-1}\log\log\Phi(t_{n+1}^{-1}))}$$

(because Φ increases). By (1.5), the numerator is equivalent to $\log n$, and, by (1.6), the denumerator to $\mathrm{e}^{n+1}\log(n+1)$. By Lemma 1.5, we thus have

$$\limsup_{t\to 0+} f(t_n)/f(t) \leq \mathrm{e} .$$

On the other hand, an application of the Borel-Cantelli to Lemma 1.5 shows that

$$\limsup_{n\to\infty} L_{t_n}/f(t_n) \leq 3 \qquad a.s.$$

and we deduce that

$$\limsup_{t\to 0+} \frac{L_t}{f(t)} \leq \left(\limsup_{n\to\infty} \frac{L_{t_n}}{f(t_n)}\right)\left(\limsup_{t\to 0+} \frac{f(t_n)}{f(t)}\right) \leq 3\mathrm{e} \qquad a.s.$$

2. To prove the lower-bound, we use the notation of Lemma 1.6 and observe that the sequence $(b_n, n \geq 2)$ decreases ultimately (by Lemma 1.6(i)). First, by Lemma 1.6(ii), we have

$$\sum \mathbb{P}\left(\sigma(b_n/3) - \sigma(b_{n+1}/3) < 2s_n/3\right) \geq \sum \mathbb{P}\left(\sigma(b_n/3) < 2s_n/3\right) = \infty ;$$

so by the Borel-Cantelli lemma for independent events,

$$\liminf_{n\to\infty} \frac{\sigma(b_n/3) - \sigma(b_{n+1}/3)}{s_n} \le \frac{2}{3}\,.$$

If we admit for a while that

$$\limsup_{n\to\infty} \frac{\sigma(b_{n+1}/3)}{s_n} \le \frac{1}{4}\,, \tag{1.8}$$

we can conclude that

$$\liminf_{n\to\infty} \frac{\sigma(b_n/3)}{s_n} < \frac{11}{12}\,.$$

This implies that the set $\{s : \sigma(f(s)/3) < s\}$ is unbounded a.s. Plainly, the same then holds for $\{s : L_s > f(s)/3\}$, and as a consequence:

$$\limsup_{t\to 0+} L_t/f(t) \ge 1/3 \qquad a.s. \tag{1.9}$$

Now we establish (1.8). The obvious inequality (which holds for any $\lambda > 0$)

$$\mathbb{P}\left(\sigma(b_{n+1}/3) > s_n/4\right) \le \left(1 - \exp\{-\lambda s_n/4\}\right)^{-1} \mathbb{E}\left(1 - \exp\{-\lambda\sigma(b_{n+1}/3)\}\right)$$

entails for the choice

$$\lambda = \varphi(2\exp\{n^2\}\log n) = \frac{2\log n}{s_n}$$

that

$$\mathbb{P}\left(\sigma(b_{n+1}/3) > s_n/4\right) \le \frac{2b_{n+1}\exp\{n^2\}\log n}{3\left(1 - \exp\{-\frac{1}{2}\log n\}\right)}\,.$$

By Lemma 1.6(i), the numerator is bounded from above for every sufficiently large n by

$$3\exp\{n^2 - (n+1)^2\}\log n \le \mathrm{e}^{-n}$$

and the denumerator is bounded away from 0. We deduce that the series

$$\sum \mathbb{P}\left(\sigma(b_{n+1}/3) > s_n/4\right)$$

converges, and the Borel-Cantelli lemma entails (1.8). The proof of (1.9) is now complete.

3. The two preceding parts show that

$$\limsup_{t\to 0+} L_t/f(t) \in [1/3, 3\mathrm{e}] \qquad a.s.$$

By the Blumenthal zero-one law, it must be a constant number c_Φ, a.s. ■

To conclude this section, we mention that the independence and homogeneity of the increments of the inverse local time are also very useful in investigating the class of lower functions for the local time. We now state the main result in that field (see e.g. Section III.4 in [1] for the proofs of variations of these results stated in terms of the subordinator σ.).

Proposition 1.7 (i) *When* $\mathtt{d} > 0$, *one has* $\lim_{t\to 0+} L_t/t = 1/\mathtt{d}$ *a.s.*

(ii) *When* $\mathtt{d} = 0$ *and* $f : [0,\infty[\to [0,\infty[$ *is an increasing function such that* $t \to f(t)/t$ *decreases, one has*

$$\liminf_{t\to 0+} L_t/f(t) \;=\; 0 \quad a.s. \quad \Longleftrightarrow \quad \int_{0+} f(x)\Pi(dx) \;=\; \infty\,.$$

Moreover, if these assertions fail, then $\lim_{t\to 0+} L_t/f(t) = \infty$ *a.s.*

1.5 Renewal theory for regenerative sets

We now turn our attention to the study of the closed range of a subordinator

$$\mathcal{R} \;=\; \{\sigma_t, t \geq 0\}^{\mathrm{cl}}\;.$$

In this direction, introduce the left and right extremities of $\mathcal{R}$ as viewed from a fixed point $t \geq 0$:

$$g_t \;=\; \sup\{s < t : s \in \mathcal{R}\} \quad \text{ and } \quad D_t \;=\; \inf\{s > t : s \in \mathcal{R}\}\;.$$

We call $(D_t : t \geq 0)$ and $(g_t : t > 0)$ the processes of first-passage and last-passage in $\mathcal{R}$, respectively. We immediately check that these processes can be expressed in terms of σ and its inverse L as follows :

$$g_t \;=\; \sigma(L_t-) \quad \text{ and } \quad D_t \;=\; \sigma(L_t) \qquad \text{for all } t \geq 0, \text{ a.s.} \tag{1.10}$$

The strong Markov property has a remarkable consequence for the range $\mathcal{R}$, which usually referred to as the *regenerative property*. Specifically, note that for every $s \geq 0$, $L_s = \inf\{t \geq 0 : \sigma_t > s\}$ is an $(\mathcal{F}_t)$-stopping time, and the sigma-fields $(\mathcal{M}_s = \mathcal{F}_{L_s})_{s\geq 0}$ thus form a filtration. An application of the Markov property at L_s shows that, conditionally on $\{L_s < \infty\}$, the shifted subordinator $\sigma' = \{\sigma_{L_s+t} - \sigma_{L_s}, t \geq 0\}$ is independent of $\mathcal{M}_s$ and has the same law as σ. It thus follows from (1.10) that conditionally on $\{D_s < \infty\}$, the shifted range

$$\mathcal{R}\circ\theta_{D_s} \;=\; \{v \geq 0 : v + D_s \in \mathcal{R}\} \;=\; \{\sigma'_t : t \geq 0\}^{\mathrm{cl}}$$

is independent of $\mathcal{M}_s$ and is distributed as $\mathcal{R}$. We stress that this regenerative property of $\mathcal{R}$ holds more generally at any $(\mathcal{M}_s)$-stopping time S which takes values in the subset of points in $\mathcal{R}$ which are not isolated on their right, a.s. on $\{S < \infty\}$. These observations have motivated many studies; see in particular [18] and also [16] and the references therein for much more on this topic.

First, we observe that the drift coefficient of σ is related to the Lebesgue measure[1] of its range $\mathcal{R}$.

[1]When the drift coefficient is zero, a remarkable result due to Fristedt and Pruitt [17] relates the local time to some Hausdorff measure on $\mathcal{R}$.

Proposition 1.8 *We have*

$$m(\mathcal{R} \cap [0,t]) = \mathtt{d} L_t \qquad \textit{a.s. for all } t \geq 0\,,$$

where $\mathtt{d}$ *is the drift coefficient and* m *the Lebesgue measure on* $[0, \infty[$*. In particular* $\mathcal{R}$ *has zero Lebesgue measure a.s. if and only if* $\mathtt{d} = 0$*, and we then say that* $\mathcal{R}$ *is light. Otherwise we say that* $\mathcal{R}$ *is heavy.*

Proof: There is no loss of generality in assuming that the killing rate is $\mathtt{k} = 0$ (i.e. $\zeta = \infty$ a.s.), because the case $\mathtt{k} > 0$ then will then follow by introducing a killing at some independent time. The canonical decomposition of the open set $\mathcal{R}^c = [0, \infty[\backslash \mathcal{R}$ is given by

$$\mathcal{R}^c = \bigcup_{s \in \mathcal{J}}]\sigma_{s-}, \sigma_s[\,, \tag{1.11}$$

where $\mathcal{J} = \{0 \leq s \leq \zeta : \Delta_s > 0\}$ denotes the set of jump times of σ. In particular, for every fixed $t \geq 0$, the Lebesgue measure of $\mathcal{R}^c \cap [0, \sigma_t]$ is $\sum_{s \leq t} \Delta_s$, and the latter quantity equals $\sigma_t - \mathtt{d}t$ by virtue of Proposition 1.3. This entails $m([0, \sigma_t] \cap \mathcal{R}) = \mathtt{d}t$ for all $t \geq 0$, a.s. Because $\mathcal{R} \cap]\sigma_{t-}, \sigma_t]$ reduces to the singleton $\{\sigma_t\}$, we have also

$$m([0, \sigma_t] \cap \mathcal{R}) = m([0, \sigma_{t-}] \cap \mathcal{R}) = \mathtt{d}t\,.$$

Replacing t by L_t and recalling that $t \in [\sigma_{L_t -}, \sigma_{L_t}]$ completes the proof. ∎

We stress that the identity (1.11) relates the lengths of the intervals in the canonical decomposition of $\mathcal{R}^c$ to the sizes of the jumps of σ, and hence the Lévy-Itô decomposition stated in Proposition 1.3 depicts the former in terms of a Poisson point process with characteristic measure Π.

Next, we turn our attention to the probability that $x \in \mathcal{R}$ for an arbitrary fixed $x > 0$, which is given by the following remarkable result.

Theorem 1.9 (i) (Kesten [20]) *If the drift is* $\mathtt{d} = 0$*, then* $\mathbb{P}(x \in \mathcal{R}) = 0$ *for every* $x > 0$*.*
(ii) (Neveu [27]) *If* $\mathtt{d} > 0$*, then the function* $u(x) = \mathtt{d}^{-1} \mathbb{P}(x \in \mathcal{R})$ *is the version of the renewal density* $dU(x)/dx$ *that is continuous and everywhere positive on* $[0, \infty[$*.*

We refer to Section III.2 in [1] for a detailed proof; let us merely present some elementary observations relating Proposition 1.8 to Theorem 1.9. First, one readily deduces from the former and Fubini's theorem that when the drift coefficient is $\mathtt{d} = 0$, then the Lebesgue measure of $\{x > 0 : \mathbb{P}(x \in \mathcal{R}) > 0\}$ is zero. Bretagnolle [12] for has an elegant argument showing that this set is in fact necessarily empty. Second, it follows again from Proposition 1.8 and Fubini's theorem that when $\mathtt{d} > 0$

$$U(x) = \mathbb{E}(L_x) = \mathbb{E}\left(\frac{1}{d}\int_0^x \mathbf{1}_{\{y \in \mathcal{R}\}} dy\right) = \frac{1}{d}\int_0^x \mathbb{P}(y \in \mathcal{R}) dy\,,$$

which explains Theorem 1.9(ii).

We next present an useful expression for the distribution of the pair (g_t, D_t) in terms of the renewal function and the tail of the Lévy measure.

Lemma 1.10 *For every real numbers a, b, t such that $0 \le a < t \le a+b$, we have*

$$\mathbb{P}\left(g_t \in da, D_t - g_t \in db\right) \;=\; \Pi(db)U(da) \quad , \quad \mathbb{P}\left(g_t \in da, D_t = \infty\right) \;=\; \mathtt{k}U(da)\,.$$

In particular, we have for $a \in [0, t[$

$$\mathbb{P}(g_t \in da) \;=\; \overline{\Pi}(t-a)U(da)\,.$$

Proof: Recall from (1.10) the identities $g_t = \sigma_{L_t -}$ and $D_t - g_t = \Delta_{L_t}$. Then observe that for any $u > 0$

$$\sigma_{L_t-} < a \;\text{ and }\; L_t = u \iff \sigma_{u-} < a \;\text{ and }\; \sigma_u \ge t\,.$$

Using the canonical expression of σ given in Proposition 1.3, we see that

$$\mathbb{P}\left(g_t < a, D_t - g_t \ge b\right) \;=\; \mathbb{E}\left(\sum \mathbf{1}_{\{\sigma_{u-} < a\}} \mathbf{1}_{\{\Delta_u \ge (t - \sigma_{u-}) \vee b\}}\right)\,,$$

where the sum in the right-hand side is taken over all the instants when the point process Δ jumps. The process $u \to \sigma_{u-}$ is left continuous and hence predictable, so the compensation formula entails that the right-hand-side in the last displayed formula equals

$$\mathbb{E}\left(\int_0^\infty \mathbf{1}_{\{\sigma_u < a\}} \overline{\Pi}\left(((t-\sigma_u) \vee b)-\right) du\right) \;=\; \int_{[0,a[} \overline{\Pi}(((t-x) \vee b)-)\, U(dx)\,.$$

This shows that for $0 \le a < t < a+b$

$$\mathbb{P}(g_t \in da, D_t - g_t \in db) \;=\; \Pi(db)U(da)\,.$$

Integrating this when b ranges over $[t-a, \infty]$ yields $\mathbb{P}(g_t \in da) = \overline{\Pi}((t-a)-)U(da)$. Since the renewal measure has no atom and the tail of the Lévy measure has at most countably many discontinuities, we may replace $\overline{\Pi}((t-a)-)$ by $\overline{\Pi}(t-a)$. ■

A possible drawback of Lemma 1.10 is that it is not expressed explicitly in terms of the Laplace exponent Φ. Considering Laplace transform easily yields the following formula.

Lemma 1.11 *For every $a, b, c, q > 0$*

$$q\int_0^\infty \mathrm{e}^{-qt}\mathbb{E}\left(\exp\{-ag_t - bL_t - c(D_t - g_t)\}\right) dt \;=\; \frac{\Phi(c+q) - \Phi(c)}{\Phi(a+q) + b}\,.$$

Proof: For the sake of simplicity, we shall assume that there is no killing (i.e. $\mathtt{k} = 0$) and we first focus on the light case, that is when the range of σ has zero Lebesgue measure a.s. (see Proposition 1.8). Because for every $t \in]\sigma_{s-}, \sigma_s[$ we have $g_t = \sigma_{s-}$, $D_t = \sigma_s$ and $L_t = s$, we deduce that

$$\begin{aligned}
& q\int_0^\infty \mathrm{e}^{-qt}\mathbb{E}\left(\exp\{-ag_t - bL_t - c(D_t - g_t)\}\right) dt \\
= \;& \mathbb{E}\left(\sum_{s \ge 0} \exp\{-a\sigma_{s-} - bs - c\Delta_s\} \int_{\sigma_{s-}}^{\sigma_s} q\mathrm{e}^{-qt} dt\right) \\
= \;& \mathbb{E}\left(\sum_{s \ge 0} \exp\{-(a+q)\sigma_{s-} - bs\} \left(\mathrm{e}^{-c\Delta_s} - \mathrm{e}^{-(c+q)\Delta_s}\right)\right)
\end{aligned}$$

To calculate the right-hand side, we recall from the Lévy-Itô decomposition (Proposition 1.3) that the jump process $\Delta = (\Delta_s, s \geq 0)$ is a Poisson point process with characteristic measure the Lévy measure Π of σ, and we apply the compensation formula. We get

$$\begin{aligned} & q \int_0^\infty \mathrm{e}^{-qt} \mathbb{E}\left(\exp\{-a g_t - b L_t - c(D_t - g_t)\}\right) dt \\ = \quad & \mathbb{E}\left(\int_0^\infty \exp\{-(a+q)\sigma_{s-} - bs\}\, ds\right) \int_{]0,\infty[} (\mathrm{e}^{-cx} - \mathrm{e}^{-(c+q)x}) \Pi(dx)\,. \end{aligned}$$

The first term in the product is equal to $1/(b+\Phi(a+q))$, and the second to $\Phi(c+q)-\Phi(c)$ by the Lévy-Khintchine formula.

This establishes the lemma in the light case. In the heavy case when $\mathtt{d} > 0$, one has to take into account an additional term, namely

$$q \int_0^\infty \mathrm{e}^{-qt} \mathbb{E}\left(\exp\{-a g_t - b L_t\}, t \in \mathcal{R}\right) dt \;=\; q \mathbb{E}\left(\int_0^\infty \mathrm{e}^{-(a+q)t} \exp\{-b L_t\} \mathbf{1}_{\{t \in \mathcal{R}\}}\right) dt\,,$$

which can be evaluated using Proposition 1.8. More precisely, we can rewrite the right-hand side as

$$\mathtt{d} q \mathbb{E}\left(\int_0^\infty \mathrm{e}^{-(a+q)t} \exp\{-b L_t\} dL_t\right)\,,$$

and then use the change of variable $L_t = s$ to get

$$\mathtt{d} q \int_0^\infty \mathrm{e}^{-bs} \mathbb{E}\left(\exp -(a+q)\sigma_s\right) ds \;=\; \frac{\mathtt{d} q}{b + \Phi(a+q)}\,.$$

This yields the desired formula in the heavy case. ■

The explicit expressions given in Lemmas 1.10 and 1.11 are the key to the following well-known limit theorems for the so-called age $t - g_t$ and residual lifetime $D_t - t$. We first state the renewal theorem.

Theorem 1.12 *Suppose that σ has finite expectation (in particular the killing rate is $\mathtt{k} = 0$),*

$$\mathbb{E}(\sigma_1) \;=\; \mathtt{d} + \int_0^\infty \overline{\Pi}(x) dx \;=\; \mathtt{d} + \int_{]0,\infty[} x \Pi(dx) \;:=\; \mu \in]0,\infty[\,.$$

Then $(t - g_t, D_t - t)$ converges in distribution as $t \to \infty$ to $(VZ, (1-V)Z)$ where the variables V and Z are independent, V is uniformly distributed on $[0,1]$ and

$$\mathbb{P}(Z \in dz) \;=\; \mu^{-1}\left(\mathtt{d}\delta_0(dz) + z\Pi(dz)\right)\,, \qquad z \geq 0\,,$$

where δ_0 stands for the Dirac point mass at 0. In particular, the probability measure

$$\mu^{-1}\left(\mathtt{d}\delta_0(dx) + \overline{\Pi}(x) dx\right)$$

on $[0,\infty[$ is the stationary law for both the age and the residual lifetime processes.

The second limit theorem determines the asymptotic behavior in distribution of the age and residual lifetime processes in the infinite mean case.

Theorem 1.13 (Dynkin [13] and Lamperti [22]) *Suppose that* $\mathbb{E}(t-g_t) \sim \alpha t$ *as* $t \to \infty$ *(or equivalently that the Laplace exponent* Φ *is regularly varying at* 0 *with index* $1-\alpha$*) for some* $\alpha \in]0,1[$*. Then for* $0 < x < 1$ *and* $y > 0$*, one has*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{t-g_t}{t} \in dx\,,\, \frac{D_t - t}{t} \in dy\right) \,=\, \frac{\alpha \sin \pi\alpha}{\pi}\,(1-x)^{\alpha-1}\,(x+y)^{-1-\alpha}\; dxdy\,.$$

In particular

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{t-g_t}{t} \in dx\right) \,=\, \frac{\sin \pi\alpha}{\pi} x^{-\alpha}\,(1-x)^{\alpha-1}\; dx\,,$$

and

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{D_t-t}{t} \in dy\right) \,=\, \frac{\sin \pi\alpha}{\pi} y^{-\alpha}(1+y)^{-1} dy\,.$$

We refer to section 3 in [3] for a proof, and results on the pathwise asymptotic behavior of the age process.

1.6 Connection with Markov processes

Write $\mathbb{D}$ for the space of càdlàg paths valued in some Polish space E, endowed with Skorohod's topology. Let $X = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \theta_t, \mathbf{P}^x)$ be a strong Markov process with sample paths in $\mathbb{D}$. As usual, $\mathbf{P}^x$ refers to its law started at x, θ_t for the shift operator and $(\mathcal{M}_t)_{t\geq 0}$ for the filtration. We refer to Blumenthal and Getoor [9] for background.

A point r of the state space is *regular for itself* if

$$\mathbf{P}^r\,(T_r = 0) \,=\, 1\,,$$

where $T_r = \inf\{t > 0 : X_t = r\}$ is the first hitting time of r. In words, r is regular for itself if the Markov process started at r, returns to r at arbitrarily small times, a.s.

Suppose now that the Markov process starts from some regular point r (i.e. we are working with the probability measure $\mathbf{P}^x$. Blumenthal and Getoor [9] have established that in this case, the closure of the set of times when X returns to r, can be identified as the closed range of some subordinator σ, i.e.

$$\mathcal{R} \,:=\, \{s \geq 0 : X_s = r\}^{\mathrm{cl}} \,=\, \{\sigma_t, t \geq 0\}^{\mathrm{cl}} \qquad \text{a.s.}$$

Moreover, the subordinator σ is essentially determined by X in the sense that if σ' is another subordinator with closed range $\mathcal{R}$, then there is some real number $c >$ such that $\sigma'_t = \sigma_{ct}$ for all $t \geq 0$. The continuous inverse L of σ defined by (1.3) is a continuous additive functional $(L_t, t \geq 0)$ that increases exactly on the set of times $t \geq 0$ for which $X_t = r$. One calls L the local time process at r.

The characteristics of the subordinator σ (killing rate $\mathtt{k}$, drift coefficient $\mathtt{d}$ and Lévy measure Π) have natural probabilistic interpretation in terms of the Markov process. First, recall that X is called transient if $\mathcal{R}$ is bounded a.s., so that

$$r \text{ is a transient state } \iff \mathtt{k} > 0 \iff L_\infty < \infty \text{ a.s.}$$

More precisely, L_∞ has then an exponential distribution with parameter $\mathtt{k}$. In the opposite case, $\mathcal{R}$ is unbounded a.s., and one says that X is recurrent. In this direction, it can be checked that positive recurrence can be characterized as follows:

$$X \text{ is positive recurrent } \iff \mathbb{E}(\sigma_1) < \infty \iff \int_0^\infty \overline{\Pi}(x)\,dx < \infty\,.$$

We conclude this section by presenting a simple criterion to decide whether a point is regular for itself, and in that case, give an explicit expression for the Laplace exponent of the inverse local time. This requires some additional assumption of duality type on the Markov process. Typically, suppose that $X = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \theta_t, \mathbf{P}^x)$ and $\widehat{X} = \left(\Omega, \widehat{\mathcal{M}}, \widehat{\mathcal{M}}_t, \widehat{X}_t, \hat{\theta}_t, \widehat{\mathbf{P}}^x\right)$ are two standard Markov processes with state space E. For every $\lambda > 0$, the λ-resolvent operators of X and $\widehat{X}$ are given by

$$V^\lambda f(x) \,=\, \mathbf{E}^x\left(\int_0^\infty f(X_t)\mathrm{e}^{-\lambda t}dt\right) \quad , \quad \widehat{V}^\lambda f(x) \,=\, \widehat{\mathbf{E}}^x\left(\int_0^\infty f(\widehat{X}_t)\mathrm{e}^{-\lambda t}dt\right)\,, \quad x \in E\,,$$

where $f \geq 0$ is a generic measurable function on E. We recall that $f \geq 0$ is called λ-excessive with respect to $\{V^\alpha\}$ if $\alpha V^{\alpha+\lambda} f \leq f$ for every $\alpha > 0$ and $\lim_{\alpha\to\infty} \alpha V^\alpha f = f$ pointwise.

We suppose that X and $\widehat{X}$ are in duality with respect to some sigma-finite measure ξ. That is, the resolvent operators can be expressed in the form

$$V^\lambda f(x) \,=\, \int_E v^\lambda(x,y) f(y)\xi(dy) \quad , \quad \widehat{V}^\lambda f(x) \,=\, \int_E v^\lambda(y,x) f(y)\xi(dy)\,.$$

Here, $v^\lambda : E \times E \to [0,\infty]$ stands for the version of the resolvent density such that, for every $x \in E$, the function $v^\lambda(\cdot, x)$ is λ-excessive with respect to the resolvent $\{V^\alpha\}$, and the function $v^\lambda(x,\cdot)$ is λ-excessive with respect to the resolvent $\{\widehat{V}^\alpha\}$. Under a rather mild hypothesis on the resolvent density, one has the following simple necessary and sufficient condition for a point to be regular for itself (see e.g. Proposition 7.3 in [10]).

Proposition 1.14 *Suppose that for every $\lambda > 0$ and $y \in E$, the function $x \to v^\lambda(x,y)$ is lower-semicontinuous. Then, for each fixed $r \in E$ and $\lambda > 0$, the following assertions are equivalent:*

(i) *r is regular for itself.*

(ii) *For every $x \in E$, $v^\lambda(x,r) \leq v^\lambda(r,r) < \infty$.*

(iii) *The function $x \to v^\lambda(x,r)$ is bounded and continuous at $x = r$.*

Finally, if these assertions hold, then the Laplace exponent Φ of the inverse local time at r is given by

$$\Phi(\lambda) \,=\, v^1(r,r)/v^\lambda(r,r)\,, \qquad \lambda > 0.$$

In the case when the semigroup of X is absolutely continuous with respect to ξ, the resolvent density can be expressed in the form

$$v^{\lambda}(x,y) = \int_0^{\infty} e^{-\lambda t} p_t(x,y) dt \,.$$

As the Laplace transform of the renewal measure U of the inverse local time at r is $1/\Phi(\lambda)$, a quantity that is proportional to $v^{\lambda}(r,r)$ by Proposition 1.14, we see by Laplace inversion that U is absolutely continuous with respect to the Lebesgue measure, with density u given by

$$u(t) = c p_t(r,r) \,, \qquad t > 0 \,.$$

Chapter 2

Lévy processes with no negative jumps

In this chapter, we develop the bases of fluctuation theory for Lévy processes with no negative jumps. We first treat the simple case of subordinators with a negative drift, and then present the extensions to the general case. Finally, we turn our attention to the so-called two-sided exit problem (i.e. exit from a finite interval).

2.1 Subordinators with a negative drift

Let $\sigma = (\sigma_t, t \geq 0)$ be a strict subordinator (i.e. the killing rate is zero) with zero drift, and consider

$$Y_t \,=\, \sigma_t - ct\,, t \geq 0\,,$$

where $c > 0$ is some constant. The process $Y = (Y_t, t \geq 0)$ is a Lévy process, i.e. it has right continuous paths with limits on the left, and its increments are independent and stationary. More precisely, Y has obviously no negative jumps and its paths have finite variation a.s. Conversely, it can be easily checked that a Lévy process Y with finite variation and no negative jumps can always be expressed in the form $Y'_t = \sigma'_t - c't$ where σ' is a strict subordinator with no drift and $c' \in \mathbb{R}$ some constant, and $c' \leq 0$ if and only if Y' is a subordinator.

Recall that Φ denotes the Laplace exponent of σ, so

$$\mathbb{E}\left(\exp\left\{-qY_t\right\}\right) \,=\, \exp\left\{t\Psi(q)\right\}\,, \qquad q, t \geq 0$$

where $\Psi(q) = cq - \Phi(q)$. We thus have the Lévy-Khintchine representation

$$\Psi(q) \,=\, cq - \int_{]0,\infty[}\left(1 - \mathrm{e}^{-qx}\right)\Pi(dx)$$

where Π is the Lévy measure of σ. Observe also from the Lévy-Itô decomposition of subordinators that the jumps of Y are given by a Poisson point process with characteristic measure Π.

The absence of negative jumps ensures that the infimum process

$$I_t = \inf_{0 \le s \le t} Y_s \,, \qquad t \ge 0$$

has continuous decreasing paths a.s. We write

$$\tau(x) = \inf \{t \ge 0 : Y_t < -x\} \,, \qquad x \ge 0$$

for the first passage times of the increasing process $-I$. Its distribution is characterized by the following.

Proposition 2.1 (i) *The function* $\Psi : [0, \infty[\to \mathbb{R}$ *is strictly convex and* $\lim_{q\to\infty} \Psi(q) = \infty$.

(ii) *Write* $\kappa(0) \ge 0$ *for the largest root* q *to the equation* $\Psi(q) = 0$ *(either* $\kappa(0) > 0$ *and then* 0 *and* $\kappa(0)$ *are the only two solutions, or* 0 *is the unique solution), and* $\kappa : [0, \infty[\to [\kappa(0), \infty[$ *for the inverse function of* Ψ*. Then* $\tau = (\tau(x), x \ge 0)$ *is a subordinator with Laplace exponent* κ*.*

Proof: (i) The convexity follows from Hölder's inequality. Then note that $\mathbb{P}(Y_1 < 0) > 0$ since otherwise Y would have decreasing paths, i.e. $-Y$ would be a subordinator, which has been excluded. This immediately implies $\lim_{\lambda\to\infty} \Psi(\lambda) = \infty$.

(ii) It should be plain that τ has increasing right-continuous paths. The absence of negative jumps ensures that $Y_{\tau(x)} = -x$ on the event $\{\tau(x) < \infty\}$, and one readily derives from the strong Markov property that τ has independent and stationary increments (cf. the argument for the Brownian motion developed at the beginning of Section 1.2). Hence τ is a subordinator.

To determine its Laplace exponent, note from the independence and stationarity of the increments of Y that the process

$$\exp\{-\lambda Y_t - \Psi(\lambda)t\} \,, \qquad t \ge 0 \,,$$

is a martingale. Then take $\lambda = \Psi(q)$ so that $\Psi(\lambda) = q$, and apply the optional sampling theorem at the bounded stopping time $\tau(x) \wedge t$, we get

$$\mathbb{E}(\exp\{-\kappa(q)Y_{\tau(x)\wedge t} - q(\tau(x) \wedge t)\}) = 1.$$

The absence of positive jumps implies that $-\kappa(q)Y_{\tau(x)\wedge t} - q(\tau(x) \wedge t)$ is bounded from above by $x\kappa(q)$, and converges as t tends to ∞ to $\kappa(q)x - q\tau(x)$ on $\{\tau(x) < \infty\}$ and to $-\infty$ on $\{\tau(x) = \infty\}$. We deduce by dominated convergence that

$$\mathbb{E}(\exp\{-q\tau(x)\}, \tau(x) < \infty) = \exp\{-x\kappa(q)\}.$$

■

Let us dwell on some consequences of Proposition 2.1. First, the killing rate of τ is $\kappa(0)$, the largest root of Ψ. As Ψ is strictly convex, $\kappa(0) > 0$ if and only if Ψ has a strictly negative (possibly infinite) derivative at 0. We thus see that

$$\kappa(0) > 0 \iff c > \int_{]0,\infty[} x\Pi(dx) \,.$$

Second, recall from the Lévy-Khintchine formula that the drift coefficient of a subordinator with Laplace exponent Φ is given by $\lim_{q\to\infty}\Phi(q)/q$. It follows that the drift coefficient of the first passage process τ is

$$\lim_{q\to\infty}\kappa(q)/q \;=\; \lim_{q\to\infty} q/\Psi(q) \;=\; 1/c\,.$$

Third, we point out that the so-called reflected Lévy process $Y-I$ has the Markov property (cf. Proposition VI.1 in [1] for a proof), and that $-I$ is a continuous increasing process which increases only on the set of times when $Y-I$ is zero. As the inverse τ of $-I$ is a subordinator, we thus see that $-I$ is a local time at level 0 for the reflected process $Y-I$.

Next, for every $t\geq 0$, we denote by $g_t\in[0,t]$ the (a.s. unique) instant at which Y reaches its overall infimum on the time interval $[0,t]$. The joint distribution of Y_t, I_t and g_t is specified by the following result.

Theorem 2.2 *Introduce T a random time independent of Y and which has an exponential distribution with parameter $q>0$. The pair of random variables (I_T,g_T) and $(Y_T-I_T,T-g_T)$ are independent. More precisely, for every $\alpha,\beta>0$ one has*

$$\mathbb{E}\left(\exp\{-\alpha g_T+\beta I_T\}\right) \;=\; \frac{\kappa(q)}{\kappa(\alpha+q)+\beta}$$

(in particular $-I_T$ has an exponential distribution with parameter $\kappa(q)$), and

$$\mathbb{E}\left(\exp\{-\alpha(T-g_T)-\beta(Y_T-I_T)\}\right) \;=\; \frac{q(\kappa(\alpha+q)-\beta)}{\kappa(q)(q+\alpha-\Psi(\beta))}\,.$$

Proof: On the one hand, as T has an exponential distribution with index q and is independent of $\tau(x)$, we have from Proposition 2.1(ii)

$$\mathbb{E}(\mathrm{e}^{-\alpha\tau(x)},\tau(x)<T) \;=\; \mathbb{E}(\mathrm{e}^{-(\alpha+q)\tau(x)}) \;=\; \mathrm{e}^{-x\kappa(\alpha+q)}\,.$$

On the other hand, the lack of memory of the exponential distribution and the hypothesis that T is independent of Y entails that for every $x\geq 0$, conditionally on the event $\{T>\tau(x)\}$, $T-\tau(x)$ is independent of $\tau(x)$ and has again an exponential distribution with parameter q. Applying the strong Markov property at time $\tau(x)$ yields for every Borel function $f:\mathbb{R}\times\mathbb{R}_+\to[0,\infty[$ and $y>x$

$$\begin{aligned}
&\mathbb{E}\left(f(Y_T-I_T,T-\tau(x))\,\mathrm{e}^{-\alpha\tau(x)},x\leq -I_T<y\right)\\
=\;&\mathbb{E}(\mathrm{e}^{-\alpha\tau(x)},\tau(x)<T)\,\mathbb{E}(f(Y_T-I_T,T),I_T>x-y)\\
=\;&\mathrm{e}^{-x\kappa(\alpha+q)}\,\mathbb{E}(f(Y_T-I_T,T),I_T>x-y)\,.
\end{aligned}$$

Then, fix $n\in\mathbb{N}$ and set $i_n=n[-I_T/n]$ where $[\cdot]$ stands for the integer part. Applying the preceding identity to $x=k/n$ and $y=(k+1)/n$ for every $k\in\mathbb{N}$, we get

$$
\begin{aligned}
& \mathbb{E}\left(f(Y_T - I_T, T-\tau(i_n))\,\mathrm{e}^{-\alpha\tau(i_n)-\beta i_n}\right) \\
= & \sum_{k=0}^{\infty} \mathbb{E}\left(f(Y_T - I_T, T-\tau(k/n))\,\mathrm{e}^{-\alpha\tau(k/n)-\beta k/n}\,, k/n \le I_T < (k+1)/n\right) \\
= & \mathbb{E}(f(Y_T - I_T, T), I_T > -1/n) \sum_{k=0}^{n} \exp\{-(\kappa(\alpha+q)+\beta)k/n\} \\
= & \frac{\mathbb{E}(f(Y_T - I_T, T), I_T > -1/n)}{1-\exp\{-(\kappa(\alpha+q)+\beta)/n\}}\,.
\end{aligned}
$$

Then let $n \to \infty$, so $i_n \to -I_T$ and $\tau(i_n) \to g_T$ a.s. We deduce from above that $(Y_T - I_T, T - g_T)$ and (I_T, g_T) are independent. On the other hand, it follows from Lemma 1.11 that

$$
\mathbb{E}\left(\exp\{-\alpha g_T + \beta I_T\}\right) \;=\; \frac{\kappa(q)}{\kappa(\alpha+q)+\beta}\,.
$$

Finally, the joint Laplace transform of $(Y_T - I_T, T - g_T)$ can be computed using the decompositions $Y_T = (Y_T - I_T) + I_T$ and $T = (T - g_T) + g_T$. More precisely, we have for every $\alpha < \kappa(q+\beta)$ the identity

$$
\mathbb{E}\left(\mathrm{e}^{-\alpha T - \beta Y_T}\right) \;=\; q\int_0^{\infty} \mathbb{E}\left(\mathrm{e}^{-\alpha t - \beta Y_t}\right)\mathrm{e}^{-qt}dt \;=\; \frac{q}{\alpha + q - \Psi(\beta)}\,.
$$

We now see, using the independence property that

$$
\begin{aligned}
& \mathbb{E}\left(\exp\{-\alpha(T-g_T) - \beta(Y_T - I_T)\}\right) \\
= & \frac{\mathbb{E}\left(\exp\{-\alpha T - \beta Y_T\}\right)}{\mathbb{E}\left(\exp\{-\alpha g_T) + \beta I_T\}\right)} \\
= & \frac{q(\kappa(\alpha+q)-\beta)}{\kappa(q)(q+\alpha-\Psi(\beta))}\,.
\end{aligned}
$$

The extension to $\beta \ge \kappa(\alpha+q)$ is obtain by the standard analytic continuation argument. ■

As an interesting consequence of Theorem 2.2, we present the following identity, due to Zolotarev, for the distribution of the first passage time $\tau(x)$.

Corollary 2.3 *The following identity holds between measures on* $[0,\infty)\times[0,\infty)$*:*

$$
t\mathbb{P}\left(\tau(x)\in dt\right)dx \;=\; x\mathbb{P}\left(-Y_t \in dx\right)dt\,.
$$

Proof: Let T be an independent random time with an exponential distribution with parameter $q > 0$. We know from the independence property stated in Theorem 2.2 that

$$
\mathbb{P}\left(Y_T \in dx\right) \;=\; \int_{y\in]-\infty,0]} \mathbb{P}\left(Y_T - I_T \in y + dx\right)\mathbb{P}\left(I_T \in dy\right)\,.
$$

Moreover, we know that $-I_T$ has then an exponential distribution with parameter $\kappa(q)$, and it follows that whenever $x < 0$, we have

$$\mathbb{P}\left(Y_T \in dx\right) \;=\; \kappa(q)\mathrm{e}^{\kappa(q)x}\mathbb{E}\left(\exp\left\{-\kappa(q)\left(Y_T - I_T\right)\right\}\right) dx\,.$$

The right-hand side can be computed by another appeal to Theorem 2.2 which entails

$$\mathbb{E}\left(\exp\left\{-\kappa(q)\left(Y_T - I_T\right)\right\}\right) \;=\; q\kappa'(q)/\kappa(q)\,,$$

so that finally for $x < 0$

$$\mathbb{P}\left(Y_T \in dx\right) \;=\; q\kappa'(q)\mathrm{e}^{\kappa(q)x}dx\,.$$

Recall from Proposition 2.1-(ii) that $q \to \exp\{\kappa(q)x\}$ is the Laplace transform of $\tau(-x)$, so

$$\int_{t\in[0,\infty[}\mathbb{P}(Y_t \in dx)\mathrm{e}^{-qt}dt \;=\; q^{-1}\mathbb{P}\left(Y_T \in dx\right) \;=\; \frac{dx}{-x}\int_{[0,\infty[}t\mathrm{e}^{-qt}\mathbb{P}\left(\tau(x) \in dt\right)\,.$$

The stated identity thus follows from Laplace inversion. Alternatively, the ballot Theorem (see the forthcoming Lemma 2.6) can be used to give a different proof.

∎

In turn, this identity of Zolotarev entails the Lévy-Khintchine formula for the function the Laplace exponent of the first passage process τ

Corollary 2.4 *Assume that Y has densities, that is that for every $t > 0$ the distribution of Y_t is absolutely continuous. Provided that we can choose a version of the density $y \to p_t(y)$ that is continuous at $y = 0$, then for every $q > 0$, we have*

$$\kappa(q) \;=\; \kappa(0) + q/c + \int_0^\infty \left(1 - \mathrm{e}^{-qt}\right)p_t(0)\frac{dt}{t}\,,$$

where $c > 0$ is the drift coefficient of Y.

Proof: The Lévy-Khintchine formula for subordinators states that

$$\kappa(q) \;=\; \kappa(0) + \mathtt{d}q + \int_{(0,\infty)}\left(1 - \mathrm{e}^{-qt}\right)\nu(dt)$$

where ν is the Lévy measure of τ and the drift coefficient given by $\mathtt{d} = \lim_{\lambda\to\infty}\kappa(\lambda)/\lambda$. It is immediate that the latter coincides with the inverse of the drift coefficient of Y .

So all that is needed is to check that $\nu(dt) = t^{-1}p_t(0)dt$. To that end, we use the fact that the Lévy measure $\nu(dt)$ is the weak limit as $\varepsilon \to 0+$ of $\varepsilon^{-1}\mathbb{P}(\tau(\varepsilon) \in dt)$; see for instance Exercise I.1 in [1]. By Corollary 2.3, the latter is given by $t^{-1}p_t(-\varepsilon)dt$, which entails the claim. ∎

Next, we turn our attention to the supremum process

$$S_t \;=\; \sup\left\{Y_s, 0 \le s \le t\right\}\,, \qquad t \ge 0\,.$$

The following simple and useful result, known as the duality lemma, enables us in particular to determine the joint distribution of S_t and Y_t using Theorem 2.2.

Lemma 2.5 *For every fixed $t > 0$, the time-reversed process*

$$\tilde{Y}_s = Y_t - Y_{(t-s)-}\,, \qquad 0 \le s \le t\,,$$

has the same law as the initial Lévy process $(Y_s, 0 \le s \le t)$. As a consequence, the following identity in distribution holds:

$$(Y_t - S_t, S_t, \gamma_t) \;\stackrel{d}{=}\; (I_t, Y_t - I_t, t - g_t)\ ,$$

where $\gamma_t \in [0,t]$ is the (a.s. unique) instant at which Y reaches its overall supremum on the time interval $[0,t]$.

Proof: The first assertion follows from the observation that the paths of $\tilde{Y}$ are right-continuous with limits on the left, and that its increments are independent and stationary and have the same law as those of Y. We deduce the second assertion from the identities $\tilde{S}_t = Y_t - I_t$, $\tilde{Y}_t - \tilde{S}_t = I_t$ and $\gamma_t = t - \tilde{g}_t$ (in the obvious notation). ■

In particular, we now see that if T is an independent exponential time, say with parameter $q > 0$, then

$$\mathbb{E}\left(\exp\{-\lambda S_T\}\right) \;=\; \frac{q(\kappa(q)-\lambda)}{\kappa(q)(q-\Psi(\lambda))}\,. \tag{2.1}$$

Note that

$$\mathbb{P}\left(S_T = 0\right) \;=\; \mathbb{P}\left(Y_T - I_T = 0\right) \;=\; \lim_{\lambda\to\infty} \mathbb{E}\left(\exp\{-\lambda(Y_T - I_T)\}\right)\ ,$$

and it follows from Theorem 2.2 that the right-hand side equals $q/(c\kappa(q)) > 0$. This shows that with probability one, the first passage time above 0,

$$\ell_1 \;=\; \inf\{t \ge 0 : Y_t > 0\}$$

is a strictly positive random variable. More precisely, its Laplace transform is given by

$$\mathbb{E}\left(\mathrm{e}^{-q\ell_1}\right) \;=\; \mathbb{P}\left(\ell_1 \le T\right) \;=\; 1 - \mathbb{P}(S_T = 0) \;=\; 1 - \frac{q}{c\kappa(q)}\,. \tag{2.2}$$

Note in particular that $\ell_1 < \infty$ a.s. if and only if $\lim_{q\to 0+} q/\kappa(q) = 0$. One easily deduce the equivalence

$$\mathbb{P}\left(Y_t > 0 \text{ for some } t \ge 0\right) \;=\; 1 \iff \int_{]0,\infty[} x\Pi(dx) \ge c\,.$$

More precisely, the joint distribution of ℓ_1, Y_{ℓ_1-} and Y_{ℓ_1} can be specified using the following version of the well-known ballot theorem (see Takács [33]).

Lemma 2.6 *For every $t > 0$, one has*

$$\mathbb{P}\left(\ell_1 > t, Y_t \in dy\right) \;=\; \frac{-y}{ct}\mathbb{P}(Y_t \in dy)\,, \qquad y \in]-\infty, 0]\,.$$

Proof: The argument relies on the easy fact that the process $(t^{-1}\sigma_t, t > 0)$ is a backwards martingale (i.e. a martingale when the time parameter t decreases from ∞ to $0+$); see Proposition III.8 in [1] for a proof). On the one hand, the fact that σ has zero drift entails

$$\mathbb{E}\left(\exp\left\{-t^{-1}\sigma_t\right\}\right) \;=\; \exp\{-t\Phi(1/t)\} \;\to\; 1 \qquad \text{as } t \to 0+\,,$$

so the a.s. limit at $0+$ of $t^{-1}\sigma_t$ is zero a.s.

On the other hand, as σ is positive and has only positive jumps as time increases, the martingale $t^{-1}\sigma_t$ is positive and has only negative jumps as time decreases. In particular, it cannot jump at times when it reaches a new maximum, so a standard application of the optional sampling theorem yields

$$\mathbb{P}\left(s^{-1}\sigma_s < c \text{ for every } 0 < s < t \mid t^{-1}\sigma_t = a\right) \;=\; 1 - a/c\,, \qquad 0 \le a < c\,.$$

This establishes our claim. ∎

Proposition 2.7 *We have for every $t > 0$, $y \le 0$ and $z > -y$*

$$\mathbb{P}\left(\ell_1 \in dt, Y_{\ell_1 -} \in dy, \Delta_{\ell_1} \in dz\right) \;=\; \frac{-y}{ct}\mathbb{P}(Y_t \in dy)\Pi(dz)dt$$

where $\Delta_t = Y_t - Y_{t-}$ and Π is the Lévy measure of σ.

Proof: Let $f : [0,\infty]\times]-\infty,0]\times[0,\infty[\to[0,\infty[$ be an arbitrary continuous function. Obviously, if ℓ_1 is finite, then it must be a jump time of the subordinator σ, so

$$\mathbb{E}(f(\ell_1, Y_{\ell_1 -}, \Delta_{\ell_1}), \ell_1 < \infty) \;=\; \mathbb{E}\left(\sum_{t\ge 0} f(t, Y_{t-}, \Delta_t)\mathbf{1}_{\{Y_s \le 0, 0\le s<t\}}\mathbf{1}_{\{\Delta_t + Y_{t-} \ge 0\}}\right)\,.$$

We can calculate the right-hand side using the fact that the jumps process of Y is a Poisson point process with characteristic measure Π. We get using the compensation formula

$$\int_0^\infty \mathbb{E}\left(g(t, Y_t), t < \ell_1\right) dt$$

where

$$g(t,y) \;=\; \int_{]-y,\infty[} f(t,y,z)\Pi(dz)\,, \qquad y \le 0\,.$$

The proof is completed by an application of Lemma 2.6. ∎

Plainly, ℓ_1 is a stopping time, so if we define by iteration

$$\ell_{k+1} \;=\; \inf\left\{t > \ell_k : Y_t > S_{\ell_k}\right\}\,, \qquad k \in \mathbb{N}\,,$$

an application of the strong Markov property shows that $((Y_{\ell_k}, \ell_k), k \in \mathbb{N})$ is a (possibly defective) random walk with values in $[0,\infty[\times[0,\infty[$. This is known as the process of strict increasing ladder points, which describes the (possibly finite) sequence of values of the successive maxima of Y and the times when these maxima occur. The distribution of the bivariate random walk is completely described by Proposition 2.7.

2.2 The unbounded variation case

We now turn our attention to the case of a Lévy process with no negative jumps with unbounded variation; the purpose of this section is to discuss the extensions of the results obtained in the preceding section when we worked with a process with bounded variation.

So we assume throughout this section that $Y = (Y_t, t \geq 0)$ be a Lévy process (i.e. a process with independent and stationary increments and right continuous with limits on the left sample paths) that has no negative jumps and unbounded variation. Then Y has a Laplace exponent $\Psi : [0, \infty[\to \mathbb{R}$, i.e.

$$\mathbb{E}\left(\exp\{-qY_t\}\right) = \exp\{t\Psi(q)\}\ , t, q \geq 0$$

and the Laplace exponent is given by the Lévy-Khintchine formula

$$\Psi(q) = aq^2 + bq + \int_{]0,\infty[} \left(\mathrm{e}^{-qx} - 1 + qx\mathbf{1}_{\{x<1\}}\right)\Pi(dx)$$

where $a \geq 0$, $b \in \mathbb{R}$ and Π is a measure on $]0, \infty[$ called the Lévy measure of Y, and such that $\int(1 \wedge x^2)\Pi(dx) < \infty$. Moreover, our assumption that Y has unbounded variation forces either $a > 0$ or $\int(1 \wedge x)\Pi(dx) = \infty$. Alternatively, this condition is equivalent to $\lim_{q\to\infty} \Psi(q)/q = \infty$. See Sections I.1 and VII.1 in [1] for details.

We now briefly review the results of the preceding section which extend *verbatim* to the present case. We still use the notation

$$I_t = \inf_{0\leq s\leq t} Y_s \quad , \quad S_t = \sup_{0\leq s\leq t} Y_s \quad , \quad \tau(x) = \inf\{t \geq 0 : Y_t < -x\}$$

for the infimum and the first passage times of Y. The argument for Proposition 2.1 also apply in the unbounded variation case, so τ is again a subordinator and its Laplace exponent κ is simply given by the inverse of the convex function Ψ. Similarly, Theorem 2.2 and Lemma 2.5 still hold in the present framework, which specifies the joint distribution of Y_t and I_t (respectively of Y_t and S_t).

The main difference with the bounded variation case is related to the supremum process S. More precisely, recall the identity (2.1) and note that now the right-hand side converges to 0 when $\lambda \to \infty$ (because $\lim_{\lambda\to\infty} \Psi(\lambda)/\lambda = \infty$). We thus have $S_T > 0$ a.s., and since T is independent of S and takes arbitrarily small values with positive probability, we deduce that a.s. Y immediately enters the positive half-line.

Recall that the Lévy process reflected at its supremum, $S - Y$, is a strong Markov process, so we have just shown that in the present framework the point 0 is regular for itself (with respect to $S - Y$). Recall also the discussion of Section 1.4. We known that then there exists a local time process $L = (L_t, t \geq 0)$ at 0, that is a continuous process that increases exactly at times when $S - Y = 0$ (i.e when Y reaches its supremum), and the inverse local time

$$L_t^{-1} = \inf\{s \geq 0 : L_s > t\}\ , \qquad t \geq 0$$

is a subordinator. Its Laplace exponent φ, that is

$$\mathbb{E}\left(\exp\left\{-qL_t^{-1}\right\}\right) = \exp\{-t\varphi(q)\}, q \geq 0$$

can be calculated as follows.

As usual, let T be an independent exponential time with parameter $q > 0$; and recall that γ_T (respectively, g_T) denotes the instant when Y reaches its overall supremum (respectively, infimum) on the time-interval $[0, T]$. We know from Lemma 1.10 that

$$\mathbb{E}(\exp\{-\lambda\gamma_T\}) = \varphi(q)/\varphi(q+\lambda).$$

On the other hand, we know from Lemma 2.5 that $T - g_T$ has the same law as γ_T, so Theorem 2.2 yields

$$\frac{\varphi(q)}{\varphi(q+\lambda)} = \frac{q\kappa(q+\lambda)}{(q+\lambda)\kappa(q)}.$$

We conclude that

$$\varphi(\lambda) = c\frac{\lambda}{\kappa(\lambda)}, \qquad \lambda \geq 0 \tag{2.3}$$

where $c > 0$ is some constant.

The constant c depends on the normalization of the local time; it is sometimes convenient to suppose that the normalization has been chosen in such a way that $c = 1$. In this direction, it can be shown that when $\varepsilon \to 0+$, the process

$$\varepsilon^{-1}\int_0^t \mathbf{1}_{\{S_s - Y_s < \varepsilon\}} ds, \qquad t \geq 0$$

converges to the local time $L = (L_t, t \geq 0)$ corresponding to $c = 1$. As a check, recall that $S - Y$ evaluated at an independent exponential time with parameter q follows an exponential distribution with parameter $\kappa(q)$ (by Theorem 2.2 and Lemma 2.5). It follows that for every $\varepsilon > 0$, one has

$$\varepsilon^{-1}\mathbb{E}\left(\int_0^\infty q\mathrm{e}^{-qt}\mathbf{1}_{\{S_t - Y_t < \varepsilon\}} dt\right) = \varepsilon^{-1}(1 - \exp\{-\varepsilon\kappa(q)\}).$$

When $\varepsilon \to 0+$, the right-hand side converges to $\kappa(q)$ and the left-hand side to

$$\begin{aligned}\mathbb{E}\left(\int_0^\infty q\mathrm{e}^{-qt} dL_t\right) &= q\mathbb{E}\left(\int_0^\infty \exp\left\{-qL_t^{-1}\right\} dt\right)\\ &= q\int_0^\infty \exp\{-t\varphi(q)\} dt = \frac{q}{\varphi(q)}.\end{aligned}$$

The deep connections between Lévy processes with no negative jumps and subordinators have many interesting applications. We now conclude this section by presenting one of such applications, namely an extension of Khintchine's law of the iterated logarithm. We refer to Chapters VI-VII in [1] for more in this vein.

Corollary 2.8 *There is a positive constant c such that*

$$\liminf_{t\to 0+} \frac{Y_t\kappa(t^{-1}\log|\log t|)}{\log|\log t|} = -c \qquad a.s.$$

This result follows readily from Theorem 1.4 and the fact that $-I_t = \sup\{-Y_s, 0 \leq s \leq t\}$ is the inverse of the first passage subordinator τ. See Corollary 8.5 in [3] for details.

2.3 The scale function

The absence of negative jumps allows us to tackle problems which have no known solution for general Lévy processes. Here, we consider the so-called two-sided-exit problem, which consists in determining the distribution of the time and the location of the first exit of Y from a finite interval. This problem has a remarkable simple solution; let us state the result concerning the event that the exit occurs at the lower boundary point and refer to Corollary 2 in [2] for the complement event.

Theorem 2.9 (Takács [33], Suprun [32]) *For every $x, y > 0$ and $q \geq 0$, we have*

$$\mathbb{E}\left(\mathrm{e}^{-q\tau(x)}, S_{\tau(x)} \leq y\right) \; = \; \frac{W^{(q)}(y)}{W^{(q)}(x+y)} \, ,$$

where $W^{(q)} : [0,\infty[\to [0,\infty[$ is the unique increasing absolutely continuous function with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\lambda x} W^{(q)}(x) dx \; = \; \frac{1}{\Psi(\lambda) - q} \, , \qquad \lambda > \kappa(q) \, .$$

Proof: Let us first assume that Y drifts to $-\infty$, i.e. $\lim_{t\to\infty} Y_t = -\infty$ a.s. Using (2.1), we see that this is equivalent to assuming that $c := \lim_{\lambda \to 0+} \Psi(\lambda)/\lambda > 0$. More precisely, the distribution of S_∞ is given by

$$\mathbb{P}\left(S_\infty \leq x\right) \; = \; c^{-1} W(x)$$

where $W : [0,\infty[\to [0,\infty[$ is the increasing right-continuous function with Laplace transform $\int_0^\infty \mathrm{e}^{-\lambda x} W(x) dx = 1/\Psi(\lambda)$. We refer to [1], Section VII.2 for an alternative expression of the function W which shows that W is absolutely continuous.

Applying the Markov property at the first passage time $\tau(x)$ of Y at $-x$, we get

$$cW(y) \; = \; \mathbb{P}\left(S_\infty \leq y\right) \; = \; \mathbb{P}(S_{\tau(x)} \leq y) \times \mathbb{P}(S_\infty \leq x + y) \, ,$$

which shows that

$$\mathbb{P}(S_{\tau(x)} \leq y) \; = \; W(y)/W(x+y) \, .$$

Now we drop the assumption that Y drifts to $-\infty$ and we fix $q > 0$. The process $\exp\{-\kappa(q) Y_t - qt\}$ is a positive martingale, and it is seen by classical arguments that if we introduce the locally equivalent probability measure

$$d\tilde{\mathbb{P}} \mid_{\mathcal{F}_t} = \; \exp\{-\kappa(q) Y_t - qt\} \; d\mathbb{P} \mid_{\mathcal{F}_t} \, ,$$

then under $\tilde{\mathbb{P}}$, Y has again independent and stationary increments, and obviously no negative jumps. In other words, Y is a $\tilde{\mathbb{P}}$-Lévy process with no negative jumps and it is immediately verified that its Laplace exponent is given by

$$\tilde{\Psi}(\lambda) \; = \; \Psi(\lambda + \kappa(q)) - q \, .$$

Note that Y drifts to $-\infty$ under $\tilde{\mathbb{P}}$ and that in the obvious notation,

$$\tilde{W}(y)/\tilde{W}(x+y) \;=\; \tilde{\mathbb{P}}\left(S_{\tau(x)} \le y\right) \;=\; \exp\{\kappa(q)x\}\mathbb{E}\left(\mathrm{e}^{-q\tau(x)}\,, S_{\tau(x)} \le y\right) .$$

We deduce that

$$\mathbb{E}\left(\mathrm{e}^{-q\tau(x)}\,, S_{\tau(x)} \le y\right) \;=\; W^{(q)}(y)/W^{(q)}(x+y)$$

where $W^{(q)}(x) \;=\; \exp\{\kappa(q)x\}\tilde{W}(x)$ is the continuous increasing function with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\lambda x} W^{(q)}(x)dx \;=\; \int_0^\infty \mathrm{e}^{-\lambda x}\mathrm{e}^{\kappa(q)x}\tilde{W}(x)dx \;=\; \frac{1}{\tilde{\Psi}(\lambda-\kappa(q))} \;=\; \frac{1}{\Psi(\lambda)-q}$$

provided that $\lambda > \kappa(q)$. This proves the theorem (the case $q=0$ being treated by letting $q>0$ decrease to $0+$). ■

The function $W = W^{(0)}$ is called the *scale function* of the Lévy process, by analogy with Feller's theory of real-valued diffusions. It has an important role in the study of Lévy processes with no negative jumps (see Section VII.2-4 in [1] for applications to the existence of so-called increase points, and to the construction of certain conditioned processes).

The simple identity

$$\frac{1}{\Psi(\lambda)-q} \;=\; \sum_{k=0}^{\infty} q^k \Psi(\lambda)^{-k-1}\,, \qquad \lambda > \kappa(q)$$

yields the following expression for $W^{(q)}(x)$ as a power series:

$$W^{(q)}(x) \;=\; \sum_{k=0}^{\infty} q^k W^{*k+1}(x) \tag{2.4}$$

where $W^{*n} = W * \cdots * W$ denotes the n-th convolution power of the function W. More precisely, the fact that the scale function increases entails by induction

$$W^{*k+1}(x) \;\le\; \frac{x^k W(x)^{k+1}}{k!}\,, \qquad x \ge 0, k \in \mathbb{N}$$

and this justifies (2.4). Observe that, by (2.1) and Laplace inversion, the distribution of S_T, the supremum of Y taken at an independent exponential time with parameter q, is given in terms of $W^{(q)}$ by

$$\mathbb{P}_0\left(S_T \in dx\right) \;=\; \frac{q}{\kappa(q)}W^{(q)}(dx) - qW^{(q)}(x)dx\,, \qquad x \ge 0.$$

The functions $W^{(q)}$ are useful to investigate the Lévy process Y killed as it exits from a finite interval, say $[-a,b]$. We refer to [32] and [2] for expressions of the resolvent density of the killed process in terms of $W^{(q)}$. In particular, it is shown in [2] that if we write $\ell = b + a$ for the length of the interval, the entire function $q \to W^{(q)}$ has a single root at

$$\sup\left\{q \in \mathbb{R} : W^{(q)}(\ell) = 0\right\} := -\rho \in]-\infty, 0[$$

and for $x \in [-a, b]$,

$$\mathbb{P}\left(Y_t \in dx, \zeta > t\right) \sim c\mathrm{e}^{-\rho t} W^{(-\rho)}(x+a) W^{(-\rho)}(b-x) dx\,, \qquad t \to \infty$$

for some constant $c > 0$, where $\zeta = \inf\{t \geq 0 : Y_t \notin [-a, b]\}$. We also refer to Lambert [25] for a study of the Lévy process conditioned to stay forever in $[-a, b]$ (i.e. on $\zeta = \infty$).

Chapter 3

Continuous state branching processes

In this chapter, we will first present a construction due to Lamperti [23] of branching processes based on time-change of left-continuous compound Poisson processes. Continuous-state branching processes (in short a CSBP's) can be viewed as a version of branching processes valued in $[0, \infty[$, and Lamperti's construction can be extended to connect CSBP to Lévy processes with no negative jumps; which enables us to shift results of the preceding chapter to CSBP's. Finally, we will present a relation between CSBP's and Bochner's subordination for subordinators.

3.1 Lamperti's construction of CSBP

Consider a continuous-time branching process in which individuals die at rate $c > 0$, independently of each others. At the end of its life, each individual give birth to a random number of children (again independent of the other individuals) which is distributed according to some probability measure ν, called the offspring distribution. So ν is a distribution on $\mathbb{N} = \{0, 1, \ldots\}$; we set $\pi(i) = \nu(i + 1)$ for $i = -1, 0, \ldots$ and for the sake of simplicity, we shall assume that $\pi(0) = 0$.

In other words, if we write $X(t, a)$ for the number of individual alive at time $t \geq 0$, when at the initial time the population has size $a = X(0, a)$, then we are dealing with a continuous time homogeneous Markov chain $(X(t, a), t \geq 0)$ with values in $\mathbb{N}$, whose dynamics can be described as follows. The chain started at $a \in \mathbb{N}$ stays at a up to time T, where T is an exponential variable with parameter ca, and at time T, it makes a jump $X_T - X_{T-}$ which is independent of T and has the distribution π. The state 0 is absorbing in the sense that $X(\cdot, 0) \equiv 0$.

We now state the fundamental property of branching processes.

Branching Property. *If $X'(\cdot,b)$ is independent of $X(\cdot,a)$ and has the same distribution as $X(\cdot,b)$, then $X(\cdot,a)+X'(\cdot,b)$ has the same law as $X(\cdot,a+b)$.*

The branching property entails that the Laplace transform of its semigroup fulfills the following identity. For every $\lambda > 0$,

$$\mathbb{E}\left(\exp\{-\lambda X(t,a)\}\right) \;=\; \exp\{-a u_t(\lambda)\} \tag{3.1}$$

for some function $u_t(\lambda)$. More precisely, it is readily seen from the verbal description of the dynamic of X that $u_t(\lambda)$ solves the differential equation

$$\frac{\partial u_t(\lambda)}{\partial t} \;=\; -\Psi(u_t(\lambda)) \quad , \quad u_0(\lambda) \;=\; \lambda \,, \tag{3.2}$$

where

$$\Psi(q) \;=\; c\sum_{k=-1}^{\infty}\left(\mathrm{e}^{-qk}-1\right)\pi(k)\,. \tag{3.3}$$

Specifically, (3.2) is a consequence of Kolmogorov's backwards equation. In the sequel, we shall refer to Ψ as the branching mechanism of the branching process X.

The fact that the jump rate of the chain is proportional to its current value is the key to the construction of branching processes based on certain compound Poisson processes. Indeed, it incites us to introduce the time substitution based on

$$C_t \;=\; \int_0^t X(a,s)\,ds\,, \qquad t\geq 0\,,$$

that is we introduce

$$\gamma(t) \;=\; \inf\{s\geq 0 : C_s > t\}\;, \qquad t\geq 0$$

and set

$$\tilde X(t,a) \;=\; X(\gamma(t),a)\,, \qquad t\geq 0\,.$$

It is immediately seen that $\left(\tilde X(t,a), t\geq 0\right)$ is again a continuous time homogeneous Markov chain with the following dynamic. The state 0 is absorbing, i.e. $\tilde X(\cdot,0)\equiv 0$. For $a\neq 0$, the first jump occurs at time aT (where T is the instant of the first jump of $X(\cdot,a)$), which has an exponential distribution with parameter c. Just as before, the jump is independent of T and has the law π. In other words, $\left(\tilde X(t,a), t\geq 0\right)$ can be viewed as a compound Poisson process with intensity measure $c\pi$ and stopped at the first instant when it reaches 0.

Alternatively, consider, $(\Delta_t, t\geq 0)$, a Poisson point process on $\{-1,0,1,\ldots\}$ with characteristic measure $c\pi$ and set $Y(t,a) = a+\sum_{0\leq s\leq t}\Delta_s$. The step process $Y(\cdot,a)$ has independent and stationary increments; one says that it is a *left-continuous* compound Poisson process to stress the property that it takes values on integers and that all its negative jumps have size -1. It is a simple case of a Lévy process, and it is seen from the exponential formula for Poisson point processes that for every $q\geq 0$

$$\mathbb{E}\left(\exp\{-q(Y(t,a)-a)\}\right) \;=\; \exp\{t\Psi(q)\}\;,$$

where Ψ is defined by (3.3). If we write $\tau(a) = \inf\{t \geq 0 : Y(t,a) = 0\}$ for the first passage time of $Y(\cdot, a)$ at 0, then $(Y(t \wedge \tau(a), a), t \geq 0)$ is a version of $\tilde{X}(\cdot, a)$, then we can make the simple observation that $X(\cdot, a)$ can be recovered from $\tilde{X}(\cdot, a)$ by the formulas

$$\gamma(t) = \int_0^t \frac{ds}{\tilde{X}(s,a)} \quad , \quad C_t = \inf\{s \geq 0 : \gamma(s) > t\} \quad , \quad X(t,a) = \tilde{X}(C_t, a) .$$

This elementary correspondence between continuous-time branching processes and left-continuous compound Poisson process can be extended to the continuous space setting. We shall merely outline the argument, and refer to Lamperti [23, 24] for a rigorous proof. First, a continuous-state branching process (CSBP) is a time-homogeneous Markov process valued in $[0, \infty[$ which enjoys the branching property. For the sake of simplicity, we now use the notation $X(t,a)$ for the value at time t of this process started from $a \in [0, \infty[$. Of course, the semigroup still fulfills the identity (3.1), and it can be checked that the function $u_t(\lambda)$ is again the solution to the differential equation (3.2) where now the branching mechanism Ψ is given by

$$\Psi(\lambda) = \alpha\lambda^2 + \beta\lambda + \int_{]0,\infty[} \left(\mathrm{e}^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x<1\}}\right) \Pi(dx)$$

for some $\alpha \geq 0$, $\beta \in \mathbb{R}$ and measure Π on $]0, \infty[$ with $\int_{]0,\infty[}(1 \wedge x^2)\Pi(dx) < \infty$. We refer to Chapter II in Le Gall [26] for details. Moreover, CSBP can be viewed as the (suitably normalized) limit of some sequence of branching processes. If we construct these branching processes using a time-substitution based on left-continuous compound Poisson processes, it can be checked that the corresponding sequence of normalized compound Poisson processes converges in distribution to a Lévy processes with no negative jumps and Laplace exponent Ψ. Putting the pieces together, we arrive at the following connection between continuous state branching processes and Lévy processes with no negative jumps.

Theorem 3.1 (Lamperti [23]) *Let $Y(\cdot, a)$ be a Lévy process with no negative jumps started from a and with Laplace exponent Ψ. If we set*

$$\gamma(t) = \int_0^{t \wedge \tau(a)} \frac{ds}{Y(s,a)} \quad , \quad C_t = \inf\{s \geq 0 : \gamma(s) > t\} ,$$

where $\tau(a)$ stands for the first passage time of $Y(\cdot, a)$ at 0, then $(Y(\gamma(t), a), t \geq 0)$ is a continuous-state branching process started at a with branching mechanism Ψ.

This construction enables us to translate results proven for Lévy processes with no negative jumps to CSBP. Let us briefly present a few of examples:

First, the solution of the two-sided exit problem in terms of the scale function immediately yields the distribution of the overall maximum of a CSBP:

$$\mathbb{P}\left(\sup_{t \geq 0} X(t,a) \leq b\right) = W(b-a)/W(b) , \qquad b \geq a$$

where $W = W^{(0)}$ is given by Theorem 2.9.

Next, the first passage time of $Y(\cdot, a)$ at 0 coincides with the total progeny of the CSBP started with initial population a, i.e.

$$\tau(a) \;=\; \int_0^\infty X(t,a)dt\,,$$

so by Proposition 2.1, we see that

$$\mathbb{E}\left(\exp\left\{-\lambda\int_0^\infty X(t,a)dt\right\}\right) \;=\; \exp\left\{-a\kappa(\lambda)\right\}\,, \qquad \lambda > 0$$

where κ is the right inverse of the branching mechanism. Alternatively, the distribution of the total progeny can be expressed in terms of the law of Y using Corollary 2.3.

Last, we point out from Corollary 2.8 the following law of the iterated logarithm:

$$\limsup_{t\to 0+} \frac{(a - X(t,a))\kappa(t^{-1}\log|\log t|)}{\log|\log t|} \;=\; c\,, \qquad a.s.$$

for some constant $c > 0$. In this vein, note also that when the underlying Lévy process Y has bounded variation, then it holds with probability one that $X(t,a) < a$ for all $t > 0$ small enough, whereas this property fails when Y has unbounded variation.

We refer to Bingham [6] and Pakes [29] for further examples of applications of Theorem 3.1.

3.2 Connection with Bochner's subordination

This section is mostly excerpt from [4]. Combining the branching property and Kolmogorov's consistency theorem, one sees that there exists a two-parameter process $(X(t,a),\ t \geq 0$ and $a \geq 0)$ such that $X(\cdot, 0) = 0$ and, for every $a, b \geq 0$, $X(\cdot, a+b) - X(\cdot, a)$ is independent of the family of processes $(X(\cdot, c), 0 \leq c \leq a)$ and has the law of $X(\cdot, b)$. In particular, for each fixed $t \geq 0$, the process $X(t, \cdot)$ has independent and homogeneous increments with values in $[0, \infty[$. We may (and will) choose its right-continuous modification which is then a subordinator. We see from (3.1) that its Laplace exponent is the function $\lambda \to u_t(\lambda)$.

We deduce from the identity (3.1) and the semigroup property that

$$u_{t+s}(\lambda) \;=\; u_t\left(u_s(\lambda)\right)\,, \tag{3.4}$$

which points out the connection with Bochner's subordination [11]. Specifically, it is easily seen that if σ and σ' are two independent subordinators, say with respective Laplace exponents Φ and Φ', then the compound process $\sigma \circ \sigma'$ has again independent and stationary (nonnegative) increments and right-continuous paths. Hence it is a subordinator, and its Laplace exponent κ is given by

$$\exp\left\{-t\kappa(q)\right\} \;=\; \mathbb{E}\left(\exp\left\{-q\sigma_{\sigma'_t}\right\}\right)\,.$$

To calculate this quantity, we condition on σ'_t the expectation in the right-hand side to get

$$\exp\{-t\kappa(q)\} \;=\; \mathbb{E}\left(\exp\{-\Phi(q)\sigma'_t\}\right) \;=\; \exp\{-t\Phi'\circ\Phi(q)\}\,.$$

Hence we have the identity $\kappa = \Phi'\circ\Phi$, and comparing with (3.4) shows that the subordinator $X(t+s,\cdot)$ has the same distribution as the compound process $X'(s, X(t,\cdot))$ where $X'(s,\cdot)$ is an independent copy of $X(s,\cdot)$. A deeper connection is described in the following statement.

Proposition 3.2 *On some probability space, there exists a process $(S^{(s,t)}(a), 0\leq s\leq t$ and $a\geq 0)$ such that:*

(i) *For every $0\leq s\leq t$, $S^{(s,t)} = \left(S^{(s,t)}(a), a\geq 0\right)$ is a subordinator with Laplace exponent $u_{t-s}(\cdot)$.*

(ii) *For every integer $p\geq 2$ and $0\leq t_1\leq\cdots\leq t_p$, the subordinators $S^{(t_1,t_2)},\ldots,S^{(t_{p-1},t_p)}$ are independent and*

$$S^{(t_1,t_p)}(a) \;=\; S^{(t_{p-1},t_p)}\circ\cdots\circ S^{(t_1,t_2)}(a)\,,\quad \forall a\geq 0\quad a.s.$$

Finally, the processes $(S^{(0,t)}(a), t\geq 0$ and $a\geq 0)$ and $(X(t,a), t\geq 0$ and $a\geq 0)$ have the same finite-dimensional marginals.

Proof: Fix $0\leq t_1\leq\cdots\leq t_p$ and consider $(p-1)$ independent subordinators $S^{(t_1,t_2)},\ldots,S^{(t_{p-1},t_p)}$, with respective Laplace exponents $u_{t_2-t_1}(\cdot),\ldots,u_{t_p-t_{p-1}}(\cdot)$. For every $a\geq 0$, we set $S_a^{(t,t)} = a$ for $t\in\{t_1,\ldots,t_p\}$, and for $1\leq i<j\leq p$

$$S^{(t_i,t_j)} \;=\; S^{(t_{j-1},t_j)}\circ\cdots\circ S^{(t_i,t_{i+1})}\,.$$

We deduce from (3.4) that for every $s\leq t$ in $\{t_1,\ldots,t_p\}$, $S^{(s,t)}$ is a subordinator with Laplace exponent $u_{t-s}(\cdot)$. Moreover, it is plain from the construction that if $0\leq s_1\leq\cdots\leq s_k$ are in $\{t_1,\ldots,t_p\}$, then the subordinators $S^{(s_1,s_2)},\ldots,S^{(s_{k-1},s_k)}$ are independent and

$$S^{(s_{k-1},s_k)}\circ\cdots\circ S^{(s_1,s_2)} \;=\; S^{(s_1,s_k)}\,.$$

By applying Kolmogorov's theorem to the laws of the $\mathbb{D}$-valued random variables $S^{(t_i,t_j)}$, $1\leq i\leq j\leq p$, we get the existence of a process $(S^{(s,t)}(a), 0\leq s\leq t$ and $a\geq 0)$ fulfilling (i) and (ii).

Let us verify the last assertion in the proposition. Fix $a\geq 0$ and write $\mathcal{F}_t$ for the sigma-field generated by the subordinators $S^{(r,s)}$ for $0\leq r\leq s\leq t$. It is plain from (ii) that $S^{(0,\cdot)}(a)$ is a homogeneous Markov process started from a whose semigroup is characterized by

$$\mathbb{E}\left(\mathrm{e}^{-\lambda S^{(0,t+s)}(a)}\mid S^{(0,s)}(a)=x\right) \;=\; \mathbb{E}\left(\mathrm{e}^{-\lambda S^{(s,t+s)}(x)}\right) \;=\; \mathrm{e}^{-xu_t(\lambda)}\,.$$

Hence the processes $S^{(0,\cdot)}(a)$ and $X(\cdot,a)$ have the same law. Next, consider an increasing sequence of times $0=t_0\leq t_1\leq\cdots$. For convenience, introduce independent subordinators $\widetilde{S}^{(t_0,t_1)},\widetilde{S}^{(t_1,t_2)},\ldots$ having the same distribution as $S^{(t_0,t_1)},S^{(t_1,t_2)},\ldots$ but independent of $\mathcal{F}_\infty$. Then, for every integer $i \geq 0$ define two processes $\rho^{(i)}$, $\sigma^{(i)}$ by

$$\begin{aligned}\rho^{(i)}(b) &= S^{(t_i,t_{i+1})}\left(b \wedge S^{(0,t_i)}(a)\right) + \widetilde{S}^{(t_i,t_{i+1})}\left((b - S^{(0,t_i)}(a))^+\right) , \qquad b \geq 0 \\ \sigma^{(i)}(b) &= S^{(t_i,t_{i+1})}\left(S^{(0,t_i)}(a) + b\right) - S^{(0,t_{i+1})}(a) , \qquad b \geq 0 .\end{aligned}$$

Since $S^{(0,t_i)}(a)$ is $\mathcal{F}_{t_i}$-measurable and $S^{(t_i,t_{i+1})}$ is independent of $\mathcal{F}_{t_i}$, the Markov property of subordinators entails that $\rho^{(i)}$ and $\sigma^{(i)}$ are independent and have the same law as $S^{(t_i,t_{i+1})}$. The pair $(\rho^{(i)}, \sigma^{(i)})$ is also independent of $\mathcal{F}_{t_i}$, and it follows by iteration that the two families of processes $\left(\rho^{(j)}, j = 0, \ldots, i\right)$ and $\left(\sigma^{(j)}, j = 0, \ldots, i\right)$ are independent. In particular, for every $a' \geq 0$, the family of variables

$$\sigma^{(j)} \circ \cdots \circ \sigma^{(0)}(a') = S^{(0,t_{j+1})}(a + a') - S^{(0,t_{j+1})}(a) , \qquad j = 0, \ldots, i$$

is independent of the processes

$$\rho^{(j)} \circ \cdots \circ \rho^{(0)}(b) = S^{(0,t_{j+1})}(b) , \quad 0 \leq b \leq a , \qquad j = 0, \ldots, i$$

and has the same law as $\left(S^{(0,t_{j+1})}(a'), j = 0, \ldots, i\right)$. This completes the proof. ■

Example: Neveu [28] considered the special case when the branching mechanism is given by

$$\Psi(u) = u \log u = cu + \int_0^\infty \left(\mathrm{e}^{-xu} - 1 + xu\mathbf{1}_{\{x \leq 1\}}\right) x^{-2} dx ,$$

where $c \in \mathbb{R}$ is a suitable constant. It is easy to verify from (3.2) that

$$u_t(\lambda) = \lambda^{\mathrm{e}^{-t}} ,$$

which implies that, for each fixed $t > 0$, $X(t, \cdot) = S^{(0,t)}$ (as well as $S^{(s,t+s)}$ for every $s \geq 0$) is a stable subordinator with index e^{-t}.

CSBP are often used to model the size of some population as time evolves. Let us now see how the preceding connection with subordinators can be used to define the genealogy in a CSBP. For the sake of simplicity, we shall henceforth focus on the most important case when the drift coefficient of the subordinator $S^{(s,t)}$ is zero for every $0 \leq s < t$.

Definition. *For every $b, c \geq 0$ and $0 \leq s < t$, we say that the individual c in the population at time t has ancestor (or is a descendant of) the individual b in the population at time s if b is a jump time of $S^{(s,t)}$ and*

$$S^{(s,t)}(b-) < c < S^{(s,t)}(b) .$$

It may be useful to comment this definition. The set of individuals in the population at time t having an ancestor at time s is the random open subset of $[0, \infty[$ with canonical decomposition $\bigcup\left(S^{(s,t)}_{b-}, S^{(s,t)}_{b}\right)$ where the union is taken over jump times of the subordinator $S^{(s,t)}$. Its complement can be identified as the closed range of $S^{(s,t)}$; the assumption that $S^{(s,t)}$ has zero drift ensures that the closed range has zero Lebesgue measure a.s. In other words, the individuals in the population at time t having no ancestor at time s form a random closed set of Lebesgue measure zero a.s. On the other hand, it is plain that an individual in the population at time t has at most one ancestor at time s.

Suppose $0 \leq r < s < t$. If the individual d in the population at time t has ancestor c in the population at time s, and if the latter has ancestor b in the population at time r, then by definition

$$S^{(s,t)}(c-) \,<\, d \,<\, S^{(s,t)}(c) \quad \text{and} \quad S^{(r,s)}(b-) \,<\, c \,<\, S^{(r,s)}(b)\,.$$

As $S^{(r,t)} = S^{(s,t)} \circ S^{(r,s)}$, we have a fortiori by monotonicity

$$S^{(r,t)}(b-) \,<\, d \,<\, S^{(r,t)}(b)\,,$$

i.e. the individual d in the population at time t has ancestor b in the population at time r (which obviously is what we expected!).

Let us stress that it is easy to visualize the genealogy: Two individuals in the population at time t have the same ancestor in the population at time s if and only if they belong to the same open interval in the canonical decomposition of the complement of the closed range of $S^{(s,t)}$. Observe also that the identity $S^{(r,t)} = S^{(s,t)} \circ S^{(r,s)}$ ensures that the closed range of $S^{(s,t)}$ contains that of $S^{(r,t)}$, which in turn confirms that two individuals in the population at time t having the same ancestor in the population at time s have also the same ancestor in the population at time r. Finally, we refer to [4] for an application of this notion to the construction of some coalescent process.

References

[1] J. Bertoin (1996). *Lévy processes.* Cambridge University Press, Cambridge.

[2] J. Bertoin (1997). Exponential decay and ergodicity of completely asymmetric Lévy processes in an interval, *Ann. Appl. Probab.* **7**, 156-169.

[3] J. Bertoin (1999). *Subordinators: Examples and Applications.* Ecole d'Ete de Probabilités de Saint-Flour XXVII. Lecture Notes in Maths. 1727, Springer.

[4] J. Bertoin and J.-F. Le Gall (2000). The Bolthausen-Sznitman coalescent and the genealogy of continuous-state branching processes. *Probab. Theory Relat. Fields* **117**, 249-266.

[5] N. H. Bingham (1975). Fluctuation theory in continuous time. *Adv. Appl. Prob.* **7**, 705-766.

[6] N. H. Bingham (1976). Continuous branching processes and spectral positivity, *Stochastic Process. Appl.* **4**, 217-242.

[7] N. H. Bingham, C. M. Goldie and J. L. Teugels (1987). *Regular variation.* Cambridge University Press, Cambridge.

[8] R. M. Blumenthal and R. K. Getoor (1961). Sample functions of stochastic processes with independent increments. *J. Math. Mech.* **10**, 493-516.

[9] R. M. Blumenthal and R. K. Getoor (1968). *Markov processes and potential theory.* Academic Press, New-York.

[10] R. M. Blumenthal and R. K. Getoor (1970). Dual processes and potential theory. *Proc. 12th Biennal Seminar, Canad. Math. Congress*, 137-156.

[11] S. Bochner (1955). *Harmonic analysis and the theory of probability.* University of California Press, Berkeley.

[12] J. Bretagnolle (1971). Résultats de Kesten sur les processus à accroissements indépendants. In: *Séminaire de Probabilités* V, Lecture Notes in Maths. 191 pp. 21-36. Springer, Berlin.

[13] E. B. Dynkin (1961). Some limit theorems for sums of independent random variables with infinite mathematical expectation. In: *Selected Translations Math. Stat. Prob.* vol. 1, pp. 171-189. Inst. Math. Statistics Amer. Math. Soc.

[14] B. E. Fristedt (1967). Sample function behaviour of increasing processes with stationary independent increments. *Pac. J. Math.* **21**, 21-33.

[15] B. E. Fristedt (1974). Sample functions of stochastic processes with stationary, independent increments. In: *Advances in Probability 3*, pp. 241-396. Dekker, New-York.

[16] B. E. Fristedt (1996). Intersections and limits of regenerative sets. In: *Random Discrete Structures* (eds. D. Aldous and R. Pemantle) pp. 121-151. Springer, Berlin.

[17] B. E. Fristedt and W. E. Pruitt (1971). Lower functions for increasing random walks and subordinators. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **18**, 167-182.

[18] J. Hoffmann-Jørgensen (1969). Markov sets. *Math. Scand.* **24**, 145-166.

[19] K. Itô (1942). On stochastic processes. I. (Infinitely divisible laws of probability). *Japan J. Math.* **18**, 261-301.

[20] H. Kesten (1969). Hitting probabilities of single points for processes with stationary independent increments. *Memoirs Amer. Math. Soc.* **93**.

[21] H. Kesten (1970). The limit points of random walk. *Ann. Math. Stat.* **41**, 1173-1205.

[22] J. Lamperti (1962). An invariance principle in renewal theory. *Ann. Math. Stat.* **33**, 685-696.

[23] J. Lamperti (1967). Continuous-state branching processes. *Bull. Amer. Math. Soc.* **73** , 382-386.

[24] J. Lamperti (1967). The limit of a sequence of branching processes. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **7**, 271-288.

[25] A. Lambert (2000). Completely asymmetric Lévy process confined in a finite interval. *Ann. Inst. Henri Poincaré* **36**, 251-274.

[26] J.-F. Le Gall (1999). *Spatial branching processes, random snakes, and partial differential equations*. Birkhäuser, Basel.

[27] J. Neveu (1961). Une généralisation des processus à accroissements positifs indépendants. *Abh. Math. Sem. Univ. Hamburg* **25**, 36-61.

[28] J. Neveu (1992). A continuous-state branching process in relation with the GREM model of spin glass theory. Rapport interne no 267, Ecole Polytechnique.

[29] A. G. Pakes (1996). A hitting time for Lévy processes, with applications to dams and branching processes. *Ann. Fac. Sci. Toulouse Math.* **(6) 5**, 521-544.

[30] D. Revuz and M. Yor (1999). *Continuous martingales and Brownian motion*, 3rd edn. Springer, Berlin.

[31] K. I. Sato (1999). *Lévy processes and infinitely divisible distributions.* Cambridge University Press, Cambridge.

[32] V. N. Suprun (1976). Problem of destruction and resolvent of terminating process with independent increments. *Ukrainian Math. J.* **28**, 39-45.

[33] L. Takács (1966). *Combinatorial methods in the theory of stochastic processes.* Wiley, New York.