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Density Transformation in Lévy Processes

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Preface

These notes are outgrowth of the Concentrated Advanced Course on Lévy Processes, MaPhySto, University of Aarhus, January 24–28, 2000. The course started with the definition of Lévy processes and discussed their elementary properties and their transformations. It was based on the book

> K. Sato, Lévy Processes and Infinitely Divisible Distributions, 1999, Cambridge University Press.

One of the subjects in the course was the density transformation of Lévy processes. This is also discussed in Chapter 6 of the book, but I treated it in a different way in the course, fully utilizing the power of the Hellinger–Kakutani inner product and distance of order α. This method was adopted by C. M. Newman in 1972–73 but it is not widely known. In pursuing this method, I found that the Lebesgue decomposition of path space measures of Lévy processes could be obtained easily. Together with the description of the Radon–Nikodým densities of the absolutely continuous parts, this clarifies the relationship of the path space measures on a finite time interval of two given Lévy processes on $\mathbb{R}^d$. The main part of these notes concentrates on this subject and gives the results with complete proofs.

The other parts of the lectures of the course are attached here as Appendix A.

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Contents

1. Introduction

A stochastic process $\{X_t\colon t \geq 0\}$ on the Euclidean space $\mathbb{R}^d$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is called a *Lévy process* if [1]

(1) it has independent increments,
(2) $X_0 = 0$ a. s.,
(3) the distribution of $X_{s+t} - X_s$ does not depend on s,
(4) it is stochastically continuous, that is, continuous in probability,
(5) there is $\Omega_0 \in \mathcal{F}$ with $P[\Omega_0] = 1$ such that, for any $\omega \in \Omega_0$, $X_t(\omega)$ is right-continuous with left limits as a function of t.

Let $\mathbf{D} = D([0,\infty), \mathbb{R}^d)$ be the space of functions $\omega\colon [0,\infty) \to \mathbb{R}^d$ right-continuous with left limits. Let $X_t(\omega) = \omega(t)$ and let $\mathcal{F}_t^0$ and $\mathcal{F}^0$ be the σ-algebras generated by $\{X_s\colon 0 \leq s \leq t\}$ and $\{X_s\colon 0 \leq s < \infty\}$, respectively. By the condition (5) above, any Lévy process on $\mathbb{R}^d$ induces a probability measure P on $(\mathbf{D}, \mathcal{F}^0)$. Thus $\{X_t\}$ on the probability space $(\mathbf{D}, \mathcal{F}^0, P)$ is identical in law with the original Lévy process. In these notes, except in the appendices, we use the following terminology. By saying that $(\{X_t\}, P)$ is a Lévy process, we mean that $\{X_t\colon t \geq 0\}$ is a Lévy process under the probability measure P on $(\mathbf{D}, \mathcal{F}^0)$. The restriction of P to $\mathcal{F}_t^0$ is denoted by $P^t = [P]_{\mathcal{F}_t^0}$.

We study the following problems. By $\ll$, $\approx$, and $\perp$ we mean "is absolutely continuous with respect to", "is mutually absolutely continuous with", and "is mutually singular with", respectively (Definition 2.1).

1. Given two Lévy processes $(\{X_t\}, P_1)$ and $(\{X_t\}, P_2)$, find necessary and sufficient conditions for $P_2^t \ll P_1^t$, for $P_2^t \approx P_1^t$, and for $P_2^t \perp P_1^t$.

2. Given two Lévy processes $(\{X_t\}, P_1)$ and $(\{X_t\}, P_2)$, describe the Lebesgue decomposition (that is, decomposition into the absolutely continuous part and the singular part) of P_2^t with respect to P_1^t.

3. In the case where $P_2^t \ll P_1^t$, find the Radon–Nikodým density of P_2^t with respect to P_1^t.

[1] The space $\mathbb{R}^d$ is the set of column d-vectors, $\langle x, y\rangle = \sum x_j y_j$ for $x = (x_j)$ and $y = (y_j)$, and $|x| = \langle x, x\rangle^{1/2}$.

4. Find the Radon–Nikodým density of the absolutely continuous part of P_2^t with respect to P_1^t.

5. Given a Lévy process $(\{X_t\}, P_1)$, construct P_2 such that $P_2^t \ll P_1^t$ and that $(\{X_t\}, P_2)$ is a second Lévy process having a prescribed property. This we call the *density transformation* from $(\{X_t\}, P_1)$ to $(\{X_t\}, P_2)$.

6. Study special cases of the density transformation such as the Cameron–Martin transformation, the exponential (Esscher) transformation, the deletion of jumps, and the truncation of supports of Lévy measures.

The problems 1 and 3 were solved by Skorokhod (1957, 60, 61), Kunita and S. Watanabe (1967), and Newman (1972, 73). As we will see in these notes, the problems 2 and 4 can be solved in their lines. We mention also Brody (1971) and Memin and Shiryayev (1985) as related papers. In Chapter 6 of [S][2], the case of mutual absolute continuity is thoroughly treated, but the other cases are not studied. Newman (1972, 73) showed that the Hellinger–Kakutani inner product and distance of order α for σ-finite measures are powerful tools in attacking these problems. In treating the problem 1, we closely follow Newman (1973), but Lemma 2.15 and its systematic use are new. The results on the problem 3 are embedded in the more general discussions by the above-mentioned authors and they are not easy to follow. Often they are based on the general theory of semimartingales. Thus it would be worth-while to give independent proofs of the results in the case of Lévy processes. Further the problems 2 and 4 have not been treated explicitly in the literature, as far as we know.

In Section 2 we give results on the Hellinger–Kakutani inner product and distance of order α for σ-finite measures on general measurable spaces. Main results on the problems 1–4 are formulated as Theorems A and B in Section 3 and their corollaries are proved. For Gaussian Lévy processes those results are special cases of the well-known dichotomy theorem for Gaussian processes. They are given in Section 4 and used in

[2][S] refers to the following book: Sato, K. (1999) *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Univ. Press., Cambridge.

the proof of our theorems. Sections 5 and 6 contain proofs of Theorems A and B, respectively. The density transformation and its special cases are studied in Section 7.

2. Hellinger–Kakutani inner product and distance

Let us introduce the notions of the Hellinger–Kakutani inner product and distance for σ-finite measures on a general measurable space and study their properties. These are basic tools in the following sections. Given a measure μ and a nonnegative measurable function f, we denote by $f\mu$ the measure defined as

$$(f\mu)(B) = \int_B f d\mu.$$

The restriction of a measure μ to a measurable set C is denoted by $[\mu]_C$, that is,

$$[\mu]_C(B) = \mu(C \cap B) = (1_C\mu)(B).$$

Definition 2.1. Given two measures ρ_1 and ρ_2 on a measurable space $(\Theta, \mathcal{B})$, we write $\rho_2 \ll \rho_1$ if ρ_2 is *absolutely continuous* with respect to ρ_1 (that is, $\rho_1(B) = 0$ implies $\rho_2(B) = 0$), and $\rho_1 \approx \rho_2$ if $\rho_1 \ll \rho_2$ and $\rho_2 \ll \rho_1$. When $\rho_1 \approx \rho_2$, we say that ρ_1 and ρ_2 are *mutually absolutely continuous* (some authors say that they are equivalent). We write $\rho_1 \perp \rho_2$ and call them *orthogonal* or *mutually singular* if ρ_1 is singular with respect to ρ_2 (equivalently, ρ_2 is singular with respect to ρ_1), that is, there is a set $B \in \mathcal{B}$ such that $\rho_1(B) = \rho_2(B^c) = 0$. Here we denote $B^c = \Theta \setminus B$.

Definition 2.2. Let $0 < \alpha < 1$. For σ-finite measures ρ_1, ρ_2 on $(\Theta, \mathcal{B})$, we define

$$H_\alpha(\rho_1, \rho_2) = \left(\frac{d\rho_1}{d\rho}\right)^\alpha \left(\frac{d\rho_2}{d\rho}\right)^{1-\alpha} \rho, \tag{2.1}$$

that is,

$$H_\alpha(\rho_1, \rho_2)(B) = \int_B \left(\frac{d\rho_1}{d\rho}\right)^\alpha \left(\frac{d\rho_2}{d\rho}\right)^{1-\alpha} d\rho, \qquad B \in \mathcal{B},$$

where ρ is a σ-finite measure such that ρ_1, $\rho_2 \ll \rho$ (for example, $\rho = \rho_1 + \rho_2$) and $d\rho_1/d\rho$ and $d\rho_2/d\rho$ are the Radon–Nikodým densities. We call $H_\alpha(\rho_1, \rho_2)$ the *Hellinger–Kakutani inner product of order α of ρ_1 and ρ_2*; it is a σ-finite measure by the remark below. Sometimes a symbolic expression

$$dH_\alpha(\rho_1, \rho_2) = (d\rho_1)^\alpha (d\rho_2)^{1-\alpha}$$

is used. The total mass of $H_\alpha(\rho_1, \rho_2)$ is written as

$$h_\alpha(\rho_1, \rho_2) = \int_\Theta dH_\alpha(\rho_1, \rho_2). \tag{2.2}$$

Kakutani (1948) made an ingenius use of $h_{1/2}(\rho_1, \rho_2)$, and also $k_{1/2}(\rho_1, \rho_2)$ to be given in Definition 2.12, in the study of equivalence and orthogonality of infinite product measures.

Remark 2.3. We have

$$H_\alpha(\rho_1, \rho_2) \le \alpha\rho_1 + (1-\alpha)\rho_2, \tag{2.3}$$

because, for f_1, $f_2 \ge 0$,

$$\begin{aligned}\int_B f_1^\alpha f_2^{1-\alpha} d\rho &\le \left(\int_B f_1 d\rho\right)^\alpha \left(\int_B f_2 d\rho\right)^{1-\alpha} \\ &\le \alpha \int_B f_1 d\rho + (1-\alpha)\int_B f_2 d\rho\end{aligned}$$

by Hölder's inequality and by the concavity of $\log x$.

Remark 2.4. $H_\alpha(\rho_1, \rho_2)$ is independent of the choice of ρ. To see this, let $H'_\alpha(\rho_1, \rho_2)$ be the one defined by using ρ' instead of ρ. Let $\widetilde{\rho} = \rho + \rho'$ and let $\widetilde{H}_\alpha(\rho_1, \rho_2)$ be the one using $\widetilde{\rho}$. Then

$$\begin{aligned}\widetilde{H}_\alpha(\rho_1, \rho_2) &= \left(\frac{d\rho_1}{d\widetilde{\rho}}\right)^\alpha \left(\frac{d\rho_2}{d\widetilde{\rho}}\right)^{1-\alpha} \widetilde{\rho} = \left(\frac{d\rho_1}{d\rho}\frac{d\rho}{d\widetilde{\rho}}\right)^\alpha \left(\frac{d\rho_2}{d\rho}\frac{d\rho}{d\widetilde{\rho}}\right)^{1-\alpha} \widetilde{\rho} \\ &= \left(\frac{d\rho_1}{d\rho}\right)^\alpha \left(\frac{d\rho_2}{d\rho}\right)^{1-\alpha} \rho = H_\alpha(\rho_1, \rho_2).\end{aligned}$$

Similarly $\widetilde{H}_\alpha(\rho_1, \rho_2) = H'_\alpha(\rho_1, \rho_2)$.

Remark 2.5. We have $\rho_1 \perp \rho_2$ if and only if $h_\alpha(\rho_1, \rho_2) = 0$. To see this, let $\rho = \rho_1 + \rho_2$. If $\rho_1 \perp \rho_2$, then choose B such that $\rho_1(B) = 0$ and $\rho_2(B^c) = 0$ and see that $d\rho_1/d\rho = 1_{B^c}$ and $d\rho_2/d\rho = 1_B$, which implies $h_\alpha(\rho_1, \rho_2) = 0$. Conversely, if $h_\alpha(\rho_1, \rho_2) = 0$, then $(d\rho_1/d\rho)^\alpha (d\rho_2/d\rho)^{1-\alpha} = 0$ ρ-a. e. and hence, for some B, $d\rho_1/d\rho = 0$ ρ-a. e. on B and $d\rho_2/d\rho = 0$ ρ-a. e. on B^c.

Example 2.6. Let μ_1 and μ_2 be Gaussian distributions on $\mathbb{R}$ with a common variance $A > 0$. Let γ_1, γ_2 be the means of μ_1, μ_2. Then

$$H_\alpha(\mu_1, \mu_2) = \big(\exp[-(2A)^{-1}\{\alpha{\gamma_1}^2 + (1-\alpha){\gamma_2}^2 - {\gamma_3}^2\}]\big)\mu_3,$$

where $\gamma_3 = \alpha\gamma_1 + (1-\alpha)\gamma_2$ and μ_3 is Gaussian on $\mathbb{R}$ with variance A and mean γ_3. For we have

$$\frac{d\mu_1}{d\mu_2} = \exp[-(2A)^{-1}(x-\gamma_1)^2 + (2A)^{-1}(x-\gamma_2)^2]$$

and

$$\begin{aligned} H_\alpha(\mu_1,\mu_2) &= (d\mu_1/d\mu_2)^\alpha \mu_2 \\ &= (2\pi A)^{-1/2}\exp[-(2A)^{-1}\alpha(x-\gamma_1)^2 - (2A)^{-1}(1-\alpha)(x-\gamma_2)^2]dx \\ &= (2\pi A)^{-1/2}\exp[-(2A)^{-1}\{x^2 - 2\gamma_3 x + \alpha{\gamma_1}^2 + (1-\alpha){\gamma_2}^2\}]dx. \end{aligned}$$

Similarly, if μ_1 and μ_2 are nondegenerate Gaussian on $\mathbb{R}^d$ with a common covariance matrix A and with means γ_1 and γ_2, respectively, then

$$\begin{aligned} &H_\alpha(\mu_1,\mu_2) \\ &\quad = \big(\exp[-2^{-1}\{\alpha\langle\gamma_1, A^{-1}\gamma_1\rangle + (1-\alpha)\langle\gamma_2, A^{-1}\gamma_2\rangle - \langle\gamma_3, A^{-1}\gamma_3\rangle\}]\big)\mu_3, \end{aligned}$$

where μ_3 is Gaussian on $\mathbb{R}^d$ with covariance A and mean $\gamma_3 = \alpha\gamma_1 + (1-\alpha)\gamma_2$.

Definition 2.7. Given σ-finite measures ρ_1, ρ_2 on $(\Theta, \mathcal{B})$, define

$$(2.4) \quad C_\rho(\rho_1) = \left\{\theta \in \Theta \colon \frac{d\rho_1}{d\rho} > 0\right\}, \quad C_\rho(\rho_2) = \left\{\theta \in \Theta \colon \frac{d\rho_2}{d\rho} > 0\right\},$$

where ρ is a σ-finite measure such that ρ_1, $\rho_2 \ll \rho$. We call $C_\rho(\rho_1)$ the *carrier of* ρ_1 *relative to* ρ and $C_\rho(\rho_2)$ the *carrier of* ρ_2 *relative to* ρ. They depend on the choice of ρ and versions of the Radon-Nikodým densities. However, we sometimes write $C_\rho(\rho_1) = C(\rho_1)$ and $C_\rho(\rho_2) = C(\rho_2)$, suppressing the dependence on ρ.

The support is defined for a Borel measure on a nice topological space. It is the smallest closed set that carries the full measure. But the carrier defined above is a measure-theoretical concept without any reference to the topology.

Example 2.8. Let $\Theta = \mathbb{R}$, ρ_1 = Lebesgue measure restricted to the interval $[0,1]$, and $\rho_2 = \delta_0$ = the unit mass at 0. If we take $\rho = \rho_1 + \rho_2$, then $C_\rho(\rho_1) = ((0,1] \setminus B_1) \cup B_2$, where, by the choice of versions of $d\rho_1/d\rho$, B_1 can be any Borel set in $(0,1]$ of Lebesgue measure 0 and B_2 can be any Borel set in $\mathbb{R} \setminus [0,1]$. If we choose $\rho = \rho_1 + \rho_2 + \rho_3$, where ρ_3 is a discrete measure concentrated on the set $\mathbb{Q}$ of rational numbers such that every point in $\mathbb{Q}$ has a positive ρ_3-measure, then $C_\rho(\rho_1) = (((0,1] \setminus B_1) \cup B_2) \setminus \mathbb{Q}$, where B_1 and B_2 are as above. If we take $\rho = \rho_1 + \rho_4$, where ρ_4 is the Lebesgue measure on $\mathbb{R}$, then $C_\rho(\rho_1) = ((0,1] \setminus B_1) \cup B_2$, where B_1 and B_2 are Borel sets of Lebesgue measure 0, $B_1 \subset (0,1]$, and $B_2 \subset \mathbb{R} \setminus [0,1]$.

Lemma 2.9. *Let ${\rho_2}^{ac}$ and ${\rho_2}^{s}$ be, respectively, the absolutely continuous part and the singular part of ρ_2 with respect to ρ_1. Then,*

$$
{\rho_2}^{ac} = 1_{C(\rho_1)}\rho_2, \tag{2.5}
$$

$$
{\rho_2}^{s} = 1_{C(\rho_1)^c}\rho_2, \tag{2.6}
$$

$$
{\rho_2}^{ac}(\Theta) = \rho_2(C(\rho_1)), \tag{2.7}
$$

$$
{\rho_2}^{s}(\Theta) = \rho_2(C(\rho_1)^c), \tag{2.8}
$$

which are independent of the choice of ρ and versions of $d\rho_1/d\rho$ and $d\rho_2/d\rho$.

Proof. Let ρ be the measure in Definition 2.7. We claim that

(2.9) $\quad 1_{C(\rho_1)}\rho_2$ is absolutely continuous with respect to ρ_1.

Indeed, let $G \in \mathcal{B}$ be such that $\rho_1(G) = 0$. Then $\rho(G \cap C(\rho_1)) = 0$, because $\rho(G \cap \{d\rho_1/d\rho > 0\}) > 0$ implies that

$$
\rho_1(G) = \int_G \frac{d\rho_1}{d\rho} d\rho = \int_{G\cap\{d\rho_1/d\rho>0\}} \frac{d\rho_1}{d\rho} d\rho > 0.
$$

Hence $\rho_2(G \cap C(\rho_1)) = 0$. Hence we have (2.9). Next notice that

(2.10) $\quad 1_{C(\rho_1)^c}\rho_2$ is singular with respect to ρ_1,

since $\rho_1(C(\rho_1)^c) = \rho_1(\{d\rho_1/d\rho = 0\}) = 0$. Since $\rho_2 = 1_{C(\rho_1)}\rho_2 + 1_{C(\rho_1)^c}\rho_2$, (2.9) and (2.10) show (2.5) and (2.6). These imply (2.7) and (2.8). $\square$

Remark 2.10. The following are consequences of Lemma 2.9.

(i) $\rho_2 \ll \rho_1$ if and only if $\rho_2(C(\rho_1)^c) = 0$.

(ii) $\rho_2 \perp \rho_1$ if and only if $\rho_2(C(\rho_1)) = 0$.

Lemma 2.11. *Suppose that ρ_1 and ρ_2 are finite. Then*

$$\lim_{\alpha\downarrow 0} h_\alpha(\rho_1,\rho_2) = \rho_2(C(\rho_1)), \tag{2.11}$$

$$\lim_{\alpha\uparrow 1} h_\alpha(\rho_1,\rho_2) = \rho_1(C(\rho_2)). \tag{2.12}$$

Proof. Choose $\rho = \rho_1 + \rho_2$. We have

$$\begin{aligned} h_\alpha(\rho_1,\rho_2) &= \int_{C_\rho(\rho_1)} \left(\frac{d\rho_1}{d\rho}\right)^\alpha \left(\frac{d\rho_2}{d\rho}\right)^{1-\alpha} d\rho \\ &\to \int_{C_\rho(\rho_1)} \frac{d\rho_2}{d\rho} d\rho = \rho_2(C_\rho(\rho_1)) \end{aligned}$$

as $\alpha \downarrow 0$ by the bounded convergence theorem. This is (2.11). The assertion (2.12) is proved similarly. In fact (2.11) and (2.12) are equivalent assertions, since $h_\alpha(\rho_1,\rho_2) = h_{1-\alpha}(\rho_2,\rho_1)$. □

Lemma 2.11 will be extended in Lemma 2.17.

Definition 2.12. Let $0 < \alpha < 1$. For σ-finite measures ρ_1, ρ_2 on $(\Theta.\mathcal{B})$, we define

$$K_\alpha(\rho_1,\rho_2) = \alpha\rho_1 + (1-\alpha)\rho_2 - H_\alpha(\rho_1,\rho_2), \tag{2.13}$$

which is a σ-finite measure by Remark 2.3. The total mass of $K_\alpha(\rho_1,\rho_2)$ is written as

$$k_\alpha(\rho_1,\rho_2) = \int_\Theta dK_\alpha(\rho_1,\rho_2), \tag{2.14}$$

which we call the *Hellinger–Kakutani distance of order α between ρ_1 and ρ_2.*

Remark 2.13. The definition of $K_\alpha(\rho_1,\rho_2)$ by (2.13) is not precise, as the right-hand side is possibly $\infty - \infty$ for some sets. The precise definition is as follows: choose ρ as in Definition 2.2 and let $d\rho_j/d\rho = f_j$. Then

$$\alpha f_1 + (1-\alpha) f_2 - f_1^\alpha f_2^{1-\alpha} \geq 0 \qquad \rho\text{-a. e.}$$

by (2.3). Define

$$K_\alpha(\rho_1,\rho_2)(B) = \int_B (\alpha f_1 + (1-\alpha) f_2 - f_1^\alpha f_2^{1-\alpha}) d\rho.$$

As in Remark 2.4, we can prove that this definition is independent of the choice of ρ.

Remark 2.14. Let $\|\rho_1 - \rho_2\|$ be the total variation norm of $\|\rho_1 - \rho_2\|$, admitting infinity. Then

$$\|\rho_1 - \rho_2\| \geq 2\, k_{1/2}(\rho_1, \rho_2). \tag{2.15}$$

If ρ_1 and ρ_2 are finite measures, then

$$\|\rho_1 - \rho_2\| \leq c\, k_{1/2}(\rho_1, \rho_2)^{1/2}, \tag{2.16}$$

where $c = 2(\rho_1(\Theta) + \rho_2(\Theta))^{1/2}$. In fact, let $\rho_1,\ \rho_2 \ll \rho$ and $d\rho_j/d\rho = f_j$ for $j = 1, 2$. Then, $\|\rho_1 - \rho_2\| = \int |f_1 - f_2| d\rho$ and

$$\begin{aligned} 2k_{1/2}(\rho_1, \rho_2) &= \int (f_1 + f_2 - 2\sqrt{f_1 f_2}) d\rho = \int (\sqrt{f_1} - \sqrt{f_2})^2 d\rho \\ &\leq \int |f_1 - f_2| d\rho. \end{aligned}$$

If ρ_1 and ρ_2 are finite, then

$$\begin{aligned} \int |f_1 - f_2| d\rho &\leq \left(\int (\sqrt{f_1} - \sqrt{f_2})^2 d\rho \right)^{1/2} \left(\int (\sqrt{f_1} + \sqrt{f_2})^2 d\rho \right)^{1/2} \\ &\leq (2k_{1/2}(\rho_1, \rho_2))^{1/2} \left(2 \int (f_1 + f_2) d\rho \right)^{1/2} \\ &= c k_{1/2}(\rho_1, \rho_2)^{1/2}. \end{aligned}$$

Lemma 2.15. *If*

$$k_\alpha(\rho_1, \rho_2) < \infty \tag{2.17}$$

for some $0 < \alpha < 1$, *then it holds for all* $0 < \alpha < 1$ *and we have*

$$\rho_2(C(\rho_1)^c) < \infty, \qquad \rho_1(C(\rho_2)^c) < \infty. \tag{2.18}$$

Proof. For two nonnegative functions $\varphi(u)$, $\psi(u)$, we say that $\varphi(u) \asymp \psi(u)$ on a set B if there exist two positive constants c_1, c_2 such that $c_1 \psi(u) \leq \varphi(u) \leq c_2 \psi(u)$ on B. Fix $0 < \alpha < 1$ and let $\varphi(u) = \alpha + (1 - \alpha) e^u - e^{(1-\alpha)u}$. Then $\varphi(0) = 0$ and $\varphi(u) > 0$ for $u \neq 0$, because e^u is strictly convex. Since $\varphi(u) = \alpha(1-\alpha)u^2/2 + o(u^2)$ as $u \to 0$, we have $\varphi(u) \asymp u^2$ on $[-1, 1]$. Obviously, $\varphi(u) \asymp e^u$ on $(1, \infty)$ and $\varphi(u) \asymp 1$ on $(-\infty, -1)$. Suppose that $k_\alpha(\rho_1, \rho_2) < \infty$. Let

ρ_1, $\rho_2 \ll \rho$ and $d\rho_j/d\rho = f_j$ for $j = 1, 2$. Since

$$k_\alpha(\rho_1, \rho_2) = \int_\Theta (\alpha f_1 + (1-\alpha) f_2 - {f_1}^\alpha {f_2}^{1-\alpha}) d\rho,$$

we see that $\rho_2(\{f_1 = 0\}) + \rho_1(\{f_2 = 0\}) < \infty$, that is, (2.18). Let $C = \{f_1 > 0 \text{ and } f_2 > 0\}$. We have $[\rho_1]_C \approx [\rho_2]_C$. Letting $d\rho_2/d\rho_1 = f = e^g$ on C, we have

$$K_\alpha(\rho_1, \rho_2)(C) = \int_C (\alpha + (1-\alpha) f - f^{1-\alpha}) d\rho_1 = \int_C \varphi(g) d\rho_1.$$

Hence

$$\int_{C \cap \{|g| \le 1\}} g^2 d\rho_1 + \int_{C \cap \{g > 1\}} e^g d\rho_1 + \int_{C \cap \{g < -1\}} d\rho_1 < \infty. \tag{2.19}$$

Conversely, if (2.18) and (2.19) are satisfied, then $k_\alpha(\rho_1, \rho_2) < \infty$. As the conditions (2.18) and (2.19) do not involve α, the assertion is proved. □

Remark 2.16. The proof of the preceding lemma shows the following. Let $\rho_2 = e^g \rho_1$ with a measurable function $g(x)$ satisfying $-\infty \le g(x) < \infty$ on Θ. Then $k_\alpha(\rho_1, \rho_2) < \infty$ for $0 < \alpha < 1$ if and only if

$$\int_{|g| \le 1} g^2 d\rho_1 + \int_{g > 1} e^g d\rho_1 + \int_{g < -1} d\rho_1 < \infty. \tag{2.20}$$

The following lemma shows the advantage of using the Hellinger–Kakutani distance $k_\alpha(\rho_1, \rho_2)$ for all orders $0 < \alpha < 1$.

Lemma 2.17. *Suppose that $k_\alpha(\rho_1, \rho_2) < \infty$. Then*

$$\lim_{\alpha \downarrow 0} k_\alpha(\rho_1, \rho_2) = \rho_2(C(\rho_1)^c) \tag{2.21}$$

and

$$\lim_{\alpha \uparrow 1} k_\alpha(\rho_1, \rho_2) = \rho_1(C(\rho_2)^c). \tag{2.22}$$

Proof. Since $k_\alpha(\rho_1, \rho_2) = k_{1-\alpha}(\rho_2, \rho_1)$, (2.21) and (2.22) say the same thing. Let us prove (2.21). Look at the proof of Lemma 2.15. There we have

$$k_\alpha(\rho_1, \rho_2) = \alpha \rho_1(C(\rho_2)^c) + (1-\alpha)\rho_2(C(\rho_1)^c) + \int_C \varphi_\alpha(g) d\rho_1,$$

where $\varphi_\alpha(u) = \alpha + (1-\alpha)e^u - e^{(1-\alpha)u}$ and C is the set defined there. Hence, in order to show (2.21), it is enough to prove that

$$\lim_{\alpha\downarrow 0}\int_C \varphi_\alpha(g)d\rho_1 = 0. \tag{2.23}$$

We claim that there are constants a_1, a_2, a_3 independent of $0<\alpha<1$ such that

$$\varphi_\alpha(u) \le a_1 u^2 \qquad \text{for } |u|\le 1, \tag{2.24}$$

$$\varphi_\alpha(u) \le a_2 e^u \qquad \text{for } u>1, \tag{2.25}$$

$$\varphi_\alpha(u) \le a_3 \qquad \text{for } u<-1. \tag{2.26}$$

Indeed, $\varphi_\alpha(0) = \varphi'_\alpha(0) = 0$ and $\varphi''_\alpha(u) = (1-\alpha)e^u - (1-\alpha)^2 e^{(1-\alpha)u}$, which is bounded with respect to $\alpha\in(0,1)$ and $u\in[0,1]$. This gives (2.24) by Taylor's theorem. (2.26) is obvious. (2.25) is also obvious, since $\varphi_\alpha(u)e^{-u} = 1-\alpha+\alpha e^{-u} - e^{-\alpha u}$ is bounded with respect to $\alpha\in(0,1)$ and $u>1$. Now we can use the bounded convergence theorem by (2.24)—(2.26) and (2.19). Thus (2.23) follows from $\lim_{\alpha\downarrow 0}\varphi_\alpha(u) = 0$. $\square$

For any signed measure σ, we denote by $|\sigma|$ the total variation measure of ρ. For $x\in\mathbb{R}^d$, $|x|$ is the Euclidean norm of x. I hope no confusion arises.

Lemma 2.18. *Let ν_1 and ν_2 be measures on $\mathbb{R}^d$ having no mass at the origin and $\int_{\mathbb{R}^d}(|x|^2\wedge 1)d\nu_j<\infty$ for $j=1,2$. If*

$$k_\alpha(\nu_1,\nu_2)<\infty, \tag{2.27}$$

then

$$\int_{|x|\le 1}|x|d|\nu_1-\nu_2|<\infty, \tag{2.28}$$

$$\int_{|x|\le 1}|x|d|\nu_j - H_\alpha(\nu_1,\nu_2)|<\infty, \qquad j=1,2. \tag{2.29}$$

Proof. By Lemma 2.15, we have $k_{1/2}(\nu_1,\nu_2)<\infty$. Let ν_1, $\nu_2\ll\nu$ and $d\nu_j/d\nu = f_j$. Write $\{|x|\le 1\} = D$. Since

$$|\nu_1-\nu_2| = |f_1-f_2|\nu = |f_1^{1/2}-f_2^{1/2}|(f_1^{1/2}+f_2^{1/2})\nu,$$

we have

$$\int_D |x| d|\nu_1 - \nu_2| \leq \left(\int_D (\sqrt{f_1} - \sqrt{f_2})^2 d\nu \int_D |x|^2 (\sqrt{f_1} + \sqrt{f_2})^2 d\nu \right)^{1/2}$$
$$\leq \left(4k_{1/2}(\nu_1, \nu_2) \int_D |x|^2 (f_1 + f_2) d\nu \right)^{1/2} < \infty,$$

as in Remark 2.14. That is, (2.28) is true. We get (2.29) from (2.28), since

$$\nu_1 - H_\alpha(\nu_1, \nu_2) = K_\alpha(\nu_1, \nu_2) + (1-\alpha)(\nu_1 - \nu_2),$$
$$\nu_2 - H_\alpha(\nu_1, \nu_2) = K_\alpha(\nu_1, \nu_2) - \alpha(\nu_1 - \nu_2).$$

Note that $\int_{|x|\leq 1} |x| dK_\alpha(\nu_1, \nu_2)$ is bounded by $k_\alpha(\nu_1, \nu_2)$. □

We prepare some more lemmas on the properties of $H_\alpha(\rho_1, \rho_2)$. We say that $\{B_n\}$ is a measurable partition of $B \in \mathcal{B}$, if $\{B_n\}$ is a finite or countably infinite family of disjoint sets in $\mathcal{B}$ such that $\bigcup_n B_n = B$. We denote by $\mathcal{P}_B$ the collection of all measurable partitions of B.

Lemma 2.19 (Brody (1971)). *Let ρ_1 and ρ_2 be σ-finite measures. For any $B \in \mathcal{B}$ and $0 < \alpha < 1$ we have*

$$H_\alpha(\rho_1, \rho_2)(B) = \inf_{\{B_n\} \in \mathcal{P}_B} \sum_n \rho_1(B_n)^\alpha \rho_2(B_n)^{1-\alpha}. \tag{2.30}$$

Proof. Let $\rho_1\, \rho_2 \ll \rho$ and $d\rho_j/d\rho = f_j$. Since

$$H_\alpha(\rho_1, \rho_2)(B_n) = \int_{B_n} f_1^\alpha f_2^{1-\alpha} d\rho \leq \rho_1(B_n)^\alpha \rho_2(B_n)^{1-\alpha}$$

by Hölder's inequality as in Remark 2.3, $H_\alpha(\rho_1, \rho_2)(B)$ is less than or equal to the right-hand side of (2.30). Let us show the reverse inequality.

Let $c > 1$ and let

$$B_{l,m} = \{\theta \in B \colon c^l \leq f_1(\theta)^\alpha < c^{l+1} \text{ and } c^m \leq f_2(\theta)^{1-\alpha} < c^{m+1}\},$$
$$B' = \{\theta \in B \colon f_1(\theta) = 0 \text{ or } f_2(\theta) = 0\}.$$

Then $\{B_{l,m}\colon l,m\in\mathbb{Z}\}\cup\{B'\}\in\mathcal{P}_B$ and

$$\begin{aligned}H_\alpha(\rho_1,\rho_2)(B_{l,m}) &\ge c^{l+m}\rho(B_{l,m}),\\ \rho_1(B_{l,m}) &\le c^{(l+1)/\alpha}\rho(B_{l,m}),\\ \rho_2(B_{l,m}) &\le c^{(m+1)/(1-\alpha)}\rho(B_{l,m}).\end{aligned}$$

Hence

$$\begin{aligned}H_\alpha(\rho_1,\rho_2)(B_{l,m}) &\ge c^l\rho(B_{l,m})^\alpha\, c^m\rho(B_{l,m})^{1-\alpha}\\ &\ge c^{-2}\rho_1(B_{l,m})^\alpha\rho_2(B_{l,m})^{1-\alpha},\end{aligned}$$

and we get

$$\begin{aligned}H_\alpha(\rho_1,\rho_2)(B) &\ge c^{-2}\sum_{l,m}\rho_1(B_{l,m})^\alpha\rho_2(B_{l,m})^{1-\alpha}\\ &\ge c^{-2}\inf_{\{B_n\}\in\mathcal{P}_B}\sum_n\rho_1(B_n)^\alpha\rho_2(B_n)^{1-\alpha},\end{aligned}$$

since $H_\alpha(\rho_1,\rho_2)(B')=0$ and $\rho_1(B')^\alpha\rho_2(B')^{1-\alpha}=0$. As $c>1$ is arbitrary, this finishes the proof. $\square$

Let $(\Theta',\mathcal{B}')$ be another measurable space and let $\varphi\colon\Theta\to\Theta'$ be a measurable mapping. For a measure ρ on Θ we denote by $\rho\varphi^{-1}$ the measure on Θ' induced by ρ through φ, that is, $(\rho\varphi^{-1})(B')=\rho(\varphi^{-1}(B'))$ for $B'\in\mathcal{B}'$.

Lemma 2.20. *We have, for any σ-finite measures ρ_1, ρ_2,*

$$h_\alpha(\rho_1\varphi^{-1},\rho_2\varphi^{-1})\ge h_\alpha(\rho_1,\rho_2)\qquad \textit{for } 0<\alpha<1. \tag{2.31}$$

Proof. By Lemma 2.19, we have

$$\begin{aligned}h_\alpha(\rho_1\varphi^{-1},\rho_2\varphi^{-1}) &= \inf_{\{B_n\}\in\mathcal{P}_{\Theta'}}\sum_n(\rho_1\varphi^{-1})(B_n)^\alpha(\rho_2\varphi^{-1})(B_n)^{1-\alpha}\\ &= \inf_{\{B_n\}\in\mathcal{P}_{\Theta'}}\sum_n\rho_1(\varphi^{-1}B_n)^\alpha\rho_2(\varphi^{-1}B_n)^{1-\alpha},\end{aligned}$$

which is bigger than or equal to $H_\alpha(\rho_1,\rho_2)(\Theta)$, since $\{\varphi^{-1}B_n\}$ is a measurable partition of Θ. $\square$

For finite Borel measures ρ and $\rho_1,\rho_2,\dots$ on a metric space, we write $\rho_n\to\rho$ if $\int f\,d\rho_n\to\int f\,d\rho$ for all bounded continuous functions f.

Lemma 2.21 (Newman (1973)). *Let Θ be a metric space and $\mathcal{B}$ be the Borel σ-algebra. Let μ_n, μ, ν_n, ν, and ρ be finite measures on $\mathcal{B}$. Fix $0 < \alpha < 1$. If $\mu_n \to \mu$, $\nu_n \to \nu$, $H_\alpha(\mu_n, \nu_n) \to \rho$, and $\inf_n h_\alpha(\mu_n, \nu_n) \geq h_\alpha(\mu, \nu)$, then $H_\alpha(\mu, \nu) = \rho$.*

Proof. For any bounded continuous function f, we have

$$\int f dH_\alpha(\mu_n, \nu_n) \leq \left(\int f d\mu_n\right)^\alpha \left(\int f d\nu_n\right)^{1-\alpha}$$

again by Hölder's inequality. Hence

$$\int f d\rho \leq \left(\int f d\mu\right)^\alpha \left(\int f d\nu\right)^{1-\alpha}.$$

It follows that, for any $B \in \mathcal{B}$, $\rho(B) \leq \mu(B)^\alpha \nu(B)^{1-\alpha}$. Hence, by Lemma 2.19, $\rho \leq H_\alpha(\mu, \nu)$. On the other hand, it follows from $H_\alpha(\mu_n, \nu_n) \to \rho$ and from $h_\alpha(\mu_n, \nu_n) \geq h_\alpha(\mu, \nu)$ that

$$\rho(\Theta) = \lim_{n\to\infty} h_\alpha(\mu_n, \nu_n) \geq h_\alpha(\mu, \nu).$$

It follows that $\rho(\Theta) = H_\alpha(\mu, \nu)(\Theta)$. Hence $\rho = H_\alpha(\mu, \nu)$. □

The following fact will not be needed in our discussion, but it is an important property of $h_\alpha(\rho_1, \rho_2)$.

Proposition 2.22 (Brody (1971)). *Let $\mathcal{B}_n$ be an increasing sequence of σ-algebras on Θ such that $\mathcal{B}$ is generated by $\bigcup_{n=1}^\infty \mathcal{B}_n$. Then, for any finite measures ρ_1 and ρ_2, we have*

$$h_\alpha(\rho_1, \rho_2) = \lim_{n\to\infty} h_\alpha([\rho_1]_{\mathcal{B}_n}, [\rho_2]_{\mathcal{B}_n}) \qquad (\textit{decreasing limit}). \tag{2.32}$$

Proof. Let φ be the identity mapping on Θ. Then, for any $n < m$, φ is a measurable mapping from $(\Theta, \mathcal{B}_m)$ to $(\Theta, \mathcal{B}_n)$. Hence, by Lemma 2.20, $h_\alpha([\rho_1]_{\mathcal{B}_n}, [\rho_2]_{\mathcal{B}_n})$ is decreasing in n. It is bigger than or equal to $h_\alpha(\rho_1, \rho_2)$ by the same reason. Hence we have (2.32) with $\leq$ in place of $=$. Let us prove that the equality actually holds.

Let $\varepsilon > 0$. By Lemma 2.19, we can find a partition $\{B_l\}$ of Θ in $\mathcal{B}$ such that

$$\sum_{l=1}^\infty \rho_1(B_l)^\alpha \rho_2(B_l)^{1-\alpha} \leq h_\alpha(\rho_1, \rho_2) + \varepsilon.$$

Choose N such that

$$\sum_{l=N+1}^{\infty} \rho_j(B_l) < \varepsilon \quad \text{for } j = 1, 2.$$

For each $l = 1, \ldots, N$, let $\delta_l > 0$ be such that $\delta_l < 2^{-l}\varepsilon$ and

$$(\rho_1(B_l) + \delta_l)^{\alpha} (\rho_2(B_l) + \delta_l)^{1-\alpha} \le \rho_1(B_l)^{\alpha} \rho_2(B_l)^{1-\alpha} + 2^{-l}\varepsilon.$$

There is n such that, we can find sets $C_l \in \mathcal{B}_n$ satisfying

$$\rho_1(B_l \bigtriangleup C_l) < \delta_l, \quad \rho_2(B_l \bigtriangleup C_l) < \delta_l \quad \text{for } l = 1, \ldots, N.$$

Here $B_l \bigtriangleup C_l = (B_l \setminus C_l) \cup (C_l \setminus B_l)$, the symmetric difference of B_l and C_l. Let

$$D_1 = C_1, \ \ D_2 = C_2 \setminus C_1, \ \ \ldots, \ \ D_N = C_N \setminus \bigcup_{l=1}^{N-1} C_l.$$

Then $D_l \in \mathcal{B}_n$ and we have

$$\begin{aligned} \rho_1(\bigcup_{l=1}^{N} D_l) &= \rho_1(\bigcup_{l=1}^{N} C_l) \ge \rho_1(\bigcup_{l=1}^{N}(C_l \cap B_l)) \ge \sum_{l=1}^{N}(\rho_1(B_l) - 2^{-l}\varepsilon) \\ &\ge \rho_1(\Theta) - 2\varepsilon \end{aligned}$$

and

$$\rho_2(\bigcup_{l=1}^{N} D_l) \ge \rho_2(\Theta) - 2\varepsilon.$$

Letting $D_{N+1} = \Theta \setminus \bigcup_{l=1}^{N} D_l$, consider the partition $\{D_l \colon l = 1, \ldots, N+1\}$. Then

$$\rho_1(D_{N+1})^{\alpha} \rho_2(D_{N+1})^{1-\alpha} \le (2\varepsilon)^{\alpha}(2\varepsilon)^{1-\alpha} = 2\varepsilon.$$

We have

$$\begin{aligned} \sum_{l=1}^{N} \rho_1(D_l)^{\alpha} \rho_2(D_l)^{1-\alpha} &\le \sum_{l=1}^{N} \rho_1(C_l)^{\alpha} \rho_2(C_l)^{1-\alpha} \\ &\le \sum_{l=1}^{N} (\rho_1(B_l) + \delta_l)^{\alpha} (\rho_2(B_l) + \delta_l)^{1-\alpha} \\ &\le \sum_{l=1}^{N} \rho_1(B_l)^{\alpha} \rho_2(B_l)^{1-\alpha} + \varepsilon \le h_{\alpha}(\rho_1, \rho_2) + 2\varepsilon. \end{aligned}$$

Hence

$$\sum_{l=1}^{N+1} \rho_1(D_l)^\alpha \, \rho_2(D_l)^{1-\alpha} \le h_\alpha(\rho_1, \rho_2) + 4\varepsilon.$$

By Lemma 2.19, we get

$$h_\alpha([\rho_1]_{\mathcal{B}_n}, [\rho_2]_{\mathcal{B}_n}) \le h_\alpha(\rho_1, \rho_2) + 4\varepsilon,$$

completing the proof. □

3. Main results

The definition of Lévy processes on $\mathbb{R}^d$ is given in Section 1. The following two theorems are basic in the theory of Lévy processes.

Theorem 3.1 (Lévy–Khintchine representation). (i) *If $\{X_t\}$ is a Lévy process on $\mathbb{R}^d$, then*

$$E[e^{i\langle z,X_t\rangle}] = \exp\left[t\left(-\frac{1}{2}\langle z,Az\rangle + i\langle\gamma,z\rangle + \int_{\mathbb{R}^d}(e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle 1_{\{|x|\le 1\}}(x))\nu(dx)\right)\right], \tag{3.1}$$

for $z \in \mathbb{R}^d$, where

$$\begin{cases} A \text{ is a symmetric nonnegative-definite } d\times d \text{ matrix,} \\ \gamma \in \mathbb{R}^d, \nu \text{ is a measure on } \mathbb{R}^d \text{ satisfying } \nu(\{0\}) = 0 \\ \text{and } \int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty. \end{cases} \tag{3.2}$$

These A, γ, and ν are uniquely determined by $\{X_t\}$.

(ii) *For any A, γ, and ν satisfying* (3.2), *there is a Lévy process $\{X_t\}$ satisfying* (3.1). *It is unique in law.*

This is a combination of [S] Theorems 7.10, 8.1, and 11.5. We call (A,ν,γ) the *generating triplet*, A the *Gaussian covariance matrix*, and ν the *Lévy measure* of the Lévy process $\{X_t\}$.

To state the second theorem, we give a definition and a proposition concerning Poisson random measures.

Definition 3.2. Let $(\Theta,\mathcal{B},\rho)$ be a σ-finite measure space. A family of random variables $\{N(B)\colon B\in\mathcal{B}\}$, defined on a probability space $(\Omega,\mathcal{F},P)$ and taking values in $\{0,1,2,\dots,+\infty\}$, is called a *Poisson random measure* with *intensity measure* ρ if

(1) for every B, $N(B)$ has Poisson distribution[3] with mean $\rho(B)$,
(2) for any n and for any disjoint $B_1,\dots,B_n$, the random variables $N(B_1),\dots,N(B_n)$ are independent,
(3) for every fixed ω, $N(B,\omega)$, $B\in\mathcal{B}$, is a measure.

[3] We use the following convention: Poisson distributions with means 0 and ∞ are, respectively, δ_0 and δ_∞. In general, we denote by δ_a the unit mass concentrated at a.

Proposition 3.3. *Let $(\Theta, \mathcal{B}, \rho)$ be a measure space with $\rho(\Theta) < \infty$ and $\{N(B)\colon B \in \mathcal{B}\}$ be a Poisson random measure with intensity measure ρ. Let g be a measurable function from Θ to $\mathbb{R}^n$ and define*

$$Y(\omega) = \int_\Theta g(\theta) N(d\theta, \omega).$$

Then Y is a random variable on $\mathbb{R}^n$ with compound Poisson distribution and

$$\begin{aligned} E[e^{i\langle z,Y\rangle}] &= \exp\left(\int_\Theta (e^{i\langle z,g(\theta)\rangle} - 1)\rho(d\theta)\right) \\ &= \exp\left(\int_{\mathbb{R}^n} (e^{i\langle z,x\rangle} - 1)(\rho g^{-1})(dx)\right) \end{aligned}$$

for $z \in \mathbb{R}^n$.

This is [S] Proposition 19.5.

Theorem 3.4 (Lévy–Itô decomposition of sample functions). *Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with generating triplet (A, ν, γ). For any $G \in \mathcal{B}_{(0,\infty)\times\mathbb{R}^d}$ and $\omega \in \Omega$ let*[4] *$J(G) = J(G, \omega)$ be the number of $s > 0$ such that $(s, X_s(\omega) - X_{s-}(\omega)) \in G$. Then $\{J(G)\colon G \in \mathcal{B}_{(0,\infty)\times\mathbb{R}^d}\}$ is a Poisson random measure with intensity measure $\widetilde{\nu}$, where $\widetilde{\nu}$ is the product measure $ds \times \nu(dx)$. We can define, a. s.,*

$$\begin{aligned} X'_t(\omega) = &\lim_{\varepsilon\downarrow 0} \int_{(0,t]\times\{\varepsilon<|x|\le 1\}} \{xJ(d(s,x),\omega) - x\widetilde{\nu}(d(s,x))\} \\ &+ \int_{(0,t]\times\{|x|>1\}} xJ(d(s,x),\omega), \end{aligned} \tag{3.3}$$

where the convergence in the right-hand side is uniform in t in any finite time interval a. s. The process $\{X'_t\}$ is a Lévy process with generating triplet $(0, \nu, 0)$. Let

$$X''_t(\omega) = X_t(\omega) - X'_t(\omega). \tag{3.4}$$

Then $\{X''_t\}$ is a Lévy process continuous in t a. s. with generating triplet $(A, 0, \gamma)$. The two processes $\{X'_t\}$ and $\{X''_t\}$ are independent.

[4] $\mathcal{B}_{(0,\infty)\times\mathbb{R}^d}$ is the class of Borel sets in $(0,\infty)\times\mathbb{R}^d$.

This is [S] Theorem 19.2.

As is explained in Section 1, when we say that $(\{X_t\}, P)$ is a Lévy process on $\mathbb{R}^d$, we mean that P is a probability measure on $(\mathbf{D}, \mathcal{F}^0)$ and $\{X_t\}$ is a Lévy process under the measure P, where $\mathbf{D}$ and $\mathcal{F}^0$, and $X_t(\omega)$ are introduced in Section 1; in particular, $X_t(\omega)$ is the coordinate mapping from $\mathbf{D}$ to $\mathbb{R}^d$.

In this section we consider two Lévy processes $(\{X_t\}, P_1)$ and $(\{X_t\}, P_2)$ on $\mathbb{R}^d$. Keep in mind that $\mathbf{D}$, $\mathcal{F}^0$, and $\{X_t\}$ are common and only the measures P_1 and P_2 are different. The generating triplets of $(\{X_t\}, P_1)$ and $(\{X_t\}, P_2)$ are denoted by (A_1, ν_1, γ_1) and (A_2, ν_2, γ_2), respectively. When $A_1 = A_2$, we write $A_1 = A_2 = A$. In this case define

$$\mathfrak{R}(A) = \{Ax \colon x \in \mathbb{R}^d\},$$

the range of the mapping of x to Ax. We write

$$P_1^t = [P_1]_{\mathcal{F}_t^0} \quad \text{and} \quad P_2^t = [P_2]_{\mathcal{F}_t^0},$$

where $\mathcal{F}_t^0$ is also introduced in Section 1.

The following Theorem A was given by Newman (1972, 73) together with Corollaries 3.6, 3.8, and 3.9.

Theorem A. (i) *Suppose that*

$$\text{(NS)} \qquad k_\alpha(\nu_1, \nu_2) < \infty, \quad A_1 = A_2, \quad \textit{and} \quad \gamma_{21} \in \mathfrak{R}(A),$$

where

$$\gamma_{21} = \gamma_2 - \gamma_1 - \int_{|x|\le 1} x d(\nu_2 - \nu_1). \tag{3.5}$$

Then

$$H_\alpha(P_1^t, P_2^t) = e^{-tL_\alpha} P_\alpha^t \quad \textit{for } t > 0,\ 0 < \alpha < 1, \tag{3.6}$$

where

$$L_\alpha = \tfrac{1}{2}\alpha(1-\alpha)\langle \eta, A\eta\rangle + k_\alpha(\nu_1, \nu_2) \tag{3.7}$$

with η satisfying $A\eta = \gamma_{21}$, and P_α is the probability measure for which $(\{X_t\}, P_\alpha)$ is the Lévy process generated by $(A, H_\alpha(\nu_1, \nu_2), \gamma_\alpha)$ with

$$\gamma_\alpha = \alpha\gamma_1 + (1-\alpha)\gamma_2 - \int_{|x|\le 1} x dK_\alpha(\nu_1, \nu_2). \tag{3.8}$$

(ii) *Suppose that* (NS) *is not satisfied, then*

$$H_\alpha(P_1^t, P_2^t) = 0 \quad \text{for } t > 0,\ 0 < \alpha < 1. \tag{3.9}$$

Notice that, by Lemma 2.15, the finiteness of $k_\alpha(\nu_1, \nu_2)$ does not depend on α, and that, if $k_\alpha(\nu_1, \nu_2) < \infty$, then γ_{21} is well-defined by virtue of (2.28) of Lemma 2.18. Also, $H_\alpha(\nu_1, \nu_2)$ can be the Lévy measure of a Lévy process by the property (2.3) in Remark 2.3. The quantity L_α does not depend on the choice of η satisfying $A\eta = \gamma_{21}$, since $A\eta = A\eta'$ implies

$$\langle \eta, A\eta \rangle = \langle \eta, A\eta' \rangle = \langle A\eta, \eta' \rangle = \langle A\eta', \eta' \rangle = \langle \eta', A\eta' \rangle$$

by the symmetry of A. We call (NS) the *nonsingularity condition.* When we say that (NS) is not satisfied, we mean that one of the following holds:

$$k_\alpha(\nu_1, \nu_2) = \infty; \tag{3.10}$$

$$k_\alpha(\nu_1, \nu_2) < \infty \text{ and } A_1 \neq A_2; \tag{3.11}$$

$$k_\alpha(\nu_1, \nu_2) < \infty, A_1 = A_2, \text{ and } \gamma_{21} \notin \mathfrak{R}(A). \tag{3.12}$$

Note that, if $k_\alpha(\nu_1, \nu_2) = \infty$, then γ_{21} may not be defined.

Remark 3.5. Suppose that $\int_{|x|\le 1} |x|\nu_1(dx) < \infty$ and $\int_{|x|\le 1} |x|\nu_2(dx) < \infty$. Then, for $j = 1, 2$, the Lévy–Khintchine representation is written as

$$E^{P_j}[e^{\langle z, X_t \rangle}] = \exp\left[t \left(-\frac{1}{2}\langle z, A_j z \rangle + i\langle \gamma_{0j}, z \rangle + \int_{\mathbb{R}^d} (e^{i\langle z, x \rangle} - 1)\nu_j(dx) \right) \right]$$

with some $\gamma_{0j} \in \mathbb{R}^d$. The vector γ_{0j} is called the *drift* of the Lévy process $(\{X_t\}, P_j)$ and we have

$$\gamma_{0j} = \gamma_j - \int_{|x|\le 1} x\nu_j(dx).$$

Hence we have

$$\gamma_{21} = \gamma_{02} - \gamma_{01}. \tag{3.13}$$

The proof of Theorem A is postponed to Section 5. Here we study some consequences of Theorem A.

Corollary 3.6. *Fix $t > 0$. P_2^t and P_1^t are not mutually singular if and only if Condition* (NS) *is satisfied.*

Proof. By Remark 2.5, $P_2^t \perp P_1^t$ if and only if (3.9) holds. By Theorem A, (NS) does not hold if and only if (3.9) holds. □

Corollary 3.7. *If $P_2^t \perp P_1^t$ for some $t > 0$, then $P_2^t \perp P_1^t$ for all $t > 0$.*

Proof. Condition (NS) does not involve t. □

Corollary 3.8. *Fix $t > 0$. If P_2^t and P_1^t are not mutually singular, then*

$$\nu_2(C(\nu_1)^c) < \infty \quad \textit{and} \quad \nu_1(C(\nu_2)^c) < \infty \tag{3.14}$$

and

$$P_2^t(C(P_1^t)) = e^{-t\nu_2(C(\nu_1)^c)} \quad \textit{and} \quad P_1^t(C(P_2^t)) = e^{-t\nu_1(C(\nu_2)^c)}. \tag{3.15}$$

Proof. We assume that P_1^t and P_2^t are not mutually singular. Then Condition (NS) holds. In particular, $k_\alpha(\nu_1, \nu_2) < \infty$. Hence (3.14) holds by Lemma 2.15, and

$$\lim_{\alpha \downarrow 0} k_\alpha(\nu_1, \nu_2) = \nu_2(C(\nu_1)^c)$$

by Lemma 2.17. Hence, as $\alpha \downarrow 0$, L_α tends to the same limit. The identity (3.6) of Theorem A implies that

$$h_\alpha(P_1^t, P_2^t) = e^{-tL_\alpha}. \tag{3.16}$$

By Lemma 2.11, $h_\alpha(P_1^t, P_2^t)$ tends to $P_2^t(C(P_1^t))$ as $\alpha \downarrow 0$. Thus we obtain the first identity of (3.15). The second is similar. □

Corollary 3.9. *Fix $t > 0$. $P_2^t \ll P_1^t$ if and only if the condition $\nu_2 \ll \nu_1$ and Condition* (NS) *are both satisfied.*

Proof. By Remark 2.10, $P_2^t \ll P_1^t$ if and only if $P_2^t(C(P_1^t)^c) = 0$, that is, $P_2^t(C(P_1^t)) = 1$. Hence, if $P_2^t \ll P_1^t$, then $\nu_2(C(\nu_1)^c) = 0$ by Corollary 3.8, that is, $\nu_2 \ll \nu_1$ by Remark 2.10. Conversely, if $\nu_2 \ll \nu_1$ and (NS) holds, then $\nu_2(C(\nu_1)^c) = 0$ and P_1^t and P_2^t are not mutually singular by Corollary 3.6, hence $P_2^t(C(P_1^t)) = 1$ by Corollary 3.8. □

Corollary 3.10. *If $P_2^t \ll P_1^t$ for some $t > 0$, then $P_2^t \ll P_1^t$ for all $t > 0$.*

Proof. Use Corollary 3.9. □

Corollary 3.11. *Fix $t > 0$. $P_2^t \approx P_1^t$ if and only if the condition $\nu_2 \approx \nu_1$ and Condition* (NS) *are both satisfied.*

Proof. Immediate from Corollary 3.9.

Corollary 3.12. *If $P_2^t \approx P_1^t$ for some $t > 0$, then $P_2^t \approx P_1^t$ for all $t > 0$.*

Proof. Immediate from Corollary 3.10 or 3.11.

Corollary 3.13 (dichotomy). *If $\nu_2 \approx \nu_1$, then either $P_2^t \approx P_1^t$ for all $t > 0$ or $P_2^t \perp P_1^t$ for all $t > 0$.*

Proof. Assume $\nu_2 \approx \nu_1$. If (NS) is satisfied, then $P_2^t \approx P_1^t$ by Corollary 3.9. Otherwise $P_2^t \perp P_1^t$ by Corollary 3.6. □

Corollary 3.8 combined with Lemma 2.9 shows that, in general, there is a case where P_1^t and P_2^t are neither mutually absolutely continuous nor mutually singular. The Lebesgue decomposition of P_2^t with respect to P_1^t will be given in Theorem B.

Let us consider P_1 and P_2 on the whole $\mathcal{F}^0$.

Corollary 3.14. *$P_2 \perp P_1$ if $P_2 \neq P_1$.*

Proof. Assume that $P_2 \neq P_1$. We claim that $h_\alpha(P_1, P_2) = 0$. Applying Lemma 2.20 to the identity mapping of $\mathbf{D}$, we get

$$h_\alpha(P_1^t, P_2^t) \geq h_\alpha(P_1, P_2).$$

If Condition (NS) does not hold, then $h_\alpha(P_1^t, P_2^t) = 0$ by Theorem A and there is nothing to prove. Assume that (NS) holds. Then $h_\alpha(P_1^t, P_2^t) = e^{-tL_\alpha}$ by Theorem A. If $L_\alpha > 0$, then $h_\alpha(P_1^t, P_2^t)$ can be arbitrarily small, which implies that $h_\alpha(P_1, P_2) = 0$. Suppose now that $L_\alpha = 0$. Then $k_\alpha(\nu_1, \nu_2) = 0$ and $\langle \eta, A\eta \rangle = 0$ with $A\eta = \gamma_{21}$. Since we can choose α as we like, let us fix it as $\alpha = 1/2$. Then

$$0 = k_{1/2}(\nu_1, \nu_2) = \int (\sqrt{f_1} - \sqrt{f_2})^2 d\nu$$

with $\nu_j \ll \nu$ and $f_j = d\nu_j/d\nu$ for $j = 1, 2$. Hence $f_1 = f_2$ ν-a. e., that is, $\nu_1 = \nu_2$. Hence $\gamma_{21} = \gamma_2 - \gamma_1$. Since A is symmetric and nonnegative-definite, there is a symmetric nonnegative-definite matrix R such that $R^2 = A$. We have $|R\eta|^2 = \langle \eta, A\eta \rangle = 0$, hence $R\eta = 0$ and $A\eta = RR\eta = 0$, hence $\gamma_2 = \gamma_1$. Thus $(A_1, \nu_1, \gamma_1) = (A_2, \nu_2, \gamma_2)$, which contradicts the assumption $P_1 \neq P_2$. This shows that the case $L_\alpha = 0$ does not occur. □

Corollary 3.15. *Suppose that P_2^t and P_1^t are not mutually singular for some $t > 0$. Then the following are true.*

(i) *If $\nu_1(\mathbb{R}^d) < \infty$, then $\nu_2(\mathbb{R}^d) < \infty$.*

(ii) *If $\nu_1(\mathbb{R}^d) = \infty$ and $\int_{|x|\le 1} |x|\nu_1(dx) < \infty$, then $\nu_2(\mathbb{R}^d) = \infty$ and $\int_{|x|\le 1} |x|\nu_2(dx) < \infty$.*

(iii) *If $\int_{|x|\le 1} |x|\nu_1(dx) = \infty$, then $\int_{|x|\le 1} |x|\nu_2(dx) = \infty$.*

Proof. (i) Assume that $\nu_1(\mathbb{R}^d) < \infty$. Let $\Lambda = \{\omega\colon \text{there are } s_n \downarrow 0 \text{ such that } X_{s_n}(\omega) \ne X_{s_n-}(\omega)\}$. That is, Λ is the event that jumping times are accumulated at 0 from above. If $\nu_2(\mathbb{R}^d) = \infty$, then $P_2^t(\Lambda) = 1$ and $P_1^t(\Lambda) = 0$ (see [S] Theorem 21.3), which implies $P_2^t \perp P_1^t$, contrary to the assumption.

(ii) We have $k_\alpha(\nu_1, \nu_2) < \infty$ by Corollary 3.6. Hence $\int_{|x|\le 1} |x| d|\nu_2 - \nu_1| < \infty$ by Lemma 2.18. Therefore, if $\int_{|x|\le 1} |x| d\nu_1 < \infty$, then

$$\int_{|x|\le 1} |x| d\nu_2 \le \int_{|x|\le 1} |x| d\nu_1 + \int_{|x|\le 1} |x| d|\nu_2 - \nu_1| < \infty.$$

Interchanging the roles of ν_1 and ν_2 and using (i), we see that, if $\nu_1(\mathbb{R}^d) = \infty$, then $\nu_2(\mathbb{R}^d) = \infty$.

(iii) The assertion is that $\int_{|x|\le 1} |x| d\nu_2 < \infty$ implies $\int_{|x|\le 1} |x| d\nu_1 < \infty$. This is a consequence of (i) and (ii), if we interchange the roles of ν_1 and ν_2. $\square$

Theorem A and its corollaries above solve the problem 1 in the introduction. The following theorem solves the problems 2, 3, and 4.

Let us denote by $(P_2^t)^{ac}$ and $(P_2^t)^s$ the absolutely continuous part and the singular part, respectively, in the Lebesgue decomposition of P_2^t with respect to P_1^t. Also, denote by ${\nu_2}^{ac}$ and ${\nu_2}^s$ the absolutely continuous part and the singular part, respectively, of ν_2 with respect to ν_1. Let $\nu = \nu_1 + \nu_2$. Choose the versions

$$\frac{d\nu_j}{d\nu} = f_j \qquad \text{for } j = 1, 2 \tag{3.17}$$

satisfying

$$f_1 \ge 0, \quad f_2 \ge 0, \quad \text{and } f_1 + f_2 = 1 \quad \text{everywhere on } \mathbb{R}^d. \tag{3.18}$$

Denote

$$\left\{\begin{aligned} C_1 &= \{f_1 = 1 \text{ and } f_2 = 0\}, \\ C_2 &= \{f_1 = 0 \text{ and } f_2 = 1\}, \\ C &= \{f_1 > 0 \text{ and } f_2 > 0\}. \end{aligned}\right. \tag{3.19}$$

Thus

$$\nu_2{}^{ac} = 1_C \nu_2 \quad \text{and} \quad \nu_2{}^{s} = 1_{C_2}\nu_2 = 1_{C_1 \cup C_2}\nu_2 \tag{3.20}$$

and $d\nu_2{}^{ac}/d\nu_1$ has the following version:

$$\frac{d\nu_2{}^{ac}}{d\nu_1} = \begin{cases} f_2/f_1 & \text{on } C \\ 0 & \text{on } C_1 \cup C_2. \end{cases} \tag{3.21}$$

Define

$$g(x) = \begin{cases} \log(f_2/f_1) & \text{on } C \\ -\infty & \text{on } C_1 \cup C_2. \end{cases} \tag{3.22}$$

$$\widetilde{g}(x) = \begin{cases} g(x) & \text{on } C \\ 0 & \text{on } C_1 \cup C_2. \end{cases} \tag{3.23}$$

As in Theorem 3.4, let $J(G,\omega)$ be the number of $s > 0$ such that $(s, X_s(\omega) - X_{s-}(\omega)) \in G$. Define $\Lambda_t \in \mathcal{F}_t^0$ by

$$\begin{aligned} \Lambda_t &= \{J((0,t] \times (C_1 \cup C_2)) = 0\} \\ &= \{X_s - X_{s-} \notin C_1 \cup C_2 \text{ for all } s \in (0,t]\}. \end{aligned} \tag{3.24}$$

Theorem B. *Suppose that P_2^t and P_1^t are not mutually singular. Then the following are true.*

(i) *For $0 < t < \infty$ the Lebesgue decomposition of P_2^t with respect to P_1^t is given by*

$$(P_2^t)^{ac} = 1_{\Lambda_t} P_2^t, \tag{3.25}$$

$$(P_2^t)^{s} = 1_{\mathbf{D}\setminus\Lambda_t} P_2^t. \tag{3.26}$$

(ii) *Define*[5]

$$V_t = \lim_{\varepsilon \downarrow 0}\Bigg(\sum_{(s, X_s - X_{s-}) \in (0,t] \times \{|x| > \varepsilon\}} \widetilde{g}(X_s - X_{s-}) \tag{3.27}$$

[5] Notice that, in (3.27), both $g(x)$ and $\widetilde{g}(x)$ are used.

$$- t \int_{|x|>\varepsilon} (e^{g(x)} - 1)\nu_1(dx) \Bigg);$$

the right-hand side exists P_1-a. s. and the convergence is uniform on any bounded time interval P_1-a. s.

(iii) *Let $\eta \in \mathbb{R}^d$ and define*

$$U_t = \langle \eta, X''_t \rangle - \frac{t}{2} \langle \eta, A\eta \rangle - t\langle \gamma_1, \eta \rangle + V_t, \tag{3.28}$$

where $\{X''_t\}$ is the continuous process derived from the process $(\{X_t\}, P_1)$ in Theorem 3.4. Then $\{U_t\}$ is, under P_1, a Lévy process on $\mathbb{R}$ with generating triplet (A_U, ν_U, γ_U) given by

$$A_U = \langle \eta, A\eta \rangle, \tag{3.29}$$

$$\nu_U(B) = \int_{\mathbb{R}^d} 1_B(g(x))\nu_1(dx) \quad \text{for } B \in \mathcal{B}_{\mathbb{R}\setminus\{0\}}, \tag{3.30}$$

$$\begin{aligned} \gamma_U = &-\tfrac{1}{2}\langle \eta, A\eta \rangle \\ &- \int_{\mathbb{R}^d} (e^{g(x)} - 1 - g(x)1_{\{|g(x)|\le 1\}}(x))\nu_1(dx). \end{aligned} \tag{3.31}$$

The processes $\{U_t\}$ and $\{J((0,t] \times (C_1 \cup C_2)) \colon t \ge 0\}$ are independent under P_1. We have $P_1(\Lambda_t) = e^{-t\nu_1(C_1)}$ and $P_2(\Lambda_t) = e^{-t\nu_2(C_2)}$.

(iv) *Choose η so that $A\eta = \gamma_{21}$. Then the Radon–Nikodým density of $(P_2^t)^{ac}$ is given by*

$$\frac{d(P_2^t)^{ac}}{dP_1^t} = e^{-t\nu_2(C_2)+U_t} 1_{\Lambda_t}. \tag{3.32}$$

Let Q be the probability measure on $(\mathbf{D}, \mathcal{F}^0)$ for which $(\{X_t\}, Q)$ is the Lévy process with generating triplet $(A, {\nu_2}^{ac}, \gamma_2 - \int_{|x|\le 1} x d{\nu_2}^s)$. Then

$$(P_2^t)^{ac} = e^{-t\nu_2(C_2)} Q^t. \tag{3.33}$$

We will prove Theorem B in Section 6.

Remark 3.16. The process $\{U_t\}$ does not depend on the choice of η satisfying $A\eta = \gamma_{21}$. If $A\eta = A\eta'$, then

$$\langle \eta, X''_t \rangle - \frac{t}{2}\langle \eta, A\eta \rangle - t\langle \gamma_1, \eta \rangle = \langle \eta', X''_t \rangle - \frac{t}{2}\langle \eta', A\eta' \rangle - t\langle \gamma_1, \eta' \rangle,$$

P_1-a. s. In fact, we have $\langle \eta, A\eta \rangle = \langle \eta', A\eta' \rangle$, as is remarked after Theorem A. To see that $\langle \eta, X''_t \rangle - t\langle \gamma_1, \eta \rangle = \langle \eta', X''_t \rangle - t\langle \gamma_1, \eta' \rangle$, P_1-a. s., note that $Y_t = \langle \eta - \eta', X''_t \rangle - t\langle \gamma_1, \eta - \eta' \rangle$ is, under P_1, a Lévy process with

$$E^{P_1}[e^{iuY_t}] = e^{-(t/2)\langle u(\eta-\eta'), uA_1(\eta-\eta')\rangle} = 1,$$

and hence $Y_t = 0$, P_1-a. s.

Remark 3.17. The proof that $\Lambda_t \in \mathcal{F}_t^0$ is as follows. Let $t_{n,j}(\omega)$ be the jth jumping time with jumping height $X_s - X_{s-}$ in $\{1/n < |x| \leq 1/(n-1)\}$ if such a jump exists. Let $t_{n,j}(\omega) = \infty$ if such a jump does not exist. Then $\{t_{n,j}(\omega)\}_{n\geq 1, j\geq 1}$ is the collection of all jumping times of ω and, possibly, ∞. Each $t_{n,j}$ is $\mathcal{F}^0$-measurable ([S] Lemma 20.9). Hence $X_{t_{n,j}}$ and $X_{t_{n,j}-}$ are also $\mathcal{F}^0$-measurable. Since

$$J((0,t] \times (C_1 \cup C_2), \omega) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} 1_{(0,t]}(t_{n,j}) 1_{C_1 \cup C_2}(X_{t_{n,j}} - X_{t_{n,j}-}),$$

$J((0,t] \times (C_1 \cup C_2), \omega)$ is $\mathcal{F}^0$-measurable. We can see that it is $\mathcal{F}_t^0$-measurable, using the fact that $\Lambda \in \mathcal{F}_t^0$ if and only if the following two conditions are satisfied:

(1) $\Lambda \in \mathcal{F}^0$,
(2) if $\omega \in \Lambda$ and $X_s(\omega) = X_s(\omega')$ for $s \in [0,t]$, then $\omega' \in \Lambda$.

See the first 9 lines of 3.2 of Itô and McKean (1965). Thus $\Lambda_t \in \mathcal{F}_t^0$.

Corollary 3.18. *Let $t > 0$. Suppose that $P_2^t \ll P_1^t$ (hence $\nu_2 \ll \nu_1$). Let*

$$\frac{d\nu_2}{d\nu_1} = e^g,$$

where g is a version satisfying $-\infty \leq g < \infty$ everywhere on $\mathbb{R}^d$. Let $C = \{-\infty < g < \infty\}$ and $C_1 = \{g = -\infty\}$. Define $\widetilde{g}$ and Λ_t by (3.23) *and* (3.24) *with $C_2 = \emptyset$, and then V_t and U_t by* (3.27) *and* (3.28) *with η satisfying $A\eta = \gamma_{21}$. Then*

$$\frac{dP_2^t}{dP_1^t} = e^{U_t} 1_{\Lambda_t}. \tag{3.34}$$

Here $\{U_t\colon t \geq 0\}$ and $\{1_{\Lambda_t}\colon t \geq 0\}$ are independent. $\{U_t\}$ is a Lévy process on $\mathbb{R}$ under P_1, and $P_1(\Lambda_t) = e^{-t\nu_1(C_1)}$, $P_2(\Lambda_t) = 1$.

Proof. We have $\nu = \nu_1 + \nu_2 = (1 + e^g)\nu_1$. Hence we can choose $f_1 = 1/(1 + e^g)$ and $f_2 = e^g/(1 + e^g)$, and Theorem B applies. □

Remark 3.19. The expression (3.34) of Corollary 3.18 can be written as

$$\frac{dP_2^t}{dP_1^t} = e^{U_t^*(\omega)}, \tag{3.35}$$

where

$$U_t^* = \langle \eta, X_t'' \rangle - \tfrac{t}{2}\langle \eta, A_1 \eta \rangle - t\langle \gamma_1, \eta \rangle$$
$$+ \lim_{\varepsilon \downarrow 0} \left(\sum_{(s, X_s - X_{s-}) \in (0,t] \times \{|x| > \varepsilon\}} g(X_s - X_{s-}) - t \int_{|x|>\varepsilon} (e^{g(x)} - 1)\nu_1(dx) \right).$$

If $X_s - X_{s-} \in C_1$ at time s, then $U_u^* = -\infty$ and $e^{U_u^*} = 0$ for all $u \geq s$.

Remark 3.20. Let us consider another expression of (3.34) of Corollary 3.18. Define the σ-algebra $\mathcal{F}_{t+}^0$ as $\mathcal{F}_{t+}^0 = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}^0$. Let

$$T = \inf\{s > 0 \colon X_s - X_{s-} \in C_1\}. \tag{3.36}$$

It follows from the definition of Λ_t that

$$\{t < T\} \subset \Lambda_t \subset \{t \leq T\}. \tag{3.37}$$

We know that $\Lambda_t \in \mathcal{F}_t^0$ by Remark 3.17. Since, for the general ω, jumping times with heights in C_1 possibly accumulate at T from above, the relation of T and $\{\mathcal{F}_t^0\}$ is delicate. We claim that

$$\{t < T\} \in \mathcal{F}_{t+}^0 \quad \text{and} \quad \{t \leq T\} \in \mathcal{F}_t^0. \tag{3.38}$$

Let us use $t_{n,j}(\omega)$ in Remark 3.17. We have $\{T < t\} \in \mathcal{F}^0$ since

$$\{T < t\} = \bigcup_{n,j} \{t_{n,j} < t \text{ and } X_{t_{n,j}} - X_{t_{n,j}-} \in C_1\}.$$

Further we have $\{T < t\} \in \mathcal{F}_t^0$, as we can check the conditions (1) and (2) of Remark 3.17. Hence $\{t \leq T\} \in \mathcal{F}_t^0$. Since

$$\{t < T\} = \bigcup_{n=n_0}^{\infty} \{t + \tfrac{1}{n} \leq T\} \qquad \text{for any } n_0 \geq 1,$$

we see $\{t < T\} \in \mathcal{F}_{t+}^0$. Thus (3.38) is proved. Theorem 3.4 says that, under P_1, $\{J((0,t] \times C_1) \colon t \geq 0\}$ is a Poisson process with parameter

$\nu_1(C_1)$. We have $\nu_1(C_1) < \infty$ by Lemma 2.15, because the assumption $P_2^t \ll P_1^t$ implies $k_\alpha(\nu_1, \nu_2) < \infty$ by Corollary 3.9. Since T is the first jumping time of the process $\{J((0,t] \times C_1)\colon t \geq 0\}$, T has exponential distribution with parameter $\nu_1(C_1)$ under P_1, if $\nu_1(C_1) > 0$. If $\nu_1(C_1) = 0$, then $T = \infty$, P_1-a. s. Hence $P(t = T) = 0$, which implies that the three sets in (3.37) differ only by sets of P_1-measure 0. Therefore, the expression (3.34) can be written as

$$P_2(B) = \int_B e^{U_t} 1_{\Lambda_t} dP_1 = \int_B e^{U_t} 1_{\{t<T\}} dP_1 = \int_B e^{U_t} 1_{\{t\leq T\}} dP_1, \quad B \in \mathcal{F}_t^0.$$

But $e^{U_t} 1_{\{t<T\}}$ does not give dP_2^t/dP_1^t, since it is beyond $\mathcal{F}_t^0$. Notice that both $\{J((0,t] \times C_1)\}$ and T are independent of $\{U_t\}$ under P_1 by Theorem 3.4, because $\{U_t\}$ is determined by $\{X_t''\}$ and by jumps of heights in C.

Example 3.21. Suppose that $0 \leq \nu_1(\mathbb{R}^d) < \infty$. Then, P_2^t and P_1^t are not mutually singular if and only if

$$0 \leq \nu_2(\mathbb{R}^d) < \infty, \quad A_2 = A_1, \quad \text{and} \quad \gamma_{21} \in \mathfrak{R}(A).$$

This follows from Corollaries 3.6 and 3.15. In this case $\gamma_{21} = \gamma_{02} - \gamma_{01}$ by (3.13), where γ_{0j}, $j = 1$, 2, are the drifts.

Suppose that $0 \leq \nu_1(\mathbb{R}^d) < \infty$ and that P_2^t and P_1^t are not mutually singular. By (3.33),

$$(P_2^t)^{ac}(\mathbf{D}) = e^{-t{\nu_2}^s(\mathbb{R}^d)}.$$

We consider two special cases.

Case 1: $\nu_2 \ll \nu_1$. We have $P_2^t \ll P_1^t$ and $P_2^t = e^{U_t} 1_{\Lambda_t} P_1^t$ with U_t and Λ_t defined by (3.24) and (3.27)–(3.28), since $\nu_2(C_2) = 0$.

Case 2: $\nu_2 \perp \nu_1$. We have $g = -\infty$ and $\widetilde{g} = 0$ on $\mathbb{R}^d$. Thus

$$\Lambda_t = \{J((0,t] \times \mathbb{R}^d) = 0\} = \{\omega\colon X_s(\omega) \text{ is continuous in } s \in [0,t]\},$$

$$V_t = t\nu_1(\mathbb{R}^d),$$
$$(P_2^t)^{ac} = e^{-t\nu_2(\mathbb{R}^d) + U_t} 1_{\Lambda_t} P_1^t,$$
$$(P_2^t)^{ac}(\mathbf{D}) = e^{-t\nu_2(\mathbb{R}^d)}.$$

If $\gamma_{21} = 0$, then

$$(P_2^t)^{ac} = e^{t(\nu_1(\mathbb{R}^d) - \nu_2(\mathbb{R}^d))} 1_{\Lambda_t} P_1^t.$$

Remark 3.22. If $\nu_2 \perp \nu_1$ and $\nu_1(\mathbb{R}^d) = \infty$, then P_2^t and P_1^t are mutually singular. Indeed, if $\nu_2 \perp \nu_1$ and if P_2^t and P_1^t are not mutually singular, then $k_\alpha(\nu_1, \nu_2) < \infty$ by Condition (NS) and $\nu_2 \perp \nu_1$ implies that

$$k_\alpha(\nu_1, \nu_2) = \alpha\nu_1(\mathbb{R}^d) + (1-\alpha)\nu_2(\mathbb{R}^d).$$

Example 3.23. Consider two scaled Poisson processes with drift. That is, for $j = 1, 2$,

$$E^{P_j}[e^{izX_t}] = \exp\left[t\left(b_j(e^{ia_jz} - 1) + i\gamma_{0j}z\right)\right], \quad z \in \mathbb{R},$$

with $b_j > 0$, $a_j \in \mathbb{R}\backslash\{0\}$, and $\gamma_{0j} \in \mathbb{R}$. Thus $\nu_j = b_j\delta_{a_j}$. This is a special case of the preceding example, but this was independently studied by Dvoretzky, Kiefer, and Wolfowitz (1953)[6]. Thus P_2^t and P_1^t are not mutually singular if and only if $\gamma_{02} = \gamma_{01}$. Suppose that $\gamma_{02} = \gamma_{01}$ and consider two cases.

Case 1: $a_2 = a_1$. We can choose $C_1 = C_2 = \emptyset$ and thus $\Lambda_t = \mathbf{D}$. We have $P_2^t \approx P_1^t$. Since $g = \widetilde{g} = \log(b_2/b_1)$, we have

$$U_t = V_t = N_t \log(b_2/b_1) - t(b_2 - b_1),$$

where $N_t = N_t(\omega)$ is the number of jumps of $X_s(\omega)$ for $s \le t$. Hence

$$P_2^t = (b_2/b_1)^{N_t} e^{-t(b_2-b_1)} P_1^t.$$

Case 2: $a_2 \neq a_1$. We can choose $C_1 = \{a_1\}$ and $C_2 = \{a_2\}$, and $g = -\infty$ and $\widetilde{g} = 0$ on $C_1 \cup C_2$. We have

$$\Lambda_t = \{X_s - X_{s-} \neq a_1, a_2 \text{ for } s \in (0, t]\}$$

and $U_t = V_t = tb_1$. Hence

$$\begin{aligned}(P_2^t)^{ac} &= e^{t(b_1-b_2)} 1_{\Lambda_t} P_1^t, \\ (P_2^t)^{ac}(\mathbf{D}) &= e^{-tb_2}, \\ (P_2^t)^{s}(\mathbf{D}) &= 1 - e^{-tb_2}.\end{aligned}$$

[6]The author thanks Goran Peskir for calling his attention to this reference.

4. Gaussian case

The following theorem is a special case of the results of Cameron and Martin. On the other hand it is a special case of the results stated in Section 3, when both $(\{X_t\}, P_1)$ and $(\{X_t\}, P_2)$ are Gaussian Lévy processes. We prove this first and use it in the proof of Theorem A. So we have to avoid to use the results in Section 3.

Theorem 4.1. *Suppose that $(\{X_t\}, P_1)$ and $(\{X_t\}, P_2)$ are Lévy processes on $\mathbb{R}^d$ with generating triplets $(A_1, 0, \gamma_1)$ and $(A_2, 0, \gamma_2)$, respectively. Fix $t > 0$.*

(i) *The dichotomy holds: either $P_2^t \approx P_1^t$ or $P_2^t \perp P_1^t$.*

(ii) *$P_2^t \approx P_1^t$ if and only if*

$$(\mathrm{NS_G}) \qquad A_1 = A_2 \quad \text{and} \quad \gamma_2 - \gamma_1 \in \mathfrak{R}(A).$$

(iii) *If $P_2^t \approx P_1^t$, then, for $0 < \alpha < 1$,*

$$H_\alpha(P_1^t, P_2^t) = e^{-tL_\alpha} P_\alpha^t, \tag{4.1}$$

where P_α is the probability measure for which $(\{X_t\}, P_\alpha)$ is the Lévy process generated by $(A, 0, \gamma_\alpha)$ with $\gamma_\alpha = \alpha\gamma_1 + (1-\alpha)\gamma_2$, and

$$L_\alpha = \tfrac{1}{2}\alpha(1-\alpha)\langle \eta, A\eta \rangle \tag{4.2}$$

with η satisfying $A\eta = \gamma_2 - \gamma_1$.

(iv) *If $P_2^t \approx P_1^t$, then*

$$\frac{dP_2^t}{dP_1^t} = e^{U_t}, \tag{4.3}$$

where

$$U_t = \langle \eta, X_t \rangle - \tfrac{1}{2}t\langle \eta, A\eta \rangle - t\langle \gamma_1, \eta \rangle \tag{4.4}$$

with η satisfying $A\eta = \gamma_2 - \gamma_1$.

Here we are continuing to write $A_1 = A_2 = A$ whenever $A_1 = A_2$.

Proof. Step 1. Let us prove that, *if* $(\mathrm{NS_G})$ *holds, then $P_2^t \approx P_1^t$ and* (4.3) *holds.* Define U_t by (4.4) with $A\eta = \gamma_2 - \gamma_1$ and set

$$Q^{(t)}(B) = \int_B e^{U_t} P_1(d\omega), \quad B \in \mathcal{F}_t^0.$$

This is a well-defined finite measure on $(\mathbf{D}, \mathcal{F}_t^0)$, since $E^{P_1}[e^{c|X_t|}] < \infty$ for any $c > 0$. For $z \in \mathbb{R}^d$ we have

$$E^{P_1}[e^{i\langle z, X_t\rangle + U_t}] = \exp\left[-\tfrac{1}{2}t\langle z, Az\rangle + it\langle \gamma_2, z\rangle\right],$$

since the left-hand side equals

$$\exp\left[-\tfrac{1}{2}t\langle \eta, A\eta\rangle - t\langle \gamma_1, \eta\rangle\right] E^{P_1}[e^{i\langle z, X_t\rangle + \langle \eta, X_t\rangle}]$$

and

$$\begin{aligned} E^{P_1}[e^{i\langle z, X_t\rangle + \langle \eta, X_t\rangle}] &= \exp\left[\tfrac{1}{2}t\langle iz + \eta, A(iz + \eta)\rangle + t\langle \gamma_1, iz + \eta\rangle\right] \\ &= \exp\left[\tfrac{1}{2}t(-\langle z, Az\rangle + 2i\langle z, A\eta\rangle + \langle \eta, A\eta\rangle) + it\langle \gamma_1, z\rangle + t\langle \gamma_1, \eta\rangle\right]. \end{aligned}$$

See Theorem A4.3 of Appendix A. Thus, letting $z = 0$, we get $E^{P_1}[e^{U_t}] = 1$, that is, $Q^{(t)}(\mathbf{D}) = 1$. If $0 < s < t$, then, for $z_1, z_2 \in \mathbb{R}^d$,

$$E^{Q^{(t)}}[e^{i\langle z_1, X_s\rangle + i\langle z_2, X_t - X_s\rangle}] = E^{Q^{(t)}}[e^{i\langle z_1, X_s\rangle}] E^{Q^{(t)}}[e^{i\langle z_2, X_{t-s}\rangle}],$$

since $(\{U_t\}, P_1)$ is a Lévy process and since

$$\begin{aligned} \text{left-hand side} &= E^{P_1}[e^{i\langle z_1, X_s\rangle + i\langle z_2, X_t - X_s\rangle + U_t}] \\ &= E^{P_1}[e^{i\langle z_1, X_s\rangle + U_s}] E^{P_1}[e^{i\langle z_2, X_t - X_s\rangle + U_t - U_s}] \\ &= E^{P_1}[e^{i\langle z_1, X_s\rangle + U_s}] E^{P_1}[e^{i\langle z_2, X_{t-s}\rangle + U_{t-s}}]. \end{aligned}$$

We can make a similar calculation for $0 < s_1 < \cdots < s_n < t$. Therefore $(\{X_s \colon 0 \le s \le t\}, Q^{(t)})$ is a stochastically continuous process with stationary independent increments, having generating triplet $(A_2, 0, \gamma_2)$. Hence $Q^{(t)}$ and P_2 coincide on $\mathcal{F}_t^0$. It means that $P_2^t \ll P_1^t$ and (4.3) holds. As the density is positive, we have $P_2^t \approx P_1^t$.

Step 2. We prove that, *if* $A_2 \ne A_1$, *then* $P_2^t \perp P_1^t$. We can find $z_0 \in \mathbb{R}^d$ such that $\langle z_0, A_2 z_0\rangle \ne \langle z_0, A_1 z_0\rangle$. Let $Y_t = \langle z_0, X_t\rangle$. Then $(\{Y_t\}, P_j)$ is a Lévy process on $\mathbb{R}$ with generating triplet $(a_j, 0, b_j)$ for each $j = 1, 2$, where $a_j = \langle z_0, A_j z_0\rangle$ and $b_j = \langle \gamma_j, z_0\rangle$. We claim that

$$\sum_{k=1}^{n} (Y_{kt/n} - Y_{(k-1)t/n})^2 \to a_j t \quad \text{in probability } (P_j) \tag{4.5}$$

as $n \to \infty$. Indeed, if $a_j > 0$, let $Z_t = a_j^{-1/2}(Y_j - b_j t)$. Then $(\{Z_t\}, P_j)$ is the Brownian motion on $\mathbb{R}$ and we have

$$E\left[\left\{\sum_{k=1}^{n}(Z_{kt/n} - Z_{(k-1)t/n})^2 - t\right\}^2\right]$$
$$= \sum_{k=1}^{n} E[\{(Z_{kt/n} - Z_{(k-1)t/n})^2 - n^{-1}t\}^2] = 2n^{-1}t^2 \to 0.$$

Hence

$$\sum_{k=1}^{n}(Z_{kt/n} - Z_{(k-1)t/n})^2 \to t \quad \text{in probability } (P_j).$$

Since

$$\sum_{k=1}^{n}(Z_{kt/n} - Z_{(k-1)t/n})^2$$
$$= a_j^{-1}\sum_{k=1}^{n}(Y_{kt/n} - Y_{(k-1)t/n})^2 - 2a_j^{-1}b_j t n^{-1} Y_t + a_j^{-1}b_j^2 n^{-1} t^2,$$

we get (4.5). If $a_j = 0$, then $Y_t = b_j t$ and (4.5) is trivial. Now it follows that, via some common subsequence $\{n'\}$, (4.5) holds in almost sure (P_j) convergence for $j = 1$ and 2. Let

$$\Lambda_j = \left\{\omega \colon \sum_{k=1}^{n'}(Y_{kt/n'} - Y_{(k-1)t/n'})^2 \to a_j t\right\}.$$

Then, $\Lambda_1 \cap \Lambda_2 = \emptyset$, $P_1[\Lambda_1] = 1$, $P_2[\Lambda_2] = 1$, and hence $P_2[\Lambda_1] = 0$. Thus we have $P_2^t \perp P_1^t$.

Step 3. Further let us see that, *if* $A_2 = A_1$, $\gamma_2 - \gamma_1 \notin \mathfrak{R}(A)$, *then* $P_2^t \perp P_1^t$. Fix $t > 0$. We have $P_j[X_t \in \mathfrak{R}(A) + t\gamma_j] = 1$ for $j = 1, 2$. It follows from $\gamma_2 - \gamma_1 \notin \mathfrak{R}(A)$ that $(\mathfrak{R}(A) + t\gamma_1) \cap (\mathfrak{R}(A) + t\gamma_2) = \emptyset$. Hence $P_2[X_t \in \mathfrak{R}(A) + t\gamma_1] = 0$. Hence $P_2^t \perp P_1^t$.

Now the assertions (i), (ii), and (iv) follow from these three steps. It remains to show (iii).

Step 4. Let us prove the assertion (iii). We assume $P_2^t \approx P_1^t$. The Lévy process $(\{X_t\}, P_\alpha)$ has generating triplet $(A, 0, \gamma_\alpha)$, and $\gamma_\alpha - \gamma_1 =$

$(1-\alpha)(\gamma_2 - \gamma_1) = (1-\alpha)A\eta$. Hence $P_\alpha^t \approx P_1^t$ by (ii), and we know the density from (iv):

$$\frac{dP_\alpha^t}{dP_1^t} = \exp\big[(1-\alpha)\langle \eta, X_t\rangle - \tfrac{1}{2}t\langle (1-\alpha)\eta, A(1-\alpha)\eta\rangle \\ - t(1-\alpha)\langle \gamma_1, \eta\rangle\big].$$

Thus we have

$$\begin{aligned} H_\alpha(P_1^t, P_2^t) &= \left(\frac{dP_2^t}{dP_1^t}\right)^{1-\alpha} P_1^t = e^{(1-\alpha)U_t} P_1^t \\ &= \exp\left[(1-\alpha)\langle \eta, X_t\rangle - \tfrac{1}{2}t(1-\alpha)\langle \eta, A\eta\rangle - t(1-\alpha)\langle \gamma_1, \eta\rangle\right] P_1^t \\ &= e^{-\frac{1}{2}t(1-\alpha)\alpha\langle \eta, A\eta\rangle} P_\alpha^t = e^{-tL_\alpha} P_\alpha^t. \end{aligned}$$

This completes the proof. □

5. Proof of Theorem A

We follow Newman (1973) in the proof of Theorem A. Let $\mathbf{D}_t = D([0,t],\mathbb{R}^d)$, the space of $\omega\colon [0,t] \to \mathbb{R}^d$ right-continuous with left limits. Let $X_s(\omega) = \omega(s)$ and $\mathcal{F}^0(\mathbf{D}_t) = \sigma(X_s\colon 0 \le s \le t)$. Since $\mathcal{F}_t^0$ and $\mathcal{F}^0(\mathbf{D}_t)$ are naturally identified, any measure on $(\mathbf{D}, \mathcal{F}_t^0)$ is identified with a measure on $(\mathbf{D}_t, \mathcal{F}^0(\mathbf{D}_t))$, and vice versa. Thus we consider P_j^t, $j = 1, 2$, as measures on $(\mathbf{D}_t, \mathcal{F}^0(\mathbf{D}_t))$. As before, $(\{X_t\}, P_j)$ is a Lévy process with generating triplet (A_j, ν_j, γ_j) for each $j = 1, 2$.

We can define the addition in $\mathbf{D}_t$:

$$(\omega + \omega')(t) = \omega(t) + \omega'(t) \qquad \text{for } \omega,\ \omega' \in \mathbf{D}_t.$$

The addition is measurable from $\mathcal{F}^0(\mathbf{D}_t) \times \mathcal{F}^0(\mathbf{D}_t)$ to $\mathcal{F}^0(\mathbf{D}_t)$. Hence we can define the convolution $\rho_1 * \rho_2$ of finite measures ρ_1 and ρ_2 on $(\mathbf{D}_t, \mathcal{F}^0(\mathbf{D}_t))$ by

$$\rho_1 * \rho_2(B) = \iint_{\mathbf{D}_t \times \mathbf{D}_t} 1_B(\omega + \omega')\rho_1(d\omega)\rho_2(d\omega').$$

Define $J(G, \omega)$ for $\omega \in \mathbf{D}_t$. That is, $J(G, \omega)$ is the number of $s \in (0, t]$ such that $(s, X_s(\omega) - X_{s-}(\omega)) \in G$ for $G \in \mathcal{B}_{(0,t]\times\mathbb{R}^d}$. Let $\nu_\alpha = H_\alpha(\nu_1, \nu_2)$ and $\widetilde{\nu}_\alpha = ds \times \nu_\alpha$. Let

$$\begin{aligned}(5.1)\qquad Y_{\varepsilon,s}(\omega) = &\int_{(0,s]\times\{\varepsilon<|x|\le 1\}} \{xJ(d(u,x),\omega) - x\widetilde{\nu}_\alpha(d(u,x))\} \\ &+ \int_{(0,s]\times\{|x|>1\}} xJ(d(u,x),\omega), \qquad 0 \le s \le t,\end{aligned}$$

$$(5.2)\qquad Z_{\varepsilon,s}(\omega) = X_s(\omega) - Y_{\varepsilon,s}(\omega), \qquad 0 \le s \le t,$$

for $0 < \varepsilon < 1$. Let $Q_{\varepsilon,j}^t$ and $R_{\varepsilon,j}^t$ denote, respectively, the probability measures on $(\mathbf{D}_t, \mathcal{F}^0(\mathbf{D}_t))$ induced by $(\{Y_{\varepsilon,s}\}, P_j^t)$ and $(\{Z_{\varepsilon,s}\}, P_j^t)$. Here we have suppressed to write the dependence on α of $Y_{\varepsilon,s}$, $Z_{\varepsilon,s}$, $Q_{\varepsilon,j}^t$, and $R_{\varepsilon,j}^t$.

Lemma 5.1. *We have*

$$(5.3)\qquad P_j^t = Q_{\varepsilon,j}^t * R_{\varepsilon,j}^t, \qquad j = 1,\ 2,$$

and

$$(5.4)\qquad H_\alpha(P_1^t, P_2^t) = H_\alpha(Q_{\varepsilon,1}^t, Q_{\varepsilon,2}^t) * H_\alpha(R_{\varepsilon,1}^t, R_{\varepsilon,2}^t).$$

This lemma remains true if we replace ν_α by any measure finite on $\{|x| > \varepsilon\}$. But this form is convenient for us.

Proof. Since The processes $\{Y_{\varepsilon,s}\}$ and $\{Z_{\varepsilon,s}\}$ are independent both under P_1^t and under P_2^t by Theorem 3.4. Since $X_s = Y_{\varepsilon,s} + Z_{\varepsilon,s}$, we have (5.3). To see (5.4), notice that the paths $Y_{\varepsilon,s}$ and $Z_{\varepsilon,s}$, $0 \le s \le t$, are determined by X_s, $0 \le s \le t$. Let

$$\mathbf{D}'_{\varepsilon,t} = \{\omega \in \mathbf{D}_t \colon X_s(\omega) = Y_{\varepsilon,s}(\omega) \text{ for } s \in [0,t]\}$$
$$\mathbf{D}''_{\varepsilon,t} = \{\omega \in \mathbf{D}_t \colon |X_s(\omega) - X_{s-}(\omega)| \le \varepsilon \text{ for } s \in (0,t]\}.$$

Let $\varphi \colon \mathbf{D}_t \to \mathbf{D}'_{\varepsilon,t} \times \mathbf{D}''_{\varepsilon,t}$ be defined by $\varphi(\omega) = (\omega', \omega'')$, where $X_s(\omega') = Y_{\varepsilon,s}(\omega)$ and $X_s(\omega'') = Z_{\varepsilon,s}(\omega)$. Then φ is bijective, measurable, and its inverse φ^{-1} is also measurable. Any finite measures ρ' and ρ'' on $\mathbf{D}'_{\varepsilon,t}$ and $\mathbf{D}''_{\varepsilon,t}$, respectively, can be considered as measures on $\mathbf{D}_t$. With this identification, we have $(\rho' \times \rho'')(\varphi B) = (\rho' * \rho'')(B)$ for $B \in \mathcal{F}^0(\mathbf{D}_t)$. Thus $P_j^t \varphi^{-1} = Q_{\varepsilon,j}^t \times R_{\varepsilon,j}^t$. Hence,

$$\begin{aligned} H_\alpha(P_1^t, P_2^t) &= H_\alpha(P_1^t\varphi^{-1}, P_2^t\varphi^{-1})\varphi = H_\alpha(Q_{\varepsilon,1}^t \times R_{\varepsilon,1}^t, Q_{\varepsilon,2}^t \times R_{\varepsilon,2}^t)\varphi \\ &= (H_\alpha(Q_{\varepsilon,1}^t, Q_{\varepsilon,2}^t) \times H_\alpha(R_{\varepsilon,1}^t, R_{\varepsilon,2}^t))\varphi \\ &= H_\alpha(Q_{\varepsilon,1}^t, Q_{\varepsilon,2}^t) * H_\alpha(R_{\varepsilon,1}^t, R_{\varepsilon,2}^t). \end{aligned}$$

This proves (5.4). $\square$

Lemma 5.2. *Let P_α be the probability measure given in Theorem* A. *Let $Q_{\varepsilon,\alpha}^t$ be the probability measure on $(\mathbf{D}_t, \mathcal{F}^0(\mathbf{D}_t))$ induced by $(\{Y_{\varepsilon,s}\}, P_\alpha^t)$. Then*

$$H_\alpha(Q_{\varepsilon,1}^t, Q_{\varepsilon,2}^t) = \exp\left[-t\int_{|x|>\varepsilon} dK_\alpha(\nu_1,\nu_2)\right] Q_{\varepsilon,\alpha}^t \tag{5.5}$$

for $0 < \alpha < 1$ and $0 < \varepsilon < 1$.

Proof. For $m = 1, 2, \ldots$, let ${\mathbf{D}'_{\varepsilon,t}}^{(m)}$ be the set of $\omega \in \mathbf{D}'_{\varepsilon,t}$ such that

$$X_s(\omega) = \sum_{l=1}^m x_l 1_{[0,\infty)}(s - s_l) - s\int_{\varepsilon<|x|\le 1} x\,d\nu_\alpha \quad \text{for } 0 \le s \le t$$

with some $0 < s_1 < \cdots < s_m \le t$ and $|x_l| > \varepsilon$, $l = 1, \ldots, m$. By the mapping $\omega \mapsto (s_1, \ldots, s_m, x_1, \ldots, x_m)$, ${\mathbf{D}'_{\varepsilon,t}}^{(m)}$ is identified with the set $\{(s_1, \ldots, s_m) \colon 0 < s_1 < \cdots < s_m \le t\} \times \{|x| > \varepsilon\}^m$. Let $M_j^{(m)}$ be the measure induced on ${\mathbf{D}'_{\varepsilon,t}}^{(m)}$ by $[(ds)^m]_{\{0<s_1<\cdots<s_m\le t\}} \times ([\nu_j]_{\{|x|>\varepsilon\}})^m$. Let ${\mathbf{D}'_{\varepsilon,t}}^{(0)}$ be the set consisting of only one path $X_s(\omega) = -s\int_{\varepsilon<|x|\le 1} x d\nu_\alpha$, $0 \le s \le t$, and let $M_j^{(0)}$ be the unit mass concentrated at this path. Then $M_j^{(m)}$ has total mass $(m!)^{-1}t^m {c_j}^m$, where $c_j = \nu_j(\{|x| > \varepsilon\})$. Under $Q^t_{\varepsilon,j}$, X_s has characteristic function

$$\exp\left[s\int_{|x|>\varepsilon}(e^{i\langle z,x\rangle}-1)\nu_j(dx) - is\left\langle\int_{\varepsilon<|x|\le 1} x d\nu_\alpha, z\right\rangle\right]$$

and we have

$$Q^t_{\varepsilon,j} = e^{-tc_j}\sum_{m=0}^{\infty} M_j^{(m)}, \tag{5.6}$$

since $X_s(\omega)$, $0 \le s \le t$, has exactly m jumps with probability

$$e^{-tc_j}(m!)^{-1}t^m {c_j}^m$$

and, given that this occurs, the conditional distribution of jumping times and jumping heights is $m!\, t^{-m}{c_j}^{-m}M_j^{(m)}$. Hence,

$$H_\alpha(Q^t_{\varepsilon,1}, Q^t_{\varepsilon,2}) = H_\alpha\left(e^{-tc_1}\sum_{m=0}^{\infty} M_1^{(m)}, e^{-tc_2}\sum_{m=0}^{\infty} M_2^{(m)}\right),$$

which is calculated in the following way. We have

$$H_\alpha(M_1^{(m)}, M_2^{(m')}) = 0 \quad \text{for } m \ne m',$$

since $M_1^{(m)}$ and $M_2^{(m')}$ have disjoint carriers. We claim that

$$H_\alpha(M_1^{(m)}, M_2^{(m)}) = M_\alpha^{(m)},$$

where $M_\alpha^{(m)}$ is the measure defined similarly, using ν_α in place of ν_j. Indeed,

$$\begin{aligned} H_\alpha(M_1^{(m)}, M_2^{(m)}) &= [(ds)^m]_{\{0<s_1<\cdots<s_m\le t\}} \times (H_\alpha([\nu_1]_{\{|x|>\varepsilon\}}, [\nu_2]_{\{|x|>\varepsilon\}}))^m \\ &= [(ds)^m]_{\{0<s_1<\cdots<s_m\le t\}} \times ([H_\alpha(\nu_1,\nu_2)]_{\{|x|>\varepsilon\}})^m \\ &= M_\alpha^{(m)}. \end{aligned}$$

In conclusion,

$$H_\alpha(Q^t_{\varepsilon,1}, Q^t_{\varepsilon,2}) = e^{-t(\alpha c_1 + (1-\alpha)c_2)}\sum_{m=0}^{\infty} H_\alpha(M_1^{(m)}, M_2^{(m)})$$

$$= e^{-t(\alpha c_1+(1-\alpha)c_2)} \sum_{m=0}^{\infty} M_\alpha^{(m)}$$
$$= e^{-t(\alpha c_1+(1-\alpha)c_2-c_\alpha)} Q_{\varepsilon,\alpha}^t$$

with $c_\alpha = \nu_\alpha(\{|x| > \varepsilon\})$. This shows (5.5). □

Assuming $k_\alpha(\nu_1, \nu_2) < \infty$, define

$$Y_{0,s}(\omega) = \lim_{\varepsilon \downarrow 0} Y_{\varepsilon,s}(\omega),$$

where the convergence is uniform in $s \in [0,t]$, P_j^t-a. s. This follows from Theorem 3.4 and Lemma 2.18 and we have

$$Y_{0,s}(\omega) = X'_{j,s}(\omega) + s \int_{|x|\le 1} x d(\nu_j - \nu_\alpha), \quad P_j^t - \text{a. } s.,$$

where $X'_{j,s}(\omega)$ is defined by (3.3) with ν_j in place of ν. Further define

$$Z_{0,s}(\omega) = X_s(\omega) - Y_{0,s}(\omega),$$
$$\overline{Y}_{\varepsilon,s}(\omega) = Z_{\varepsilon,s}(\omega) - Z_{0,s}(\omega) = Y_{0,s}(\omega) - Y_{\varepsilon,s}(\omega),$$

and let $Q_{0,j}^t$, $R_{0,j}^t$, and $\overline{Q}_{\varepsilon,j}^t$ be the distributions of $\{Y_{0,s}\}$, $\{Z_{0,s}\}$, and $\{\overline{Y}_{\varepsilon,s}\}$ under P_j^t.

Lemma 5.3. *Assume that $k_\alpha(\nu_1, \nu_2) < \infty$. Then*

$$P_j^t = Q_{0,j}^t * R_{0,j}^t$$
$$= Q_{\varepsilon,j}^t * \overline{Q}_{\varepsilon,j}^t * R_{0,j}^t \quad \textit{for } j = 1,\, 2, \tag{5.7}$$

and

$$\begin{aligned} H_\alpha(P_1^t, P_2^t) &= H_\alpha(Q_{0,1}^t, Q_{0,2}^t) * H_\alpha(R_{0,1}^t, R_{0,2}^t) \\ &= H_\alpha(Q_{\varepsilon,1}^t, Q_{\varepsilon,2}^t) * H_\alpha(\overline{Q}_{\varepsilon,1}^t, \overline{Q}_{\varepsilon,2}^t) * H_\alpha(R_{0,1}^t, R_{0,2}^t) \\ &= H_\alpha(\overline{Q}_{\varepsilon,1}^t, \overline{Q}_{\varepsilon,2}^t) * H_\alpha(Q_{\varepsilon,1}^t * R_{0,1}^t, Q_{\varepsilon,2}^t * R_{0,2}^t). \end{aligned} \tag{5.8}$$

Proof. We have $X_s = Y_{0,s} + Z_{0,s} = Y_{\varepsilon,s} + \overline{Y}_{\varepsilon,s} + Z_{0,s}$, sums of independent terms under P_j^t. Therefore we get (5.7). Similarly to the proof of Lemma 5.1, consider

$$\mathbf{D}_{0,t} = \{\omega \in \mathbf{D}_t \colon Y_{\varepsilon,s}(\omega) \text{ converges uniformly in } s \in [0,t] \text{ as } \varepsilon \downarrow 0\},$$

$$\mathbf{D}'_{0,t} = \{\omega \in \mathbf{D}_{0,t}\colon X_s(\omega) = \lim_{\varepsilon\downarrow 0} Y_{\varepsilon,s}(\omega) \text{ for } 0 \le s \le t\},$$
$$\mathbf{D}''_{0,t} = \{\omega \in \mathbf{D}_{0,t}\colon X_s(\omega) \text{ is continuous in } s\}.$$

We can prove the first equality in (5.8) in the same way as in the proof of Lemma 5.1. Furthermore, by Lemma 5.1 itself,

$$H_\alpha(Q^t_{0,1}, Q^t_{0,2}) = H_\alpha(Q^t_{\varepsilon,1}, Q^t_{\varepsilon,2}) * H_\alpha(\overline{Q}^t_{\varepsilon,1}, \overline{Q}^t_{\varepsilon,2})$$

and

$$H_\alpha(Q^t_{\varepsilon,1} * R^t_{0,1}, Q^t_{\varepsilon,2} * R^t_{0,2}) = H_\alpha(Q^t_{\varepsilon,1}, Q^t_{\varepsilon,2}) * H_\alpha(R^t_{0,1}, R^t_{0,2}).$$

Hence we get the second and third equalities in (5.8). $\square$

Proof of Theorem A. *Step* 1. Let us show that, *if* $k_\alpha(\nu_1, \nu_2) = \infty$, *then* $P^t_2 \perp P^t_1$. We have

$$h_\alpha(P^t_1, P^t_2) = h_\alpha(Q^t_{\varepsilon,1}, Q^t_{\varepsilon,2}) h_\alpha(R^t_{\varepsilon,1}, R^t_{\varepsilon,2}) \tag{5.9}$$

by Lemma 5.1. Remark 2.3 shows that $h_\alpha(R^t_{\varepsilon,1}, R^t_{\varepsilon,2}) \le 1$, while Lemma 5.2 shows that

$$h_\alpha(Q^t_{\varepsilon,1}, Q^t_{\varepsilon,2}) = \exp\left[-t \int_{|x|>\varepsilon} dK_\alpha(\nu_1, \nu_2)\right],$$

which tends to 0 as $\varepsilon \downarrow 0$. Hence $h_\alpha(P^t_1, P^t_2) = 0$, that is, $P^t_2 \perp P^t_1$.

Step 2. Next we see that, *if* $k_\alpha(\nu_1, \nu_2) < \infty$ *and* $A_2 \ne A_1$, *then* $P^t_2 \perp P^t_1$. In fact, the process under $R^t_{0,j}$ has generating triplet $(A_j, 0, \gamma_j - \int_{|x|\le 1} x d(\nu_j - \nu_\alpha))$. Since $A_1 \ne A_2$, we have $h_\alpha(R^t_{0,1}, R^t_{0,2}) = 0$ by Theorem 4.1. Hence $h_\alpha(P^t_1, P^t_2) = 0$ by (5.8) of Lemma 5.3.

Step 3. *If* $k_\alpha(\nu_1, \nu_2) < \infty$, $A_2 = A_1$, *and* $\gamma_{21} \notin \mathfrak{R}(A_1)$, *then* $P^t_2 \perp P^t_1$, because we can use Theorem 4.1 again, observing that

$$\left(\gamma_2 - \int_{|x|\le 1} x d(\nu_2 - \nu_\alpha)\right) - \left(\gamma_1 - \int_{|x|\le 1} x d(\nu_1 - \nu_\alpha)\right) = \gamma_{21} \notin \mathfrak{R}(A).$$

These three steps prove the part (ii) of the theorem.

Step 4. *Suppose that* (NS) *is satisfied. Let us prove* (3.6). This will prove the part (i). We have $\overline{Y}_{\varepsilon,s} \to 0$, and hence $Y_{\varepsilon,s} + Z_{0,s} \to X_s$, uniformly in $s \in [0,t]$ P^t_j-a.s. as $\varepsilon \downarrow 0$. Hence these converge in the Skorohod metric. Hence we have, as $\varepsilon \downarrow 0$, $Q^t_{\varepsilon,j} * R^t_{0,j} \to P^t_j$ as the weak convergence of probability measures on the metric space $\mathbf{D}_t$. See

Section 14 of Billingsley (1968) for relevant material; especially, $\mathcal{F}^0(\mathbf{D}_t)$ is identical with the σ-algebra generated by the open sets. It follows from the third expression of $H_\alpha(P_1^t, P_2^t)$ in (5.8) that

$$h_\alpha(P_1^t, P_2^t) \le h_\alpha(Q_{\varepsilon,1}^t * R_{0,1}^t, Q_{\varepsilon,2}^t * R_{0,2}^t).$$

Hence, if we show that $H_\alpha(Q_{\varepsilon,1}^t * R_{0,1}^t, Q_{\varepsilon,2}^t * R_{0,2}^t)$ tends to some ρ as $\varepsilon \downarrow 0$, then $\rho = H_\alpha(P_1^t, P_2^t)$ by Lemma 2.21. Now, by Lemma 5.2 and Theorem 4.1,

$$\begin{aligned} H_\alpha(Q_{\varepsilon,1}^t * R_{0,1}^t, Q_{\varepsilon,2}^t * R_{0,2}^t) &= H_\alpha(Q_{\varepsilon,1}^t, Q_{\varepsilon,2}^t) * H_\alpha(R_{0,1}^t, R_{0,2}^t) \\ &= \exp\left[-t \int_{|x|>\varepsilon} dK_\alpha(\nu_1, \nu_2)\right] Q_{\varepsilon,\alpha}^t * \exp\left[-\frac{1}{2} t\alpha(1-\alpha)\langle \eta, A\eta\rangle\right] R_\alpha^t, \end{aligned}$$

where R_α^t corresponds to the Gaussian Lévy process generated by $(A, 0, \gamma_\alpha)$ with

$$\begin{aligned} \gamma_\alpha &= \alpha\left(\gamma_1 - \int_{|x|\le 1} x d(\nu_1 - \nu_\alpha)\right) + (1-\alpha)\left(\gamma_2 - \int_{|x|\le 1} x d(\nu_2 - \nu_\alpha)\right) \\ &= \alpha\gamma_1 + (1-\alpha)\gamma_2 - \int_{|x|\le 1} x dK_\alpha(\nu_1, \nu_2) \end{aligned}$$

and $A\eta = \gamma_{21}$. As $\varepsilon \downarrow 0$, the measure $Q_{\varepsilon,\alpha}^t$ tends to the measure $Q_{0,\alpha}^t$ that has triplet $(0, \nu_\alpha, 0)$, since $Q_{\varepsilon,\alpha}^t$ is the distribution of $\{Y_{\varepsilon,s}\}$ seen under P_α^t. Noticing that $Q_{0,\alpha}^t * R_\alpha^t = P_\alpha^t$, which has triplet $(A, \nu_\alpha, \gamma_\alpha)$, we see that $H_\alpha(Q_{\varepsilon,1}^t * R_{0,1}^t, Q_{\varepsilon,2}^t * R_{0,2}^t)$ tends to

$$\rho = \exp\left[-tk_\alpha(\nu_1, \nu_2) - \tfrac{1}{2} t\alpha(1-\alpha)\langle \eta, A\eta\rangle\right] P_\alpha^t$$

as $\varepsilon \downarrow 0$. This completes the proof. $\square$

6. Proof of Theorem B

Let us prove Theorem B. We can now use the assertions of Theorem A and Corollaries 3.6–3.14 freely. We prepare two lemmas. They are [S] Lemmas 33.6 and 33.7, but we give their proofs here, as they are essential parts of the proof of Theorem B. As in Theorem 3.4, let $J(G,\omega)$ be the number of $s>0$ such that $(s, X_s(\omega)-X_{s-}(\omega)) \in G$ for $G \in \mathcal{B}_{(0,\infty)\times\mathbb{R}^d}$ and let $\widetilde{\nu}_1 = ds \times \nu_1(dx)$. We continue to assume that we are given two Lévy processes $(\{X_t\}, P_1)$ and $(\{X_t\}, P_2)$ with generating triplets (A_1, ν_1, γ_1) and (A_2, ν_2, γ_2), respectively. Recall that, if $A_1 = A_2$, then we write $A_1 = A_2 = A$.

Lemma 6.1. *Consider only the process $(\{X_t\}, P_1)$ with generating triplet (A_1, ν_1, γ_1). Let $\widetilde{g}(x) = g(x)$ be a finite-valued measurable function on $\mathbb{R}^d$ satisfying*

$$\int_{|g|\le 1} g^2 d\nu_1 + \int_{g>1} e^g d\nu_1 + \int_{g<-1} d\nu_1 < \infty. \tag{6.1}$$

Then the right-hand side of (3.27) exists P_1-a. s. and the convergence is uniform on any bounded time interval P_1-a. s. Let $\eta \in \mathbb{R}^d$ and define U_t by (3.28) with A_1 replacing A. Then $(\{U_t\}, P_1)$ is a Lévy process on $\mathbb{R}$, that is, $\{U_t\}$ is, under P_1, a Lévy process on $\mathbb{R}$. Its generating triplet (A_U, ν_U, γ_U) is given by (3.29)–(3.31). We have

$$E^{P_1}[e^{U_t}] = 1. \tag{6.2}$$

Proof. Let

$$B_{\varepsilon 0} = \{x \colon |x| > \varepsilon \text{ and } |g(x)| \le 1\},$$
$$B_{\varepsilon 1} = \{x \colon |x| > \varepsilon \text{ and } |g(x)| > 1\}$$

and let

$$\begin{aligned} V_t^{\varepsilon 0} &= \sum_{(s, X_s - X_{s-}) \in (0,t]\times B_{\varepsilon 0}} g(X_s - X_{s-}) - t\int_{B_{\varepsilon 0}} g(x) d\nu_1 \\ &= \int_{(0,t]\times B_{\varepsilon 0}} g(x)(J(d(s,x)) - \widetilde{\nu}_1(d(s,x))), \end{aligned}$$

$$V_t^{\varepsilon 1} = \sum_{(s, X_s - X_{s-}) \in (0,t]\times B_{\varepsilon 1}} g(X_s - X_{s-})$$

$$= \int_{(0,t]\times B_{\varepsilon 1}} g(x) J(d(s,x)),$$

$$c_\varepsilon = -\int_{|x|>\varepsilon} (e^{g(x)} - 1 - g(x) 1_{[-1,1]}(g(x)))\nu_1(dx).$$

Then (3.27) is written as

$$V_t = \lim_{\varepsilon \downarrow 0} (V_t^{\varepsilon 0} + V_t^{\varepsilon 1} + tc_\varepsilon). \tag{6.3}$$

Since $\{J(G) \colon G \in \mathcal{B}_{(0,\infty)\times\mathbb{R}^d}\}$ is, under P_1, a Poisson random measure with intensity measure $\widetilde{\nu}_1$ by Theorem 3.4, we have

$$E^{P_1}[e^{iuV_t^{\varepsilon 0}}] = \exp\left[t \int_{B_{\varepsilon 0}} (e^{iug(x)} - 1 - iug(x))\nu_1(dx) \right],$$

$$E^{P_1}[e^{iuV_t^{\varepsilon 1}}] = \exp\left[t \int_{B_{\varepsilon 1}} (e^{iug(x)} - 1)\nu_1(dx) \right]$$

for $u \in \mathbb{R}$ by Proposition 3.3. Since $\nu_1(\{|g(x)| > 1\})$ is finite, $J((0,t] \times \{|g(x)| > 1\})$ is finite P_1-a. s. Hence $V_t^{\varepsilon 1}$ converges uniformly on any bounded interval P_1-a. s. as $\varepsilon \downarrow 0$. Since

$$\int_{|g(x)|\le 1} |e^{iug(x)} - 1 - iug(x)|\, \nu_1(dx) < \infty$$

by (6.1), the term $V_t^{\varepsilon 0}$ also converges uniformly on any bounded interval P_1-a. s. as $\varepsilon \downarrow 0$, which can be verified in a way similar to the proof of Theorem 3.4 (see [S] Lemmas 20.6 and 20.7). The term tc_ε is nonrandom and tends to a finite limit as $\varepsilon \downarrow 0$, by virtue of (6.1). Therefore the right-hand side of (6.3) is convergent uniformly on any bounded interval P_1-a. s. and

$$\begin{aligned} E^{P_1}[e^{iuV_t}] = \exp\Bigg[t \Bigg(&\int_{\mathbb{R}^d} (e^{iug(x)} - 1 - iug(x) 1_{[-1,1]}(g(x)))\nu_1(dx) \\ &- iu \int_{\mathbb{R}^d} (e^{g(x)} - 1 - g(x) 1_{[-1,1]}(g(x)))\nu_1(dx) \Bigg) \Bigg]. \end{aligned} \tag{6.4}$$

Since $\{X_t''\}$ and $\{V_t^{\varepsilon 0} + V_t^{\varepsilon 1}\}$ are independent under P_1 by Theorem 3.4, $\{X_t''\}$ and $\{V_t\}$ are independent under P_1. Thus, noticing that

$$E^{P_1}[e^{iu\langle \eta, X_t''\rangle}] = e^{t(-\frac{1}{2}\langle u\eta, A(u\eta)\rangle + i\langle \gamma_1, u\eta\rangle)},$$

we get

$$E^{P_1}[e^{iuU_t}] = e^{t(-\frac{1}{2}u^2\langle\eta,A\eta\rangle - \frac{iu}{2}\langle\eta,A\eta\rangle)}E^{P_1}[e^{iuV_t}]. \tag{6.5}$$

Since $\{V_t\}$ is right-continuous with left limits, has independent increments, and satisfies (6.4), it is a Lévy process. Hence $\{U_t\}$ is a Lévy process. The formulas (6.4) and (6.5) show that the generating triplet (A_U, ν_U, γ_U) has expression (3.29)–(3.31). Since $\int_{|y|>1} e^y(\nu_1 g^{-1})(dy) < \infty$, we see, by Theorem A4.3 in Appendix A, that $E^{P_1}[e^{U_t}]$ is finite and equal to $e^{\Psi_U(1)}$, where

$$\Psi_U(w) = \frac{w^2}{2}A_U + \int_{\mathbb{R}}(e^{wx} - 1 - wx1_{[-1,1]}(x))\nu_U(dx) + w\gamma_U.$$

Hence $\Psi_U(1) = 0$ and (6.2) holds. □

Lemma 6.2. *Let $t > 0$. Suppose that $P_2^t \approx P_1^t$ (hence $\nu_2 \approx \nu_1$). Choose f_1, f_2 of* (3.17) *and* (3.18) *satisfying $C = \mathbb{R}^d$ and $C_1 = C_2 = \emptyset$. Define $\widetilde{g} = g$ by* (3.22)*. Then the right-hand side of* (3.27) *exists P_1-a. s. and the convergence is uniform on any bounded time interval P_1-a. s. Let $\eta \in \mathbb{R}^d$ be such that $A\eta = \gamma_{21}$ and define $\{U_t\}$ by* (3.28)*, using this η. Then*

$$P_2(B) = E^{P_1}[e^{U_t}1_B] \quad \text{for } B \in \mathcal{F}_t^0,\ t \in [0,\infty). \tag{6.6}$$

Proof. We have $\nu_2 = e^g\nu_1$ and $k_\alpha(\nu_1,\nu_2) < \infty$. Hence, by Remark 2.16, g satisfies (6.1). Hence, by Lemma 6.1, V_t can be defined by (3.27) with the convergence being uniform on any bounded interval P_1-a. s.

Define $X_t^*(\omega) = (X_{t,j}^*(\omega))_{1\le j\le d+1} \in \mathbb{R}^{d+1}$ for $\omega \in \mathbf{D}$ by $(X_{t,j}^*(\omega))_{1\le j\le d} = X_t(\omega)$ and $X_{d+1}^*(\omega) = U_t(\omega)$. Then $(\{X_t^*\}, P_1)$ is a Lévy process on $\mathbb{R}^{d+1}$. In fact, the proof of Lemma 6.1 shows that it satisfies the defining conditions of a Lévy process in Section 1. Let us calculate its characteristic function. Let $B_0 = \{|x| \le 1\}$, $B_1 = \{|x| > 1\}$, $C_0 = \{x\colon |g(x)| \le 1\}$, and $C_1 = \{x\colon |g(x)| > 1\}$. We claim that, for $z \in \mathbb{R}^d$ and $u \in \mathbb{R}$,

$$\begin{aligned} E^{P_1}[e^{i\langle z,X_t\rangle + iuU_t}] = \exp\Bigg[t\Bigg(&F - \frac{1}{2}\langle z+u\eta, A(z+u\eta)\rangle \\ &+ i\langle\gamma_1, z\rangle - \frac{i}{2}\langle\eta, A\eta\rangle u \end{aligned} \tag{6.7}$$

$$-iu\int_{\mathbb{R}^d}(e^{g(x)}-1-g(x)1_{[-1,1]}(g(x)))\nu_1(dx)\Big)\Big],$$

where

$$F=\int_{\mathbb{R}^d}(e^{i\langle z,x\rangle+iug(x)}-1-i\langle z,x\rangle 1_{B_0}(x)-iug(x)1_{[-1,1]}(g(x)))\nu_1(dx).$$

Let us denote

$$F_B=\int_B|e^{i\langle z,x\rangle+iug(x)}-1-i\langle z,x\rangle 1_{B_0}(x)-iug(x)1_{[-1,1]}(g(x))|\nu_1(dx)$$

for $B\in\mathcal{B}_{\mathbb{R}^d}$. Let us check that $F_{\mathbb{R}^d}<\infty$. Using (6.1), we have

$$\begin{aligned}
F_{B_0\cap C_0}&\le\frac{1}{2}\int_{B_0\cap C_0}(\langle z,x\rangle+ug(x))^2\nu_1(dx)\\
&\le|z|^2\int_{B_0}|x|^2\nu_1(dx)+u^2\int_{C_0}g(x)^2\nu_1(dx)<\infty,\\
F_{B_0\cap C_1}&=\int_{B_0\cap C_1}|(e^{i\langle z,x\rangle}-1-i\langle z,x\rangle)e^{iug(x)}+(e^{iug(x)}-1)\\
&\qquad\qquad+i\langle z,x\rangle(e^{iug(x)}-1)|\nu_1(dx)\\
&\le\frac{1}{2}|z|^2\int_{B_0}|x|^2\nu_1(dx)+2\int_{C_1}\nu_1(dx)\\
&\quad+2|z|\left(\int_{B_0}|x|^2\nu_1(dx)\right)^{1/2}\left(\int_{C_1}\nu_1(dx)\right)^{1/2}<\infty,\\
F_{B_1\cap C_0}&=\int_{B_1\cap C_0}|e^{i\langle z,x\rangle}(e^{iug(x)}-1-iug(x))+(e^{i\langle z,x\rangle}-1)\\
&\qquad\qquad+(e^{i\langle z,x\rangle}-1)iug(x)|\nu_1(dx)\\
&\le\frac{1}{2}u^2\int_{C_0}g(x)^2\nu_1(dx)+2\int_{B_1}\nu_1(dx)\\
&\quad+2|u|\left(\int_{C_0}g(x)^2\nu_1(dx)\right)^{1/2}\left(\int_{B_1}\nu_1(dx)\right)^{1/2}<\infty,\\
F_{B_1\cap C_1}&\le 2\int_{B_1}\nu_1(dx)<\infty.
\end{aligned}$$

Hence $F_{\mathbb{R}^d} < \infty$ and we can define F in (6.7). To prove (6.7), use the notation in the proof of Lemma 6.1 and let

$$X_t^\varepsilon = X_t'' + \sum_{(s, X_s - X_{s-}) \in (0,t] \times B_\varepsilon} (X_s - X_{s-}) - t \int_{\varepsilon < |x| \le 1} x \nu_1(dx),$$

$$U_t^\varepsilon = \langle \eta, X_t'' \rangle - \frac{t}{2} \langle \eta, A\eta \rangle - t\langle \gamma_1, \eta \rangle + V_t^{\varepsilon 0} + V_t^{\varepsilon 1} + tc_\varepsilon,$$

where $B_\varepsilon = \{|x| > \varepsilon\} = B_{\varepsilon 0} \cup B_{\varepsilon 1}$. Then, $X_t^\varepsilon \to X_t$ and $U_t^\varepsilon \to U_t$, P_1-a. s., as $\varepsilon \downarrow 0$. We have

$$E^{P_1}[e^{i\langle z, X_t^\varepsilon \rangle + iuU_t^\varepsilon}] = k\, l_\varepsilon m_\varepsilon,$$

where

$$k = E^{P_1}[e^{i\langle z+u\eta, X_t'' \rangle}],$$

$$l_\varepsilon = E^{P_1}\left[\exp\left(i \int_{(0,t]\times B_\varepsilon} (\langle z, x\rangle + ug(x)) J(d(s,x)) \right)\right],$$

$$m_\varepsilon = \exp\left(-it \int_{\varepsilon < |x| \le 1} \langle z, x \rangle \nu_1(dx) + itu \left(c_\varepsilon - \tfrac{1}{2}\langle \eta, A\eta\rangle - \langle \gamma_1, \eta \rangle \right) \right.$$
$$\left. -itu \int_{B_{\varepsilon 0}} g(x) \nu_1(dx) \right).$$

We have

$$k = \exp(-\tfrac{t}{2}\langle z+u\eta, A(z+u\eta)\rangle + it\langle \gamma_1, z+u\eta\rangle)$$

and, by Proposition 3.3,

$$l_\varepsilon = \exp\left(t \int_{B_\varepsilon} (e^{i(\langle z,x\rangle + ug(x))} - 1) \nu_1(dx) \right).$$

Letting $\varepsilon \downarrow 0$, we get (6.7). The identity (6.7) shows that the generating triplet (A^*, ν^*, γ^*) of the Lévy process $(\{X_t^*\}, P_1)$ on $\mathbb{R}^{d+1}$ is such that

$$\langle z^*, A^* z^* \rangle = \langle z + u\eta, A(z+u\eta)\rangle$$

for $z^* = (z, u) \in \mathbb{R}^{d+1}$ with $z \in \mathbb{R}^d$ and

$$\nu^* = \nu_1 f^{-1},$$

where f is a mapping from $\mathbb{R}^d$ into $\mathbb{R}^{d+1}$ defined as $f(x) = (x, g(x))$ for $x \in \mathbb{R}^d$. We apply Theorem A4.3 of Appendix A to the $(d+1)$-dimensional Lévy process $\{X_t^*\}$. Since $E^{P_1}[e^{\langle z^*, X_t^* \rangle}] = E^{P_1}[e^{U_t}] < \infty$

with $z^* = (0, \ldots, 0, 1) \in \mathbb{R}^{d+1}$ by Lemma 6.1, the result is

$$E^{P_1}[e^{i\langle z, X_t\rangle + U_t}] = \exp\left[t\left(\widetilde{F} + \frac{1}{2}\langle iz + \eta, A(iz+\eta)\rangle + i\langle \gamma_1, z\rangle \right.\right.$$
$$\left.\left. -\frac{1}{2}\langle \eta, A\eta\rangle - \int_{\mathbb{R}^d} (e^{g(x)} - 1 - g(x)1_{[-1,1]}(g(x)))\nu_1(dx)\right)\right]$$

with

$$\widetilde{F} = \int_{\mathbb{R}^d} (e^{i\langle z, x\rangle + g(x)} - 1 - i\langle z, x\rangle 1_{B_0}(x) - g(x)1_{[-1,1]}(g(x)))\nu_1(dx).$$

The finiteness of this integral is checked as before. Now we have

$$E^{P_1}[e^{i\langle z, X_t\rangle + U_t}] = \exp\left[t\left(-\frac{1}{2}\langle z, Az\rangle \right.\right.$$
$$+ \int_{\mathbb{R}^d} (e^{i\langle z, x\rangle} - 1 - i\langle z, x\rangle 1_{B_0}(x))e^{g(x)}\nu_1(dx)$$
$$\left.\left. + i\ \langle \gamma_1, z\rangle + i\langle A\eta, z\rangle + i\int_{B_0} \langle z, x\rangle (e^{g(x)} - 1)\nu_1(dx)\right)\right].$$

That is,

$$E^{P_1}[e^{i\langle z, X_t\rangle + U_t}] = \exp\left[t\left(-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma_2, z\rangle \right.\right.$$
$$\left.\left. + \int_{\mathbb{R}^d} (e^{i\langle z, x\rangle} - 1 - i\langle z, x\rangle 1_{B_0}(x))e^{g(x)}\nu_1(dx)\right)\right], \tag{6.8}$$

since $A\eta = \gamma_{21} = \gamma_2 - \gamma_1 - \int_{|x|\le 1} x(e^{g(x)} - 1)\nu_1(dx)$. The process $\{X_s\colon 0 \le s \le t\}$ under the probability measure $e^{U_t}P_1$ has independent increments and satisfies (6.8). It follows that $\{X_s\colon 0 \le s \le t\}$ under $e^{U_t}P_1$ has stationary increments and its generating triplet equals (A, ν_2, γ_2). Therefore

$$P_2(B) = E^{P_1}[e^{U_t}1_B] \quad \text{for } B \in \mathcal{F}_t^0,$$

which completes the proof of the lemma. □

Our proof of Theorem B will be made in two steps. The first is a proof in the case where $P_2^t \ll P_1^t$. The second is a proof of the general case by the reduction to the first step.

Proof of Theorem B *under the condition that* $P_2^t \ll P_1^t$. We have $\nu_2 \ll \nu_1$ by Corollary 3.9. Hence $\nu_2(C_2) = 0$. By (3.19), $\nu_2(C_1) = \nu_1(C_2) = 0$. Hence $P_2(\Lambda_t) = 1$ by Theorem 3.4, and the assertion (i) and the identity (3.33) are trivially true.

It follows from $k_\alpha(\nu_1, \nu_2) < \infty$ that $\nu_1(C_1) < \infty$ by Lemmas 2.9 and 2.15. Let

$$\widetilde{\nu}_2 = e^{\widetilde{g}}\nu_1 = \nu_2 + 1_{C_1}\nu_1.$$

Then $\widetilde{\nu}_2 \approx \nu_1$, since $\widetilde{g}$ is finite-valued. We have

$$k_\alpha(\nu_1, \widetilde{\nu}_2) = \int (\alpha + (1-\alpha)e^{\widetilde{g}} - e^{(1-\alpha)\widetilde{g}})d\nu_1 = k_\alpha(\nu_1, \nu_2) - \alpha\nu_1(C_1),$$

which is finite. Hence

$$\int_{|\widetilde{g}|\le 1} \widetilde{g}^2 d\nu_1 + \int_{\widetilde{g}>1} e^{\widetilde{g}} d\nu_1 + \int_{\widetilde{g}<-1} d\nu_1 < \infty \tag{6.9}$$

by Remark 2.16. Hence we can apply Lemma 6.1. Namely, we have

$$\widetilde{V}_t = \lim_{\varepsilon\downarrow 0}\Bigg(\sum_{(s,X_s-X_{s-})\in(0,t]\times\{|x|>\varepsilon\}} \widetilde{g}(X_s - X_{s-}) - t\int_{|x|>\varepsilon}(e^{\widetilde{g}(x)} - 1)\nu_1(dx)\Bigg), \tag{6.10}$$

where the convergence is uniform on any bounded interval P_1-a. s. Let

$$\widetilde{U}_t = \langle \eta, X_t''\rangle - \tfrac{t}{2}\langle \eta, A\eta\rangle - t\langle \gamma_1, \eta\rangle + \widetilde{V}_t. \tag{6.11}$$

Then $(\{\widetilde{U}_t\}, P_1)$ is a Lévy process with generating triplet

$$A_{\widetilde{U}} = \langle \eta, A\eta\rangle, \tag{6.12}$$

$$\nu_{\widetilde{U}}(B) = \nu_1(\widetilde{g}^{-1}B) \quad \text{for } B \in \mathcal{B}_{\mathbb{R}\setminus\{0\}}, \tag{6.13}$$

$$\gamma_{\widetilde{U}} = -\tfrac{1}{2}\langle \eta, A\eta\rangle - \int_{\mathbb{R}^d}(e^{\widetilde{g}(x)} - 1 - \widetilde{g}(x)\, 1_{\{|\widetilde{g}(x)|\le 1\}}(x))\,\nu_1(dx). \tag{6.14}$$

Since

$$\int_{|x|>\varepsilon}(e^{\widetilde{g}(x)} - 1)\nu_1(dx) = \int_{|x|>\varepsilon}(e^{g(x)} - 1)\nu_1(dx) + \int_{|x|>\varepsilon} 1_{C_1}(x)\nu_1(dx), \tag{6.15}$$

the right-hand side of (3.27) also exists P_1-a. s. and the convergence is uniform on any finite time interval P_1-a. s. We have

$$V_t = \widetilde{V}_t + t\nu_1(C_1), \qquad U_t = \widetilde{U}_t + t\nu_1(C_1). \tag{6.16}$$

Hence $(\{U_t\}, P_1)$ is also a Lévy process and $A_U = A_{\widetilde{U}}$, $\nu_U = \nu_{\widetilde{U}}$, and $\gamma_U = \gamma_{\widetilde{U}} + \nu_1(C_1)$. It follows that, for $B \in \mathcal{B}_{\mathbb{R}\setminus\{0\}}$,

$$\nu_U(B) = \int 1_B(\widetilde{g}(x))\nu_1(dx) = \int 1_B(g(x))\nu_1(dx) = \nu_1(g^{-1}B)$$

and

$$\begin{aligned}\gamma_U &= -\tfrac{1}{2}\langle \eta, A\eta\rangle - \int_{\mathbb{R}^d} (e^{\widetilde{g}(x)} - 1 - \widetilde{g}(x)1_{\{|\widetilde{g}(x)|\le 1\}}(x))\nu_1(dx) + \nu_1(C_1)\\ &= -\tfrac{1}{2}\langle \eta, A\eta\rangle - \int_{\mathbb{R}^d} (e^{g(x)} - 1 - g(x)1_{\{|g(x)|\le 1\}}(x))\nu_1(dx).\end{aligned}$$

These prove (3.29)—(3.31). The independence of $\{U_t\}$ and $\{J((0,t]\times (C_1 \cup C_2))\}$ under P_1 follows from Theorem 3.4. From the properties of Poisson random measures we have $P_j(\Lambda_t) = e^{-t\nu_j(C_j)}$ for $j = 1, 2$. Thus we have proved (ii) and (iii).

Let us prove (iv). We use $\widetilde{\nu}_2$, $\widetilde{V}_t$, and $\widetilde{U}_t$ defined above. We choose η so that

$$A\eta = \gamma_{21} = \gamma_2 - \gamma_1 - \int_{|x|\le 1} xd(\nu_2 - \nu_1).$$

Denote

$$c = \int_{|x|\le 1} x1_{C_1}(x)d\nu_1.$$

It follows that

$$A\eta = \gamma_2 - \gamma_1 - \int_{|x|\le 1} xd(\widetilde{\nu}_2 - \nu_1) + c.$$

Choose

$$\widetilde{\gamma}_2 = \gamma_2 + c$$

and let $(\{X_t\}, \widetilde{P}_2)$ be the Lévy process with generating triplet $(A, \widetilde{\nu}_2, \widetilde{\gamma}_2)$. The relation between $(\{X_t\}, P_1)$ and $(\{X_t\}, \widetilde{P}_2)$ is exactly that which is treated in Lemma 6.2. By this lemma we have

$$\widetilde{P}_2(B) = \int_B e^{\widetilde{U}_t} dP_1 \quad \text{for } B \in \mathcal{F}_t^0, \tag{6.17}$$

where $\widetilde{U}_t$ is given by (6.10) and (6.11). That is, by (6.16),

$$\widetilde{P}_2(B) = e^{-t\nu_1(C_1)} \int_B e^{U_t} dP_1 \quad \text{for } B \in \mathcal{F}_t^0. \tag{6.18}$$

Next we consider the relation between $(\{X_t\}, \widetilde{P}_2)$ and $(\{X_t\}, P_2)$. The Lévy measure of the latter is a truncation of that of the former, that is, $\nu_2 = 1_{C_1{}^c}\widetilde{\nu}_2 = 1_C\widetilde{\nu}_2$. Let Λ_t be defined by (3.24). We claim that

$$E^{\widetilde{P}_2}[e^{i\langle z, X_t\rangle} 1_{\Lambda_t}] = e^{-t\nu_1(C_1)} E^{P_2}[e^{i\langle z, X_t\rangle}] \quad \text{for } z \in \mathbb{R}^d. \tag{6.19}$$

Indeed, let

$$Y_t = \lim_{\varepsilon \downarrow 0} \left(\sum_{(s, X_s - X_{s-}) \in (0,t] \times G_\varepsilon} (X_s - X_{s-}) - t \int_{\varepsilon < |x| \le 1} x d\nu_2 \right),$$

$$Z_t = \lim_{\varepsilon \downarrow 0} \left(\sum_{(s, X_s - X_{s-}) \in (0,t] \times H_\varepsilon} (X_s - X_{s-}) - t \int_{\varepsilon < |x| \le 1} x 1_{C_1}(x) d\nu_1 \right),$$

where $G_\varepsilon = \{|x| > \varepsilon\} \cap C$ and $H_\varepsilon = \{|x| > \varepsilon\} \cap (C_1 \cup C_2)$. Then $Y_t + Z_t$ is the jump part of $(\{X_t\}, \widetilde{P}_2)$ in Theorem 3.4. Let $W_t = X_t - Y_t - Z_t$. Then W_t is the continuous part of $(\{X_t\}, \widetilde{P}_2)$ in Theorem 3.4. By their independence,

$$E^{\widetilde{P}_2}[e^{i\langle z, X_t\rangle} 1_{\Lambda_t}] = E^{\widetilde{P}_2}[e^{i\langle z, Y_t + W_t\rangle}] E^{\widetilde{P}_2}[e^{i\langle z, Z_t\rangle} 1_{\Lambda_t}].$$

But $Z_t = -tc$ on Λ_t and $\widetilde{P}_2(\Lambda_t) = e^{-t\nu_1(C_1)}$. Hence, writing

$$r(z, x) = e^{i\langle z, x\rangle} - 1 - i\langle z, x\rangle 1_{\{|x| \le 1\}}(x),$$

we get

$$E^{\widetilde{P}_2}[e^{i\langle z, X_t\rangle} 1_{\Lambda_t}] = \exp\Bigg[t \bigg(-\tfrac{1}{2}\langle z, Az\rangle + i\langle \widetilde{\gamma}_2, z\rangle + \int r(z, x) \nu_2(dx) - i\langle c, z\rangle - \nu_1(C_1) \bigg) \Bigg].$$

Since $\widetilde{\gamma}_2 - c = \gamma_2$, this means (6.19). Next, let us show that, for $0 < s < t$,

$$E^{\widetilde{P}_2}[e^{i\langle z, X_s\rangle + i\langle w, X_t - X_s\rangle} 1_{\Lambda_t}] = e^{-t\nu_1(C_1)} E^{P_2}[e^{i\langle z, X_s\rangle + i\langle w, X_t - X_s\rangle}] \tag{6.20}$$

for $z, w \in \mathbb{R}^d$. Indeed,

$$
\begin{aligned}
&\text{left-hand side of (6.20)}\\
&\quad = E^{\widetilde{P}_2}[e^{i\langle z, Y_s+W_s\rangle + i\langle w, Y_t+W_t-Y_s-W_s\rangle}] E^{\widetilde{P}_2}[e^{i\langle z, Z_s\rangle + i\langle w, Z_t - Z_s\rangle} 1_{\Lambda_t}]\\
&\quad = \exp\Bigg[s\Big(-\tfrac{1}{2}\langle z, Az\rangle + i\langle \widetilde{\gamma}_2, z\rangle + \int r(z,x)\nu_2(dx)\Big)\\
&\qquad\qquad + (t-s)\Big(-\tfrac{1}{2}\langle w, Aw\rangle + i\langle \widetilde{\gamma}_2, w\rangle + \int r(w,x)\nu_2(dx)\Big)\\
&\qquad\qquad - is\langle c, z\rangle - i(t-s)\langle c, w\rangle - t\nu_1(C_1)\Bigg]\\
&\quad = \exp\Bigg[s\Big(-\tfrac{1}{2}\langle z, Az\rangle + i\langle \gamma_2, z\rangle + \int r(z,x)\nu_2(dx)\Big)\\
&\qquad\qquad + (t-s)\Big(-\tfrac{1}{2}\langle w, Aw\rangle + i\langle \gamma_2, w\rangle + \int r(w,x)\nu_2(dx)\Big)\\
&\qquad\qquad - t\nu_1(C_1)\Bigg]\\
&\quad = \text{right-hand side of (6.20)}.
\end{aligned}
$$

In the same way we can prove that, for $0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = t$,

$$
\begin{aligned}
(6.21)\qquad & E^{\widetilde{P}_2}\left[1_{\Lambda_t} \prod_{j=1}^{n+1} \exp(i\langle z_j, X_{s_j} - X_{s_{j-1}}\rangle)\right]\\
&\qquad = e^{-t\nu_1(C_1)} E^{P_2}\left[\prod_{j=1}^{n+1} \exp(i\langle z_j, X_{s_j} - X_{s_{j-1}}\rangle)\right]
\end{aligned}
$$

for $z_1, \ldots, z_{n+1} \in \mathbb{R}^d$. It follows that

$$
\begin{aligned}
(6.22)\qquad & E^{\widetilde{P}_2}\left[1_{\Lambda_t} \prod_{j=1}^{n+1} \exp(i\langle z_j, X_{s_j}\rangle)\right]\\
&\qquad = e^{-t\nu_1(C_1)} E^{P_2}\left[\prod_{j=1}^{n+1} \exp(i\langle z_j, X_{s_j}\rangle)\right]
\end{aligned}
$$

for $z_1, \dots, z_{n+1} \in \mathbb{R}^d$. Therefore

$$\widetilde{P}_2(B \cap \Lambda_t) = e^{-t\nu_1(C_1)} P_2(B) \quad \text{for all } B \in \mathcal{F}_t^0. \tag{6.23}$$

Hence

$$P_2(B) = e^{t\nu_1(C_1)} \widetilde{P}_2(B \cap \Lambda_t) = \int_{B \cap \Lambda_t} e^{U_t} dP_1.$$

The last equality is by (6.18) and by the fact $\Lambda_t \in \mathcal{F}_t^0$ in Remark 3.17. Thus we have obtained (3.32) with $(P_2^t)^{ac} = P_2^t$.

Proof of Theorem B *in the general case.* Condition (NS) of Theorem A holds by Corollary 3.6 from the assumption that P_2^t and P_1^t are not mutually singular. Thus $k_\alpha(\nu_1, \nu_2) < \infty$, which implies $\nu_1(C_1) < \infty$ and $\nu_2(C_2) < \infty$ by Lemmas 2.9 and 2.15. Since ${\nu_2}^{ac} = e^g \nu_1$, the function g satisfies (6.1) by (2.19) in the proof of Lemma 2.15. Hence the function $\widetilde{g}$ defined by (3.23) satisfies (6.9). Apply Lemma 6.1 to the function $\widetilde{g}$. Then we see that we can define $\widetilde{V}_t$ and $\widetilde{U}_t$ formally by (6.10) and (6.11). Since we have (6.15), this means that the assertion (ii) on V_t is true and that

$$V_t = \widetilde{V}_t + t\nu_1(C_1) \quad \text{and} \quad U_t = \widetilde{U}_t + t\nu_1(C_1),$$

where V_t and U_t are defined by (3.27) and (3.28). The process $\{U_t\}$ is, under P_1, a Lévy process on $\mathbb{R}$. Its generating triplet is expressed by (3.29)–(3.31) by the same argument as in the case where $P_2^t \ll P_1^t$. The assertion (iii) is thus proved.

Let us prove (i) and (iv). Let Q be the probability measure on $(\mathbf{D}, \mathcal{F}^0)$ for which $(\{X_t\}, Q)$ is the Lévy process on $\mathbb{R}^d$ with generating triplet $(A, {\nu_2}^{ac}, \gamma_Q)$, where

$$\gamma_Q = \gamma_2 - \int_{|x| \le 1} x d{\nu_2}^s.$$

Let us compare Q with P_1 on the one hand and with P_2 on the other. Choose η so that $A\eta = \gamma_{21}$. We have

$$\begin{aligned} k_\alpha(\nu_1, {\nu_2}^{ac}) &= \int_{\mathbb{R}^d} (\alpha f_1 + (1-\alpha) 1_C f_2 - f_1^\alpha f_2^{1-\alpha} 1_C) d\nu \\ &= \int_C dK_\alpha(\nu_1, \nu_2) + \alpha \nu_1(C_1) < \infty \end{aligned}$$

and

$$\gamma_Q - \gamma_1 - \int_{|x|\le 1} xd({\nu_2}^{ac} - \nu_1) = \gamma_2 - \gamma_1 - \int_{|x|\le 1} xd(\nu_2 - \nu_1) = A\eta.$$

Hence, applying the theorem in the absolutely continuous case, we have

$$\frac{dQ^t}{dP_1^t} = e^{U_t} 1_{\Lambda_t}, \tag{6.24}$$

where U_t and Λ_t are the same as those defined by (3.24), (3.27), and (3.28) (note that $\nu_1(C_2) = 0$, so that the deletion of ${\nu_2}^s$ does not affect U_t and Λ_t when we look at them under P_1). On the other hand,

$$k_\alpha(\nu_2, {\nu_2}^{ac}) = \int_{\mathbb{R}^d} (\alpha f_2 + (1-\alpha)1_C f_2 - f_2 1_C) d\nu = \alpha\nu_2(C_2) < \infty$$

and

$$\gamma_Q - \gamma_2 - \int_{|x|\le 1} xd({\nu_2}^{ac} - \nu_2) = 0.$$

Hence, letting

$$h(x) = \begin{cases} 0 & \text{on } C \\ -\infty & \text{on } C_1 \cup C_2 \end{cases}$$

and $\widetilde{h}(x) = 0$ on $\mathbb{R}^d$ and again applying the theorem in the absolutely continuous case, we get

$$\frac{dQ^t}{dP_2^t} = e^{U_t^*} 1_{\Lambda_t}, \tag{6.25}$$

where

$$U_t^* = -t \int_{\mathbb{R}^d} (e^{h(x)} - 1)\nu_2(dx) = t\nu_2(C_2).$$

It follows from (6.24) and (6.25) that, if $B \in \mathcal{F}_t^0$ satisfies $B \subset \Lambda_t$, then

$$\int_B e^{U_t} dP_1 = e^{t\nu_2(C_2)} P_2(B).$$

Therefore, $1_{\Lambda_t} P_2^t$ is absolutely continuous with respect to P_1^t. We already know that $(P_2^t)^{ac}$ has total mass $e^{-t\nu_2(C_2)}$ by Corollary 3.8 combined with Lemma 2.9. Since $\{J(G) \colon G \in \mathcal{B}_{(0,\infty)\times\mathbb{R}^d}\}$ is a Poisson random measure under P_2 with intensity measure $dt\, \nu_2(dx)$, we have

$$P_2^t(\Lambda_t) = P_2(\Lambda_t) = e^{-t\nu_2(C_2)}.$$

Hence, $(P_2^t)^{ac} = 1_{\Lambda_t} P_2^t$, which proves the assertion (i). Since (6.25) says that $Q^t = e^{t\nu_2(C_2)} 1_{\Lambda_t} P_2^t$, we have (3.33). Now (3.32) follows from (6.24). The proof is complete.

7. Density transformation

We start from one Lévy process $(\{X_t\}, P_1)$ on $\mathbb{R}^d$ with generating triplet (A_1, ν_1, γ_1). Suppose that we are given a measurable function $g(x)$ with values $-\infty \le g(x) < \infty$ and a vector $\eta \in \mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} (e^{g(x)/2} - 1)^2 \nu_1(dx) < \infty. \tag{7.1}$$

Define (A_2, ν_2, γ_2) by

$$A_2 = A_1, \quad \nu_2(dx) = e^{g(x)} \nu_1(dx), \quad \gamma_2 = \gamma_1 + \int_{|x|\le 1} x d(\nu_2 - \nu_1) + A_1 \eta.$$

Notice that (7.1) means that $k_{1/2}(\nu_1, \nu_2) < \infty$ and hence, by Lemma 2.15, $k_\alpha(\nu_1, \nu_2) < \infty$ for all $0 < \alpha < 1$. The condition (7.1) is equivalent to the property that

$$\int_{|g|\le 1} g^2 d\nu_1 + \int_{g>1} e^g d\nu_1 + \int_{g<-1} d\nu_1 < \infty \tag{7.2}$$

by Remark 2.16. We have

$$\int (1 \wedge |x|^2) \nu_2(dx) < \infty, \tag{7.3}$$

since (7.2) implies that

$$\int (1 \wedge |x|^2) e^g d\nu_1 \le e \int_{g\le 1} (1 \wedge |x|^2) d\nu_1 + \int_{g>1} e^g d\nu_1 < \infty.$$

It follows that (A_2, ν_2, γ_2) is the generating triplet of a Lévy process $(\{X_t\}, P_2)$ on $\mathbb{R}^d$. Define C_1, C_2, C, g, $\widetilde{g}$, V_t, U_t, and Λ_t by (3.19)–(3.24) and (3.27)–(3.28) with $C_2 = \emptyset$ and $A = A_1$. Define

$$Q^{(t)}(B) = \int_B e^{U_t} 1_{\Lambda_t} dP_1 \quad \text{for } B \in \mathcal{F}_t^0. \tag{7.4}$$

Then $Q^{(t)}$ is a probability measure on $\mathcal{F}_t^0$. The family $\{Q^{(t)} \colon t \in (0,\infty)\}$ is compatible in the sense that, if $s < t$ and $B \in \mathcal{F}_s^0$, then $Q^{(s)}(B) = Q^{(t)}(B)$. There exists a unique probability measure on $\mathcal{F}^0$ such that its restriction to $\mathcal{F}_t^0$ coincides with $Q^{(t)}$. Indeed, Theorem B shows that this probability measure is just P_2 and that $Q^{(t)}$ equals the restriction P_2^t of P_2 to $\mathcal{F}_t^0$. We call this procedure to get $(\{X_t\}, P_2)$ from $(\{X_t\}, P_1)$ the *density transformation*.

Example 7.1 (Deletion of jumps). When can we make $g(x) \equiv -\infty$, that is, $\nu_2 = 0$? A necessary and sufficient condition in order that we can make $A_2 = A_1$ and $\nu_2 = 0$ with some γ_2 through density transformation is that $\nu_1(\mathbb{R}^d) < \infty$. This is because

$$k_\alpha(\nu_1, 0) = \alpha \nu_1(\mathbb{R}^d).$$

The admissible γ_2 is of the form

$$\gamma_2 = \gamma_1 - \int_{|x| \le 1} x d\nu_1 + A_1 \eta$$

with $\eta \in \mathbb{R}^d$. Hence, if A_1 has full rank, then γ_2 can be any vector in $\mathbb{R}^d$. But, if A_1 is degenerate, then not all γ_2 are admissible. In any case, $\gamma_2 = \gamma_1 - \int_{|x| \le 1} x d\nu_1$ is possible. We have

$$U_t = \langle \eta, X_t'' \rangle - \tfrac{t}{2} \langle \eta, A_1 \eta \rangle - t \langle \gamma_1, \eta \rangle + t \nu_1(\mathbb{R}^d)$$

and T of (3.36) is the first jumping time. The distribution of T is exponential with parameter $\nu_1(\mathbb{R}^d)$.

Example 7.2 (Truncation of the support of Lévy measure). For what set $C = \mathbb{R}^d \setminus C_1$ can we make $\nu_2 = 1_C \nu_1$, that is, throw away $1_{C_1} \nu_1$? We have

$$k_\alpha(\nu_1, 1_C \nu_1) = \int (\alpha + (1 - \alpha) 1_C - 1_C) d\nu_1 = \alpha \nu_1(C_1).$$

Hence, in order that we can make $A_2 = A_1$ and $\nu_2 = 1_C \nu_1$ by density transformation, it is necessary and sufficient that $\nu_1(C_1) < \infty$. The admissible γ_2 is

$$\gamma_2 = \gamma_1 - \int_{|x| \le 1} x 1_{C_1}(x) d\nu_1 + A_1 \eta$$

with $\eta \in \mathbb{R}^d$. This time

$$U_t = \langle \eta, X_t'' \rangle - \tfrac{t}{2} \langle \eta, A_1 \eta \rangle - t \langle \gamma_1, \eta \rangle + t \nu_1(C_1)$$

and T of (3.36) is the first jumping time with jump height in C_1.

Example 7.3 (Esscher transformation). Consider the density transformation of a given Lévy process $(\{X_t\}, P_1)$ on $\mathbb{R}^d$ using the function

$g(x) = \langle \xi, x \rangle$ with $\xi \in \mathbb{R}^d \setminus \{0\}$. The vector ξ must be chosen to satisfy (7.1) or, equivalently, (7.2). Since

$$\int_{|g|\le 1} g^2 d\nu_1 \le |\xi|^2 \int_{|x|\le 1} |x|^2 d\nu_1 + \int_{|x|>1} d\nu_1 < \infty,$$

and

$$\int_{g<-1} d\nu_1 \le \int_{|x|\ge 1/|\xi|} d\nu_1 < \infty,$$

the condition (7.2) is equivalent to

$$\int_{\langle \xi, x\rangle > 1} e^{\langle \xi, x\rangle} \nu_1(dx) < \infty. \tag{7.5}$$

This condition is equivalent to

$$\int_{|x|>1} e^{\langle \xi, x\rangle} \nu_1(dx) < \infty, \tag{7.6}$$

since $\int_{\langle \xi, x\rangle > 1, |x|\le 1} e^{\langle \xi, x\rangle} d\nu_1$ and $\int_{\langle \xi, x\rangle \le 1, |x|>1} e^{\langle \xi, x\rangle} d\nu_1$ are finite. Further, by Theorem A4.3 of Appendix A, the condition (7.6) is equivalent to

$$E^{P_1}[e^{\langle \xi, X_t\rangle}] < \infty \tag{7.7}$$

for some $t > 0$ (or, equivalently, for all $t > 0$). Now assume that ξ is chosen so that (7.6) is satisfied and, letting $\eta = \xi$, consider the density transformation of $(\{X_t\}, P_1)$. Since

$$\begin{aligned}
&\langle \xi, X_t'' \rangle \\
&\quad + \lim_{\varepsilon \downarrow 0} \left(\sum_{(s, X_s - X_{s-}) \in (0,t] \times \{|x|>\varepsilon\}} \langle \xi, X_s - X_{s-} \rangle - t \int_{|x|>\varepsilon} (e^{\langle \xi, x\rangle} - 1) \nu_1(dx) \right) \\
&\qquad = \langle \xi, X_t \rangle - t \int_{\mathbb{R}^d} (e^{\langle \xi, x\rangle} - 1 - \langle \xi, x\rangle 1_{\{|x|\le 1\}}(x)) \nu_1(dx)
\end{aligned}$$

by Theorem 3.4, we have

$$U_t = \langle \xi, X_t \rangle - t\Psi_1(\xi), \tag{7.8}$$

where

$$\begin{aligned}
\Psi_1(\xi) = {} & \tfrac{1}{2}\langle \xi, A_1 \xi\rangle + \langle \gamma_1, \xi \rangle \\
& + \int_{\mathbb{R}^d} (e^{\langle \xi, x\rangle} - 1 - \langle \xi, x\rangle 1_{\{|x|\le 1\}}(x)) \nu_1(dx).
\end{aligned} \tag{7.9}$$

We have $\Lambda_t = \mathbf{D}$. The second Lévy process $(\{X_t\}, P_2)$ is obtained by

$$P_2(B) = e^{-t\Psi_1(\xi)} \int_B e^{\langle \xi, X_t \rangle} dP_1, \quad B \in \mathcal{F}_t^0. \tag{7.10}$$

It has generating triplet (A_2, ν_2, γ_2), where

$$A_2 = A_1, \tag{7.11}$$

$$\nu_2 = e^{\langle \xi, x \rangle} \nu_1, \tag{7.12}$$

$$\gamma_2 = \gamma_1 + \int_{|x| \le 1} x (e^{\langle \xi, x \rangle} - 1) \nu_1(dx) + A_1 \xi. \tag{7.13}$$

Its distribution at time t is such that

$$P_2(X_t \in B) = e^{-t\Psi_1(\xi)} \int_B e^{\langle \xi, x \rangle} P_1(X_t \in dx), \quad B \in \mathcal{B}_{\mathbb{R}^d}. \tag{7.14}$$

The identity (6.2) in this case is no other than

$$E^{P_1}[e^{\langle \xi, X_t \rangle}] = e^{t\Psi_1(\xi)},$$

which is given also in Theorem A4.3.

This transformation is called *Esscher transformation* or *exponential transformation*. It was introduced by Esscher (1932) in compound Poisson processes on $\mathbb{R}$ and utilized by Cramér (1938) in the study of large deviations. It was formulated in Sato (1990) for general Lévy processes on $\mathbb{R}^d$.

Example 7.4 (Esscher transformation in subordinators). Let the original process $(\{X_t\}, P_1)$ be a subordinator, that is, an increasing Lévy process on $\mathbb{R}$ (see Theorem A2.1). Thus, $A_1 = 0$, $\nu_1((-\infty, 0)) = 0$, $\int_{(0,1]} x \nu_1(dx) < \infty$, and $\gamma_{01} \ge 0$. Let

$$\xi < 0.$$

Then ξ satisfies (7.6) obviously. Let $(\{X_t\}, P_2)$ be the new Lévy process on $\mathbb{R}$ obtained by Esscher transformation from $(\{X_t\}, P_1)$ using this ξ. Then (7.11)–(7.13) show that $(\{X_t\}, P_2)$ is again a subordinator. Its drift γ_{02} is identical with γ_{01}. Let us consider their Laplace transforms: for $j = 1, 2$,

$$E^{P_j}[e^{-uX_t}] = e^{t\Psi_j(-u)}, \qquad u \ge 0,$$

with

$$\Psi_j(w) = \int_{(0,\infty)} (e^{wx} - 1)\, \nu_j(dx) + \gamma_{0j} w.$$

We have

$$\Psi_2(-u) = \Psi_1(-u+\xi) - \Psi_1(\xi), \tag{7.15}$$

which follows from (7.14). As in the general Esscher transformation, we have the notable relations

$$\nu_2 = e^{\xi x} \nu_1$$

and

$$P_2(X_t \in B) = e^{-t\Psi_1(\xi)} \int_B e^{\xi x} P_1(X_t \in dx), \quad B \in \mathcal{B}_{\mathbb{R}}.$$

In particular, suppose that $(\{X_t\}, P_1)$ is a strictly α-stable subordinator. Then, $0 < \alpha < 1$ and

$$E^{P_1}[e^{-uX_t}] = e^{-tc' u^\alpha}, \qquad u \ge 0$$

with some $c' > 0$ (see Example following Theorem A2.1). Thus

$$E^{P_2}[e^{-uX_t}] = e^{-tc'((u-\xi)^\alpha - u^\alpha)}, \qquad u \ge 0$$

in this case. If $\alpha = 1/2$, then

$$P_1(X_t \in B) = \frac{tc'}{2\sqrt{\pi}} \int_{B\cap(0,\infty)} e^{-(tc')^2/(4x)} x^{-3/2} dx,$$

$$P_2(X_t \in B) = \frac{tc'}{2\sqrt{\pi}} e^{tc'(-\xi)^{1/2}} \int_{B\cap(0,\infty)} e^{\xi x - (tc')^2/(4x)} x^{-3/2} dx$$

for $B \in \mathcal{B}_{\mathbb{R}}$. The distribution of X_t under P_2 is called *inverse Gaussian.*

Example 7.5 (Drift transformation). Given a Lévy process $(\{X_t\}, P_1)$ on $\mathbb{R}^d$ with generating triplet (A_1, ν_1, γ_1), how can we change the term γ_1 by density transformation while retaining the Gaussian covariance matrix and the Lévy measure unchanged? We should consider the density transformation with $g = 0$. Thus we get

$$\gamma_2 = \gamma_1 + A_1\eta \quad \text{with } \eta \in \mathbb{R}^d. \tag{7.16}$$

The transformation is given by $\Lambda_t = \mathbf{D}$, $V_t = 0$, and

$$U_t = \langle \eta, X_t'' \rangle - \tfrac{t}{2}\langle \eta, A_1\eta \rangle - t\langle \gamma_1, \eta \rangle \tag{7.17}$$

as in (3.28). Thus we have

$$P_2^t = e^{U_t} P_1^t. \tag{7.18}$$

We call this procedure the *drift transformation*. Notice that here we are using the continuous part $(\{X_t''\}, P_1)$ of the process $(\{X_t\}, P_1)$ while, in the case of the Esscher transformation, we use $\{X_t\}$ itself in (7.10).

If $\mathfrak{R}(A_1) = \mathbb{R}^d$, that is, if A_1 has full rank, then we can make γ_2 equal to any vector in $\mathbb{R}^d$. If $A_1 = 0$, then γ_2 cannot be other than γ_1. These are consequences of (7.16). In the case of the Brownian motion, this transformation is a special case of the absolutely continuous change of measures discovered by Cameron and Martin (1944).

Appendix A. A course on Lévy processes

The following are lectures of the Concentrated Advanced Course on Lévy Processes, MaPhySto, January 24–28, 2000, with some material added. The section on density transformation is deleted.

A.1. **Characterization of Lévy and stable processes.**

Lévy processes are processes with stationary independent increments satisfying stochastic continuity. In other words, they are time homogeneous Markov processes with spatial homogeneity. They are basic in the study of stochastic processes admitting jumps.

Brownian motion, Poisson process, Cauchy process, and Γ-process are examples of Lévy processes. Stable processes are also examples. The class of Lévy processes includes more irregular (in some sense) processes such as a process whose distribution at time t is continuous and singular for all $t > 0$ or a process whose distribution at time t is continuous and singular until some time t_0 and absolutely continuous after t_0.

Definition. ([S] Def.1.6) A stochastic process $\{X_t\colon t \geq 0\}$ on $\mathbb{R}^d$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is a *Lévy process* if

(1) it has independent increments, that is, for any $n \geq 1$ and for any choice of $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent,
(2) $X_0 = 0$ a. s.,
(3) $\mathcal{L}(X_{s+t} - X_s)$, the distribution of $X_{s+t} - X_s$, does not depend on s (time homogeneity),
(4) it is stochastically continuous, that is, $\lim_{t\to 0} P[|X_{s+t} - X_s| > \varepsilon] = 0$ for every $\varepsilon > 0$,
(5) there is $\Omega_0 \in \mathcal{F}$ with $P[\Omega_0] = 1$ such that, for every $\omega \in \Omega_0$, $X_t(\omega)$ is right-continuous with left limits as a function of t.

$\{X_t\}$ is a *Lévy process in law* if (1), (2), (3), and (4) are satisfied.
$\{X_t\}$ is an *additive process* if (1), (2), (4), and (5) are satisfied.
$\{X_t\}$ is an *additive process in law* if (1), (2), and (4) are satisfied.

The convolution of two finite measures ρ_1, ρ_2 on $\mathbb{R}^d$ is the finite measure ρ defined by $\rho(B) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} 1_B(x+y)\rho_1(dx)\rho_2(dy)$, $B \in \mathcal{B}_{\mathbb{R}^d}$, denoted as $\rho = \rho_1 * \rho_2$.

Definition. ([S] Def.7.1) A probability measure μ on $\mathbb{R}^d$ is *infinitely divisible* if, for every positive integer n, there is a probability measure μ_n on $\mathbb{R}^n$ such that $\mu = {\mu_n}^n = \mu_n * \ldots * \mu_n$ (n times).

Remark. If $\{X_t\}$ is a Lévy process in law, then for every $t \geq 0$ $\mathcal{L}(X_t)$, the distribution of X_t, is infinitely divisible. This is a consequence of (1), (2), and (3).

Denote the characteristic function of μ by $\widehat{\mu}(z)$, $z \in \mathbb{R}^d$. Thus $\widehat{\mu}(z) = \int_{\mathbb{R}^d} e^{i\langle z,x\rangle}\mu(dx)$ and the characteristic function of $\mathcal{L}(X)$ is $E[e^{i\langle z,X\rangle}]$. See [S] Prop.2.5 for basic properties of characteristic functions. They are continuous functions. If $\mu = \mu_1 * \mu_2$, then $\widehat{\mu}(z) = \widehat{\mu_1}(z)\widehat{\mu_2}(z)$. Denote the weak convergence of a sequence of probability measures μ_n, $n = 1, 2, \ldots$, to a probability measure μ by $\mu_n \to \mu$. We have $\mu_n \to \mu$ if and only if $\widehat{\mu_n}(z) \to \widehat{\mu}(z)$ on $\mathbb{R}^d$.

Lemma. ([S] Lem.7.5) *If μ is infinitely divisible, then $\widehat{\mu}(z) \neq 0$ for every $z \in \mathbb{R}^d$.*

Lemma. ([S] Lem.7.6) *Let $\varphi(z)$ be a complex-valued continuous function on $\mathbb{R}^d$ with $\varphi(0) = 1$ and $\varphi(z) \neq 0$ for every z. Then,*

(1) *there is a unique complex-valued continuous function $f(z)$ on $\mathbb{R}^d$ (called the distinguished logarithm of φ) such that $f(0) = 0$ and $e^{f(z)} = \varphi(z)$;*

(2) *for each positive integer n there is a unique complex-valued continuous function $g_n(z)$ on $\mathbb{R}^d$ (called the distinguished n-th root of φ) such that $g_n(0) = 1$ and $g_n(z)^n = \varphi(z)$.*

We denote the distinguished logarithm f of φ by $f = \log \varphi$. Note that $g_n(z) = e^{(1/n)\log \varphi(z)}$. By the two lemmas above, for the characteristic function $\widehat{\mu}(z)$ of an infinitely divisible distribution μ, we can define its distinguished logarithm.

Corollary. *If μ is infinitely divisible, then the n-th root μ_n of μ in convolution sense is unique.*

Lemma. ([S] Lem.7.8) *If μ_n are infinitely divisible, μ is a probability measure, and $\mu_n \to \mu$, then μ is infinitely divisible.*

Corollary. ([S] Lem.7.9) *If μ is infinitely divisible, then, for every $t \geq 0$, there is a unique probability measure μ_t such that $\widehat{\mu_t}(z) = e^{t \log \widehat{\mu}(z)}$. This μ_t is infinitely divisible. We denote μ_t by μ^t, and $\widehat{\mu_t}(z)$ by $\widehat{\mu}(z)^t$.*

Theorem A1.1. ([S] Th.7.10) (i) *If $\{X_t\}$ is a Lévy process in law on $\mathbb{R}^d$ with $\mathcal{L}(X_1) = \mu$, then μ is infinitely divisible and $\mathcal{L}(X_t) = \mu^t$.*

(ii) *If μ is infinitely divisible on $\mathbb{R}^d$, then there is a Lévy process in law $\{X_t\}$ on $\mathbb{R}^d$ such that $\mathcal{L}(X_1) = \mu$. This $\{X_t\}$ is unique in law.*

Remark. ([S] Rem.7.11) If $\{X_t\}$ satisfies (1), (2), and (3) in the definition of a Lévy process in law, $\mathcal{L}(X_t)$ is infinitely divisible, but it is not necessarily equal to μ^t unless the stochastic continuity (4) is assumed.

Theorem A1.2. ([S] Th.11.5) (Regularity of sample functions) *If $\{X_t\}$ is a Lévy process in law on $\mathbb{R}^d$, then there is a Lévy process $\{X'_t\}$ such that, for every $t \geq 0$, $P[X'_t = X_t] = 1$.*

Thus there is a correspondence between infinitely divisible distributions μ on $\mathbb{R}^d$ and Lévy processes $\{X_t\}$ on $\mathbb{R}^d$ by the relation $\mathcal{L}(X_1) = \mu$. Here $\{X_t\}$ is determined by μ uniquely in law.

The following result is fundamental in the theory of Lévy processes.

Theorem A1.3. ([S] Th.8.1) (Lévy–Khintchine representation) (i) *If μ is infinitely divisible on $\mathbb{R}^d$, then*

$$\text{(A1.1)}\quad \widehat{\mu}(z) = \exp\Big[-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle 1_{\{|x|\leq 1\}}(x))\nu(dx)\Big],$$

where

$$\text{(A1.2)}\quad \begin{cases} A \text{ is a symmetric nonnegative-definite } d \times d \text{ matrix}, \\ \gamma \in \mathbb{R}^d, \nu \text{ is a measure on } \mathbb{R}^d \text{ satisfying } \nu(\{0\}) = 0, \\ \text{and } \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty. \end{cases}$$

These A, γ, and ν are unique.

(ii) *For any A, γ, and ν satisfying* (A1.2), *there is an infinitely divisible distribution μ on $\mathbb{R}^d$ satisfying* (A1.1).

Definition. (A, ν, γ) in the theorem above is called the *generating triplet* of μ or of the corresponding Lévy process. The matrix A is called its *Gaussian covariance matrix*, and ν is called its *Lévy measure.*

Remark. Note that

$$e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle 1_{\{|x|\le 1\}}(x) = \begin{cases} O(|x|^2) & \text{as } x \to 0, \\ O(1) & \text{as } |x| \to \infty \end{cases}$$

for fixed z. Hence the left-hand side is ν-integrable. If $\int_{|x|\le 1} |x|\nu(dx) < \infty$, then (A1.1) is written as

$$\text{(A1.3)} \qquad \widehat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma_0, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\nu(dx)\right],$$

with $\gamma_0 = \gamma - \int_{|x|\le 1} x\nu(dx)$. This γ_0 is called the *drift* of μ or of the corresponding Lévy process. We write the new triplet as $(A, \nu, \gamma_0)_0$. If $\int_{|x|>1} |x|\nu(dx) < \infty$, then (A1.1) is written as

$$\text{(A1.4)} \qquad \widehat{\mu}(z) = \exp\Big[-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma_1, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle)\nu(dx)\Big],$$

with $\gamma_1 = \gamma + \int_{|x|>1} x\nu(dx)$. This γ_1 is called the *center* of μ or of the corresponding Lévy process. We write the new triplet as $(A, \nu, \gamma_1)_1$. In this case we can prove that $\int_{\mathbb{R}^d} |x|\mu(dx) < \infty$ and $\gamma_1 = \int_{\mathbb{R}^d} x\mu(dx)$, the mean of μ. ([S] Ex.25.12)

Interpretation of the Lévy–Khintchine representation of (the infinitely divisible distribution corresponding to) a Lévy process in terms of its sample function behavior is given by the Lévy-Itô decomposition of its sample functions. This is Theorem 3.4 in the main part of these lecture notes.

It is convenient to introduce the function

$$\text{(A1.5)} \qquad \Psi(w) = \frac{1}{2}\langle w, Aw\rangle + \langle \gamma, w\rangle + \int_{\mathbb{R}^d} (e^{\langle w,x\rangle} - 1 - \langle w, x\rangle 1_{\{|x|\le 1\}}(x))\nu(dx)$$

for $w \in \mathbb{C}^d$ such that[7] $\int_{|x|>1} |e^{\langle w,x\rangle}|\nu(dx) < \infty$. Here $\langle w, v\rangle = \sum_{j=1}^d w_j v_j$ for $w = (w_j)$ and $v = (v_j)$ in $\mathbb{C}^d$. Thus

$$E[e^{i\langle z,X_t\rangle}] = e^{t\Psi(iz)} \quad \text{for } z \in \mathbb{R}^d.$$

Definition. A Lévy process $\{X_t\}$ with generating triplet (A, ν, γ) is called of *type* A if $A = 0$ and $\nu(\mathbb{R}^d) < \infty$; of *type* B if $A = 0$, $\nu(\mathbb{R}^d) = \infty$, and $\int_{|x|\le 1} |x|\nu(dx) < \infty$; of *type* C if $A \neq 0$ or $\int_{|x|\le 1} |x|\nu(dx) = \infty$. $\{X_t\}$ is called *Gaussian* if $\nu = 0$; *purely non-Gaussian* if $A = 0$.

Examples.

1. *Brownian motion.* $E[e^{i\langle z,X_t\rangle}] = \exp(-\frac{t}{2}|z|^2)$. Generating triplet is $(I, 0, 0)$, where I is the identity matrix. $X_t(\omega)$ is continuous in t, a. s.

2. *Poisson process* with parameter $c > 0$ is on $\mathbb{R}$ with $E[e^{izX_t}] = \exp(tc(e^{izx} - 1))$. The generating triplet is $(0, c\delta_1, 0)_0$. $X_t(\omega)$ is a step function of t with jump height 1, a. s., and the first jumping time $T(\omega)$ has exponential distribution with parameter c, that is, $P[T > t] = e^{-ct}$.

3. *Compound Poisson process* on $\mathbb{R}^d$ is a Lévy process with generating triplet $(0, c\sigma, 0)_0$, where $c > 0$ and σ is a probability measure on $\mathbb{R}^d$ with $\sigma(\{0\}) = 0$. That is,

$$E[e^{i\langle z,X_t\rangle}] = \exp\left(tc\int_{\mathbb{R}^d}(e^{i\langle z,x\rangle} - 1)\sigma(dx)\right).$$

$X_t(\omega)$ is a step function of t, a. s., and the first jumping time $T(\omega)$ has exponential distribution with parameter c. Jumping times and jumping heights are independent. Jumping heights have distribution σ. $\mathcal{L}(X_t)$ is called a *compound Poisson distribution*.

4. Γ*-process* with parameter $q > 0$ corresponds to the exponential distribution with parameter q, that is, Γ-distribution with parameters 1, q. Its distribution at time t is Γ-distribution with parameters t, q. See Exercise 1.3.

Remark. ([S] Th.21.1, 21.3, 21.9) Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ with (A, ν, γ). Its sample functions are continuous a. s. if and only if it is Gaussian. Its sample functions are of finite variation on every finite time interval, a. s., if and only if it is of type A or B. Its sample

[7] $\mathbb{C}$ is the set of complex numbers.

functions are of infinite variation on every nonempty time interval, a. s., if and only if it is of type C. The number of jumps is finite on every finite time interval, a. s., if and only if $\nu(\mathbb{R}^d) < \infty$. Jumping times are dense in $[0, \infty)$ if and only if $\nu(\mathbb{R}^d) = \infty$.

Definition. A distribution μ on $\mathbb{R}^d$ is *stable* if, for every positive integer n, there are $b > 0$ and $c \in \mathbb{R}^d$ such that $\widehat{\mu}(z)^n = \widehat{\mu}(bz)e^{i\langle c,z\rangle}$. It is *strictly stable* if c can be chosen to be 0. If μ is stable, then it is infinitely divisible and, for every $a > 0$, there are $b > 0$ and $c \in \mathbb{R}^d$ such that $\widehat{\mu}(z)^a = \widehat{\mu}(bz)e^{i\langle c,z\rangle}$. Here c can be chosen to be 0 for every a if μ is strictly stable. A Lévy process $\{X_t\}$ is a *stable process* if it corresponds to a stable distribution. It is a *strictly stable process* if it corresponds to a strictly stable distribution.

In other words, strictly stable processes are *selfsimilar* Lévy processes.

We say that a distribution μ on $\mathbb{R}^d$ is *trivial* if $\mu = \delta_c$ for some $c \in \mathbb{R}^d$, that is, concentrated at c. A Lévy process $\{X_t\}$ is *trivial* if $\mathcal{L}(X_t)$ is trivial for every t or, equivalently, if $X_t = tc$ a. s. for some $c \in \mathbb{R}^d$.

Remark. If μ is stable and nontrivial, then b and c in the definition are uniquely determined by a.

Theorem A1.4. ([S] Prop.13.5, Th.13.11, Th.13.15) *If μ is stable and nontrivial on $\mathbb{R}^d$, then there is a unique $\alpha \in (0, 2]$ such that, for every $a > 0$, there is $c \in \mathbb{R}^d$ satisfying $\widehat{\mu}(z)^a = \widehat{\mu}(a^{1/\alpha}z)e^{i\langle c,z\rangle}$. If μ is strictly stable and nontrivial, then there is a unique $\alpha \in (0, 2]$ such that, for every $a > 0$, $\widehat{\mu}(z)^a = \widehat{\mu}(a^{1/\alpha}z)$.*

This α is called the *index* of a (strictly) stable distribution or process. Sometimes we say (strictly) α-stable instead of (strictly) stable with index α.

Stable distributions in one dimension are described as follows.

Theorem A1.5. ([S] Th.14.1, 14.3, 14.15) *Let μ be infinitely divisible and nontrivial on $\mathbb{R}$ with generating triplet (A, ν, γ).*

(i) *μ is 2-stable if and only if $\nu = 0$, that is, μ is Gaussian.*

(ii) *Let $0 < \alpha < 2$. The following three statements are equivalent.*

(1) *μ is α-stable.*

(2) *$A=0$ and $\nu=(c_1 1_{(0,\infty)}(x)+c_2 1_{(-\infty,0)}(x))|x|^{-1-\alpha}dx$ with $c_1\ge 0$, $c_2\ge 0$, and $c_1+c_2>0$.*

(3) *Either*

$$\alpha\neq 1 \quad \text{and} \quad \widehat{\mu}(z)=\exp\left[-c|z|^\alpha(1-i\beta\tan\tfrac{\pi\alpha}{2}\operatorname{sgn} z)+i\tau z\right],$$

or

$$\alpha=1 \quad \text{and} \quad \widehat{\mu}(z)=\exp\left[-c(|z|+i\beta\tfrac{2}{\pi}z\log|z|)+i\tau z\right],$$

where $c>0$, $-1\le\beta\le 1$, and $\tau\in\mathbb{R}$.

Here $\operatorname{sgn} z=-1,\ 0,\ 1$ according as $z<0,\ =0,\ >0$. The parameters in (3) are uniquely determined by μ. We have $\beta=(c_1-c_2)/(c_1+c_2)$. Hence β represents a degree of nonsymmetry of ν

Examples. See Exercises 1.1 and 1.2 at the end of this section.

The preceding theorem is a special case of the following result for a general d. Denote $S=\{\xi\in\mathbb{R}^d\colon |\xi|=1\}$, the unit sphere in $\mathbb{R}^d$.

Theorem A1.6. ([S] Th.14.1, 14.3, 14.10) *Let μ be infinitely divisible and nontrivial on $\mathbb{R}^d$ with generating triplet (A,ν,γ).*

(i) *μ is 2-stable if and only if μ is Gaussian.*

(ii) *Let $0<\alpha<2$. The following three statements are equivalent.*

(1) *μ is α-stable.*

(2) *$A=0$ and*

$$\nu(B)=\int_S\lambda(d\xi)\int_0^\infty 1_B(r\xi)r^{-\alpha-1}dr, \qquad B\in\mathcal{B}_{\mathbb{R}^d},$$

where λ is a uniquely determined finite nonzero measure on S.

(3) *Either $\alpha\neq 1$ and*

$$\Psi(iz)=-\int_S|\langle z,\xi\rangle|^\alpha\big(1-i\tan\tfrac{\pi\alpha}{2}\operatorname{sgn}\langle z,\xi\rangle\big)\lambda_1(d\xi)+i\langle\tau,z\rangle$$

or $\alpha=1$ and

$$\Psi(iz)=-\int_S\big(|\langle z,\xi\rangle|+i\tfrac{2}{\pi}\langle z,\xi\rangle\log|\langle z,\xi\rangle|\big)\lambda_1(d\xi)+i\langle\tau,z\rangle,$$

where λ_1 is a uniquely determined finite nonzero measure on S.

The measures λ and λ_1 are constant multiples of each other as follows ([S] E18.8):

$$\lambda_1 = \pi^{1/2} 2^{-\alpha} \frac{\Gamma((2-\alpha)/2)}{\alpha\Gamma((1+\alpha)/2)} \lambda.$$

The parameter τ in Theorems A1.5 and A1.6 is the drift if $0 < \alpha < 1$ and the center if $1 < \alpha < 2$.

The conditions for strict stability of stable distributions are as follows.

Theorem A1.7. ([S] Th.14.2, 14.7) *Let μ be α-stable and nontrivial on $\mathbb{R}^d$ with $0 < \alpha \leq 2$. Then, μ is strictly α-stable if and only if $\{\alpha = 2$ and $\gamma = 0\}$ or $\{\alpha = 1$ and $\int_S \xi\lambda(d\xi) = 0\}$ or $\{\alpha \neq 1, 2$ and $\tau = 0\}$.*

If $d = 1$, then the condition $\{\alpha = 1$ and $\int_S \xi\lambda(d\xi) = 0\}$ can be replaced by $\{\alpha = 1$ and $\beta = 0\}$.

Definition. A distribution μ on $\mathbb{R}^d$ is *selfdecomposable* if, for every $b > 1$, there is a probability measure ρ_b on $\mathbb{R}^d$ such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho_b}(z)$.

Stable distributions and selfdecomposable distributions are important in the theory of limit distributions for sums of independent random variables. ([S] Th.15.3, 15.7)

Theorem A1.8. (Prop.15.5, Th.15.10) *If μ is selfdecomposable, then it is infinitely divisible. An infinitely divisible distribution μ on $\mathbb{R}^d$ is selfdecomposable if and only if its Lévy measure ν is expressed as*

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) k_\xi(r) r^{-1} dr \quad \text{for } B \in \mathcal{B}_{\mathbb{R}^d \setminus \{0\}},$$

where λ is a finite measure on S and $k_\xi(r)$ is a nonnegative function measurable in $\xi \in S$ and decreasing[8] *in $r > 0$.*

If $d = 1$, then $S = \{-1, 1\}$ and the condition above is that $\nu(dx) = k(x)|x|^{-1}dx$ with $k(x)$ being decreasing on $(0, \infty)$ and increasing on $(-\infty, 0)$. Selfdecomposable distributions can have any Gaussian covariance matrices A.

[8]We say that $k(r)$ is decreasing if $k(r) \geq k(s)$ for $r < s$; increasing if $k(r) \leq k(s)$ for $r < s$.

Definition. A Lévy process corresponding to a selfdecomposable distribution is called a *selfdecomposable process.*

Selfdecomposable processes are important in connection with self-similar additive processes and with processes of Ornstein–Uhlenbeck type. ([S] Th.16.1, 17.5)

Examples. Stable distributions on $\mathbb{R}^d$ are selfdecomposable ([S] Ex. 15.2). Exponential, Γ-, and two-sided exponential ([S] Ex.15.14) distributions on $\mathbb{R}$ are selfdecomposable and not stable.

Generalization of stability in another direction is semi-stability. An infinitely divisible distribution μ on $\mathbb{R}^d$ is called *semi-stable* if there is $a > 1$ such that $\widehat{\mu}(z)^a = \widehat{\mu}(bz)e^{i\langle c,z\rangle}$ for some $b > 0$ and $c \in \mathbb{R}^d$. A notion which extends both selfdecomposability and semi-stability is semi-selfdecomposability. It will be introduced in Section A4. See [S] Chapter 3 for their exposition.

Exercises.

1.1. A Cauchy distribution μ on $\mathbb{R}^d$ is

$$\mu(B) = c\,\pi^{-(d+1)/2}\Gamma((d+1)/2)\int_B (|x-\gamma|^2+c^2)^{-(d+1)/2}dx$$

for $B \in \mathcal{B}_{\mathbb{R}^d}$, where $c > 0$ and $\gamma \in \mathbb{R}^d$. Show that μ is strictly 1-stable,

$$\widehat{\mu}(z) = e^{-c|z|+i\langle\gamma,z\rangle}, \quad z \in \mathbb{R}^d,$$

and its Lévy measure is

$$\nu(B) = c\,2^{\alpha}\frac{\Gamma((\alpha+d)/2)}{\Gamma(d/2)\Gamma((2-\alpha)/2)}\int_S \lambda_0(d\xi)\int_0^\infty 1_B(r\xi)r^{-2}dr$$

for $B \in \mathcal{B}_{\mathbb{R}^d\setminus\{0\}}$, where λ_0 is the uniform probability measure on the unit sphere S. ([S] Ex.2.12, Th.14.14, E18.8)

1.2. Let

$$\mu = c(2\pi)^{-1/2}e^{-c^2/(2x)}x^{-3/2}1_{(0,\infty)}(x)dx$$

with $c > 0$. Show that it has Laplace transform

$$L_\mu(u) = \int_{(0,\infty)} e^{-ux}\mu(dx) = e^{-c(2u)^{1/2}}, \quad u \geq 0,$$

and characteristic function

$$\widehat{\mu}(z) = e^{-c|z|^{1/2}(1-i\,\mathrm{sgn}\,z)}, \quad z \in \mathbb{R}.$$

Show that it is strictly $\frac{1}{2}$-stable and its Lévy measure is

$$\nu = c(2\pi)^{-1/2} x^{-3/2} 1_{(0,\infty)}(x)dx.$$

([S] Ex.2.13, 8.11)

1.3. Let μ be a Γ-distribution with parameters $c > 0$, and $q > 0$, that is,

$$\mu = \frac{q^c}{\Gamma(c)} e^{-qx} x^{c-1} 1_{(0,\infty)}(x)dx.$$

Show that it has Laplace transform

$$L_\mu(u) = (1 + q^{-1}u)^{-c}, \quad u \geq 0$$

and characteristic function

$$\widehat{\mu}(z) = (1 - iq^{-1}z)^{-c} = \exp[-c\log(1 - iq^{-1}z)], \quad z \in \mathbb{R},$$

where log is the principal value (that is, the imaginary part is in $(-\pi, \pi]$). Show that it is selfdecomposable with generating triplet $(A, \nu, \gamma_0)_0$ being $A = 0$,

$$\nu = ce^{-qx} x^{-1} 1_{(0,\infty)} dx,$$

and $\gamma_0 = 0$. ([S] Ex.2.15, 8.10)

1.4. A distribution μ on $\mathbb{R}^d$ is called *symmetric* if $\mu(B) = \mu(-B)$ for $B \in \mathcal{B}_{\mathbb{R}^d}$, where $-B = \{-x \colon x \in B\}$. Let μ be a nontrivial α-stable distribution on $\mathbb{R}^d$ with $0 < \alpha < 2$. Show that μ is symmetric if and only if

$$\widehat{\mu}(z) = \exp\left[-\int_S |\langle z, \xi\rangle|^\alpha \lambda_1(d\xi)\right], \quad z \in \mathbb{R}^d$$

with a symmetric finite measure λ_1 on S. Show that μ is rotation invariant if and only if

$$\widehat{\mu}(z) = e^{-c|z|^\alpha}, \quad z \in \mathbb{R}^d$$

with $c > 0$. ([S] Th.14.13, 14.14)

A.2. **Subordination.** In the theory of stochastic processes, the word subordinator is used in the following meaning. This is connected to the transformation that we discuss here.

Definition. $\{Z_t\}$ is a *subordinator* if it is a Lévy process on $\mathbb{R}$ having increasing sample functions a. s. or, equivalently, if it is a Lévy process on $\mathbb{R}$ corresponding to an infinitely divisible distribution λ supported on $[0, \infty)$.

Theorem A2.1. ([S] Th.21.5, 24.11) *Let $\{Z_t\}$ be a Lévy process on $\mathbb{R}$ with Lévy measure ρ. Then it is a subordinator if and only if it is of type* A *or* B*, ρ is supported on $[0, \infty)$, and the drift β is nonnegative. In this case*

$$E[e^{izZ_t}] = \exp\left[t\int_{(0,\infty)}(e^{izs}-1)\rho(ds) + i\beta z\right], \qquad z \in \mathbb{R},$$

$$E[e^{-uZ_t}] = \exp\left[t\int_{(0,\infty)}(e^{-us}-1)\rho(ds) - \beta u\right], \qquad u \geq 0.$$

These are the characteristic function and the Laplace transform of $\lambda^t = \mathcal{L}(Z_t)$, respectively.

Example. $\{Z_t\}$ is an α-stable subordinator, or an α-stable process with increasing sample functions, if and only if

$$E[e^{-uZ_t}] = e^{-t(c'u^\alpha + \beta u)}, \qquad u \geq 0$$

with $0 < \alpha < 1$, $c' > 0$, and $\beta \geq 0$. It is a strictly α-stable subordinator if and only if $\beta = 0$. The connection of c' and β with c_1, c_2, c, and τ in Theorem A1.5 is as follows: $c' = \alpha^{-1}\Gamma(1-\alpha)c_1 = (\cos(\pi\alpha/2))^{-1}c$, $c_2 = 0$, and $\beta = \tau$. ([S] Ex.24.12)

Lemma. *Let $\{Z_t\}$ be a subordinator with Lévy measure ρ and drift β. Then, for $w \in \mathbb{C}$ with $\operatorname{Re} w \leq 0$, we can define*

$$\Psi(w) = \int_{(0,\infty)}(e^{ws}-1)\rho(ds) + \beta w \tag{A2.1}$$

and

$$E[e^{wZ_t}] = e^{t\Psi(w)}.$$

Subordination is a random time change by an independent subordinator. It was introduced by Bochner [41][9] in 1949. It transforms a Markov process to a Markov process and a Lévy process to a Lévy process.

Theorem A2.2. ([S] Th.30.1) *Let $\{Z_t\}$ be a subordinator with Lévy measure ρ, drift β, and $\mathcal{L}(Z_1) = \lambda$. Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ with generating triplet (A, ν, γ) and $\mathcal{L}(X_1) = \mu$. Assume that $\{X_t\}$ and $\{Z_t\}$ are independent. Define*

$$Y_t(\omega) = X_{Z_t(\omega)}(\omega).$$

Then $\{Y_t\}$ is a Lévy process on $\mathbb{R}^d$ and

$$P(Y_t \in B) = \int_{[0,\infty)} \mu^s(B)\lambda^t(ds), \qquad B \in \mathcal{B}_{\mathbb{R}^d},$$
$$E[e^{i\langle z,Y_t\rangle}] = e^{t\Psi(\log \widehat{\mu}(z))}, \qquad z \in \mathbb{R}^d.$$

The generating triplet $(A^\sharp, \nu^\sharp, \gamma^\sharp)$ of $\{Y_t\}$ is:

$$A^\sharp = \beta A,$$
$$\nu^\sharp(B) = \beta\nu(B) + \int_{(0,\infty)} \mu^s(B)\rho(ds), \qquad B \in \mathcal{B}_{\mathbb{R}^d\setminus\{0\}},$$
$$\gamma^\sharp = \beta\gamma + \int_{(0,\infty)} \rho(ds) \int_{|x|\le 1} x\mu^s(dx).$$

Definition. The transformation from $\{X_t\}$ to $\{Y_t\}$ is called *subordination by* $\{Z_t\}$. Any Lévy process identical in law with $\{Y_t\}$ is said to be *subordinate to* $\{X_t\}$.

Remark. ([S] Th.30.1) If $\beta = 0$ and $\int_{(0,1]} s^{1/2}\rho(ds) < \infty$, then $\{Y_t\}$ is of type A or B and has drift 0.

Example. Let $\{X_t\}$ be the Brownian motion on $\mathbb{R}^d$ and let $\{Z_t\}$ be a strictly α-stable subordinator, $0 < \alpha < 1$. Then the process $\{Y_t\}$ subordinate to $\{X_t\}$ by $\{Z_t\}$ is a rotation invariant 2α-stable process. Indeed, $\widehat{\mu}(z) = \exp(-\frac{1}{2}z^2)$, $\Psi(-u) = -c'u^\alpha$ for $u \ge 0$ with $c' > 0$, and

$$E[e^{i\langle z,Y_t\rangle}] = e^{t\Psi(\log \widehat{\mu}(z))} = e^{-tc'\,2^{-\alpha}|z|^{2\alpha}}.$$

[9] The number in square brackets indicates the reference in [S].

Proof of Theorem A2.2 uses the following lemma.

Lemma. ([S] Lem.30.3) *Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$. For any $\varepsilon > 0$, there is $C = C(\varepsilon)$ such that, for any t,*

$$P(|X_t| > \varepsilon) \leq Ct.$$

There are C_1, C_2, and C_3 such that, for any t,

$$E[|X_t|^2; |X_t| \leq 1] \leq C_1 t,$$
$$|E[X_t; |X_t| \leq 1]| \leq C_2 t,$$
$$E[|X_t|; |X_t| \leq 1] \leq C_3 t^{1/2}.$$

Theorem A2.3. ([S] Th.30.4) *Let $\{Z_1(t)\}$ and $\{Z_2(t)\}$ be independent subordinators.*

(i) *Let $Z_3(t) = Z_1(Z_2(t))$. Then $\{Z_3(t)\}$ is a subordinator and we have $\Psi_3(w) = \Psi_2(\Psi_1(w))$. Here the function $\Psi(w)$ of* (A2.1) *for $\{Z_j(t)\}$ is denoted by $\Psi_j(w)$.*

(ii) *Let $\{X_t\}$, $\{Y_t\}$, and $\{W_t\}$ be Lévy processes on $\mathbb{R}^d$. If $\{Y_t\}$ is subordinate to $\{X_t\}$ by the subordinator $\{Z_1(t)\}$ and $\{W_t\}$ is subordinate to $\{Y_t\}$ by $\{Z_2(t)\}$, then $\{W_t\}$ is subordinate to $\{X_t\}$ by $\{Z_3(t)\}$.*

Example. If $\{Z_1(t)\}$ and $\{Z_2(t)\}$ are strictly α_1- and α_2-stable subordinators, respectively, then $\{Z_3(t)\}$ is a strictly $\alpha_1\alpha_2$-stable subordinator.

Definition. Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ with $\mu = \mathcal{L}(X_1)$. For $q \geq 0$ let

$$V^q(B) = \int_0^\infty e^{-qt}\mu^t(B)dt = E\left[\int_0^\infty e^{-qt}1_B(X_t)dt\right], \quad B \in \mathcal{B}_{\mathbb{R}^d}.$$

This V^q is called the *q-potential measure* of $\{X_t\}$. V^0 is written as V and called *potential measure* of $\{X_t\}$. Note that $V^q(\mathbb{R}^d) = 1/q$ for $q > 0$ and $V^0(\mathbb{R}^d) = \infty$.

Theorem A2.4. ([S] Th.30.10, Prop.37.4) *For any $q > 0$, qV^q is infinitely divisible, purely non-Gaussian, with Lévy measure $\nu_q^\sharp$ equal to*

$$\nu_q^\sharp(B) = \int_0^\infty e^{-qt}\mu^t(B)\frac{dt}{t}, \qquad B \in \mathcal{B}_{\mathbb{R}^d\setminus\{0\}}$$

and satisfying $\int(|x| \wedge 1)\nu_q^\sharp(dx) < \infty$. The drift of qV^q is 0.

In fact, if we consider the process $\{Y_t\}$ that is subordinate to $\{X_t\}$ by the Γ-process with parameter $q > 0$, then $qV^q = \mathcal{L}(Y_1)$.

Exercises.

2.1. Let $\{Z_t\}$ be the Γ-process with parameter $q > 0$. Let $\{Y_t\}$ be a Lévy process on $\mathbb{R}$ subordinate by $\{Z_t\}$ to a symmetric α-stable process $\{X_t\}$ with $\widehat{\mu}(z) = e^{-|z|^\alpha}$, $0 < \alpha \leq 2$. Show that the characteristic function of Y_t is as follows:

$$E[e^{izY_t}] = (1 + q^{-1}|z|^\alpha)^{-t}.$$

The distribution $\mathcal{L}(Y_1)$ is called Linnik distribution or geometric stable distribution. ([S] Ex.30.8)

2.2. Let $\{Y_t\}$ be a Lévy process on $\mathbb{R}^d$ subordinate to the Brownian motion by a selfdecomposable subordinator $\{Z_t\}$. Show that $\{Y_t\}$ is a selfdecomposable process. ([S] E34.3)

A.3. **Recurrence and transience.** Let us consider long time behavior of Lévy processes.

Definition. A Lévy process $\{X_t\}$ on $\mathbb{R}^d$ is *recurrent* if

$$\liminf_{t\to\infty} |X_t| = 0 \qquad \text{a. s.;}$$

it is *transient* if

$$\lim_{t\to\infty} |X_t| = \infty \qquad \text{a. s.}$$

$\{S_n\}$ is a *random walk* on $\mathbb{R}^d$ if $S_0 = 0$ and $S_n = Z_1 + \cdots + Z_n$ for $n = 1, 2, \ldots$, where $\{Z_n\}$ is a family of independent identically distributed random variables on $\mathbb{R}^d$. A random walk $\{S_n\}$ is *recurrent* if

$$\liminf_{n\to\infty} |S_n| = 0 \qquad \text{a. s.;}$$

transient if

$$\lim_{n\to\infty} |S_n| = \infty \qquad \text{a. s.}$$

We define

$$W(B) = \sum_{n=1}^{\infty} P[S_n \in B] = E\left[\sum_{n=1}^{\infty} 1_B(S_n)\right],$$

an analogue of the potential measure V. Denote, in the following, the open disc with center 0 and radius a by

$$B_a = \{x \in \mathbb{R}^d : |x| < a\}.$$

Theorem A3.1. ([S] Th.35.3) *Let $\{S_n\}$ be a random walk on $\mathbb{R}^d$. Then the following are true.*

(i) *It is either recurrent or transient.*

(ii) *It is recurrent if and only if*

$$W(B_a) = \infty \qquad \text{for all } a > 0.$$

(iii) *It is transient if and only if*

$$W(B_a) < \infty \qquad \text{for all } a > 0.$$

Lemma (Kingman [260]). ([S] Lem.35.5) *Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$. Then there is a function $c(\varepsilon)$ satisfying $c(\varepsilon) \to 1$ as $\varepsilon \downarrow 0$ such that, for every $t > 0$ and every $a > 0$,*

$$P\left[\int_t^\infty 1_{B_{2a}}(X_s)ds > \varepsilon\right] \geq c(\varepsilon)P[|X_{t+s}| < a \text{ for some } s > 0].$$

Theorem A3.2. ([S] Th.35.4) *Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$. Then the following are true.*

(i) *It is either recurrent or transient.*

(ii) *It is recurrent if and only if*

$$V(B_a) = \infty \qquad \text{for all } a > 0. \tag{A3.1}$$

(iii) *It is recurrent if and only if*

$$\int_0^\infty 1_{B_a}(X_t)dt = \infty \quad \text{a. s.} \qquad \text{for all } a > 0. \tag{A3.2}$$

(iv) *It is transient if and only if*

$$V(B_a) < \infty \qquad \text{for all } a > 0. \tag{A3.3}$$

(v) *It is transient if and only if*

$$\int_0^\infty 1_{B_a}(X_t)dt < \infty \quad \text{a. s.} \qquad \text{for all } a > 0. \tag{A3.4}$$

(vi) *Fix $h > 0$ arbitrarily. $\{X_t\}$ is recurrent if and only if the random walk $\{X_{nh} \colon n = 0, 1, \dots\}$ is recurrent.*

Outline of proof. Step 1. Fix $a > 0$. It is proved that the following five statements are equivalent:

(a) $$P[\text{there are } t_n \uparrow \infty \text{ such that } X_{t_n} \in B_a] = 1;$$

(b) $$P\left[\int_0^\infty 1_{B_{2a}}(X_t)dt = \infty\right] = 1;$$

(c) $$V(B_{2a}) = \infty;$$

(d) $$\sum_{n=1}^{\infty} P[X_{nh} \in B_{3a}] = \infty \text{ for all sufficiently small } h > 0;$$

(e) $\{X_{nh} \colon n = 0, 1, \dots\}$ is recurrent for all sufficiently small $h > 0$.

Step 2. The condition (e) does not involve a. Hence, each of the conditions (a)–(d) is independent of a. But the condition (a) for all $a > 0$ is equivalent to the recurrence of $\{X_t\}$. Hence each of (b)–(e) is equivalent to the recurrence of $\{X_t\}$. Also, if (A3.1) does not hold, then (A3.3) holds.

Step 3. Proof of the theorem by the reduction to Step 2.

Example. Let $\{X_t\}$ be the Brownian motion on $\mathbb{R}^d$. If $d = 1$ or 2, then it is recurrent. If $d \geq 3$, then it is transient. We can check (A3.1) for $d \leq 2$ and (A3.3) for $d \geq 3$.

Theorem A3.3 (Chung–Fuchs type criterion)**.** ([S] Th.37.5) *Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ with $\mu = \mathcal{L}(X_1)$, $\Psi(iz) = \log \widehat{\mu}(z)$. Fix $a > 0$. Then $\{X_t\}$ is recurrent if and only if*

(A3.5) $$\limsup_{q \downarrow 0} \int_{B_a} \mathrm{Re}\left(\frac{1}{q - \Psi(iz)}\right) dz = \infty.$$

Corollary. ([S] Cor.37.6) *If $\displaystyle\int_{B_a} \frac{dz}{|\Psi(iz)|} < \infty$, then $\{X_t\}$ is transient.*

Definition. For any bounded measurable function f and $q > 0$, let

$$(U^q f)(x) = \int_{\mathbb{R}^d} f(x+y)V^q(dy) = E\left[\int_0^\infty e^{-qt} f(x + X_t)dt\right]$$

$$= \int_0^\infty e^{-qt}dt \int_{\mathbb{R}^d} f(x+y)\mu^t(dy).$$

For any integrable function f, let

$$(Ff)(z) = \int_{\mathbb{R}^d} e^{i\langle z,x\rangle} f(x)dx.$$

U^q is the *q-potential operator.* F is the *Fourier transform.*

Lemma. ([S] Prop37.4) *If f is continuous and integrable on $\mathbb{R}^d$ and if Ff is integrable on $\mathbb{R}^d$, then, for $q > 0$,*

$$(U^q f)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} (Ff)(-z) \frac{e^{i\langle x,z\rangle}}{q - \Psi(iz)} dz.$$

Definition. A measure μ on $\mathbb{R}^d$ is *degenerate* if, for some $a \in \mathbb{R}^d$ and some proper linear subspace L of $\mathbb{R}^d$, the support of μ is included in $a + L$. A process is *degenerate* if, for all t, $\mathcal{L}(X_t)$ is degenerate.

Theorem A3.4. ([S] Th.37.8) *If $d \geq 3$, then any nondegenerate Lévy process $\{X_t\}$ on $\mathbb{R}^d$ is transient.*

Proof is given by application of the Chung–Fuchs type criterion.

In the next theorem, the sufficiency of (A3.6) for recurrence is an easy consequence of Theorem A3.3, but the necessity of (A3.6) is a hard result proved by Spitzer [438], Ornstein [328], Stone [446], and Port and Stone [348].

Theorem A3.5 (Spitzer type criterion)**.** *Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$. Fix $a > 0$. Then $\{X_t\}$ is recurrent if and only if*

$$\int_{B_a} \mathrm{Re}\left(\frac{1}{-\Psi(iz)}\right) dz = \infty. \tag{A3.6}$$

Other results for $d = 1$. ([S] Th.35.8, 36.5, 36.6, 36.7, Prop.37.10) Let $\{X_t\}$ be a Lévy process on $\mathbb{R}$ with $\mu = \mathcal{L}(X_1)$.

1. If $\int_{\mathbb{R}} |x|\mu(dx) < \infty$ and $\int_{\mathbb{R}} x\mu(dx) \neq 0$, then $\{X_t\}$ is transient.
2. If $\int_{\mathbb{R}} |x|\mu(dx) < \infty$ and $\int_{\mathbb{R}} x\mu(dx) = 0$, then $\{X_t\}$ is recurrent.
3. More generally than 2, if $n^{-1}X_n \to 0$ in probability as $n = 0, 1, \ldots \to \infty$, then $\{X_t\}$ is recurrent.

4. If $\int_{(0,\infty)} x\mu(dx) = \infty$ and $\int_{(-\infty,0)} x\mu(dx) > -\infty$ (or if $\int_{(0,\infty)} x\mu(dx) < \infty$ and $\int_{(-\infty,0)} x\mu(dx) = -\infty$), then $\{X_t\}$ is transient.

5. Suppose that $\{X_t\}$ is non-zero[10]. Then it satisfies one of the following:

(d$^+$) $\quad X_t \to \infty$ as $t \to \infty$, a. s. (*drifting to* ∞);

(d$^-$) $\quad X_t \to -\infty$ as $t \to \infty$, a. s. (*drifting to* $-\infty$);

(o) $\quad \limsup_{t\to\infty} X_t = \infty$ and $\liminf_{t\to\infty} X_t = -\infty$, a. s. (*oscillating*).

If $\{X_t\}$ is recurrent and non-zero, then it is oscillating. But the converse is not true. Thus, if $\{X_t\}$ is oscillating and transient, then

$$P(\text{the set of limit points of } X_t \text{ as } t \to \infty \text{ equals } \{\infty, -\infty\}) = 1.$$

If $\{X_t\}$ is recurrent, then

$$P(\text{the set of limit points of } X_t \text{ as } t \to \infty \text{ equals } \Sigma) = 1,$$

where Σ is the smallest closed set satisfying $P(X_t \in \Sigma \text{ for all } t) = 1$.

A criterion for the three cases in 5 was given by Spitzer [436] and Rogozin [379].

Theorem A3.6. ([S] Th.48.1) *Let $\{X_t\}$ be a non-zero Lévy process on $\mathbb{R}$. Let*

$$I^+ = \int_1^\infty t^{-1} P[X_t > 0]dt \quad \text{and} \quad I^- = \int_1^\infty t^{-1} P[X_t < 0]dt.$$

Then (d$^+$) *holds if and only if $I^- < \infty$;* (d$^-$) *holds if and only if $I^+ < \infty$;* (o) *holds if and only if $I^+ = \infty$ and $I^- = \infty$.*

The following theorem is a related result by Kesten [249] and Erickson [115].

Theorem A3.7. ([S] Rem.37.13) *Let $\{X_t\}$ be a Lévy process on $\mathbb{R}$ with $\mu = \mathcal{L}(X_1)$ satisfying*

$$\text{(A3.7)} \qquad \int_{(0,\infty)} x\mu(dx) = \infty \quad \text{and} \quad \int_{(-\infty,0)} x\mu(dx) = -\infty.$$

Then it satisfies one of the following three:

(dd$^+$) $\quad$ *$t^{-1}X_t \to \infty$ as $t \to \infty$, a. s.;*

(dd$^-$) $\quad$ *$t^{-1}X_t \to -\infty$ as $t \to \infty$, a. s.;*

[10]A Lévy process is called *non-zero* unless it is identically zero a. s.

(oo) $\limsup_{t\to\infty} t^{-1}X_t = \infty$ *and* $\liminf_{t\to\infty} t^{-1}X_t = -\infty$, *a. s.*

Further, let ν *be its Lévy measure and define*

$$K^+ = \int_{(2,\infty)} x\left(\int_{-x}^{-1} \nu((-\infty, y))dy\right)^{-1} \nu(dx),$$

$$K^- = \int_{(-\infty,-2)} |x|\left(\int_{1}^{|x|} \nu((y, \infty))dy\right)^{-1} \nu(dx).$$

Then $K^+ + K^- = \infty$ *and the following equivalences are true:* (dd$^+$) *holds if and only if* $\{K^+ = \infty$ *and* $K^- < \infty\}$*;* (dd$^-$) *holds if and only if* $\{K^+ < \infty$ *and* $K^- = \infty\}$*;* (oo) *holds if and only if* $\{K^+ = \infty$ *and* $K^- = \infty\}$*.*

One consequence is that, under the condition (A3.7), the properties (dd$^+$), (dd$^-$), and (oo) are respectively equivalent to (d$^+$), (d$^-$), and (o). We can say, in terms of ν, whether the condition (A3.7) holds or not. When (A3.7) does not hold, we can express $\int_{\mathbb{R}} x\mu(dx)$ by ν and γ (see Section A4). When (A3.7) holds, we can now distinguish (d$^+$), (d$^-$), and (o) by looking at ν, using Theorem A3.7.

Within the class of Lévy processes on $\mathbb{R}$ satisfying (o), we have no general criterion of recurrence and transience in terms of the generating triplets. But, in symmetric case, Shepp's results [423, 424] have analogues formulated in the following two theorems by Sato [407].

Definition. ([S] Def.38.1) A symmetric measure on $\mathbb{R}$ finite outside of any neighborhood of the origin is *quasi-unimodal* if there is $x_0 > 0$ such that $\rho((x, \infty))$ is convex for $x > x_0$. It has a *bigger tail* than a symmetric measure ρ' if there is $x_0 > 0$ such that $\rho((x, \infty)) \geq \rho'((x, \infty))$ for $x > x_0$.

Theorem A3.8. ([S] Th.38.2) *Suppose that both* $\{X_t\}$ *and* $\{Y_t\}$ *are symmetric*[11] *Lévy processes on* $\mathbb{R}$*. Let* ν_X *and* ν_Y *be their respective Lévy measures.*

(i) *If* $\int_{(0,\infty)} x^2|\nu_X - \nu_Y|(dx) < \infty$*, then recurrence of* $\{X_t\}$ *is equivalent to that of* $\{Y_t\}$*.*

[11]A process $\{X_t\}$ is called *symmetric* if $\{X_t\}$ and $\{-X_t\}$ are identical in law.

(ii) *If* ν_Y *has a bigger tail than* ν_X *and if* ν_Y *is quasi-unimodal, then transience of* $\{X_t\}$ *implies that of* $\{Y_t\}$.

Theorem A3.9. ([S] Th.38.3, 38.4) (i) *Let* $\{X_t\}$ *be a symmetric Lévy process on* $\mathbb{R}$ *with quasi-unimodal Lévy measure* ν_X. *Let* $c > 0$ *be fixed. Then* $\{X_t\}$ *is recurrent if and only if*

$$\int_c^\infty \left(\int_0^r x\nu_X((x \vee 1, \infty))dx \right)^{-1} dr = \infty.$$

(ii) *For any symmetric Lévy process* $\{X_t\}$ *with Lévy measure* ν_X, *there exists a recurrent symmetric Lévy process* $\{Y_t\}$ *such that its Lévy measure* ν_Y *has a bigger tail than* ν_X.

Some results for $d = 2$**.** ([S] Th.36.5, 37.14) Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^2$ with $\mu = \mathcal{L}(X_1)$.

1. If $\int_{\mathbb{R}^2} |x|\mu(dx) < \infty$ and $\int_{\mathbb{R}^2} x\mu(dx) \neq 0$, then $\{X_t\}$ is transient.
2. If $\int_{\mathbb{R}^2} |x|^2\mu(dx) < \infty$ and $\int_{\mathbb{R}^2} x\mu(dx) = 0$, then $\{X_t\}$ is recurrent.

Remark. (Sato [406]) Let $L_a(\omega) = \sup\{t \colon X_t(\omega) \in B_a\}$, the last exit time from the disc B_a for a Lévy process $\{X_t\}$ on $\mathbb{R}^d$. Transience means that $B_a < \infty$ a.s. for every $a > 0$. The process $\{X_t\}$ is called *strongly transient* if $E[L_a] < \infty$ for every $a > 0$. If $d \geq 5$, then every nondegenerate Lévy process on $\mathbb{R}^d$ is strongly transient.

Exercises.

3.1. Let $\{X_t\}$ be a nontrivial α-stable process on $\mathbb{R}$. Show the following. If $1 \leq \alpha \leq 2$, then, in order that $\{X_t\}$ be recurrent, it is necessary and sufficient that it is strictly α-stable. If $0 < \alpha < 1$, then $\{X_t\}$ is transient. ([S] Cor.37.17)

3.2. Let $\{X_t\}$ be a nondegenerate α-stable process on $\mathbb{R}^2$. Show that $\{X_t\}$ is recurrent if and only if it is strictly 2-stable. ([S] Th.37.18)

3.3. Let $\{X_t\}$ be a Cauchy process on $\mathbb{R}$, that is, the Lévy process corresponding to a Cauchy distribution on $\mathbb{R}$. Let $\{Y_t\}$ be a Lévy process on $\mathbb{R}$ such that $\{X_t\}$ and $\{Y_t\}$ are independent and $E|Y_t| < \infty$ for $t > 0$. Show that $\{X_t + Y_t\}$ is recurrent. ([S] E39.8)

A.4. **Distributional properties.** It is important to study the properties of the distribution of a Lévy process at a fixed time in relation to the properties of its generating triplet. In the case of moments, we can establish the following relationship.

Definition. Let $g(x)$ be a nonnegative measurable function on $\mathbb{R}^d$. We call $\int g(x)\mu(dx)$ the *g-moment* of a measure μ on $\mathbb{R}^d$. We call $E[g(X)]$ the *g-moment* of a random variable X on $\mathbb{R}^d$.

Definition. A function $g(x)$ on $\mathbb{R}^d$ is called *submultiplicative* if it is nonnegative and if there is a constant $a > 0$ such that

$$g(x+y) \leq a g(x) g(y) \quad \text{for } x, y \in \mathbb{R}^d.$$

Theorem A4.1 (Kruglov). ([S] Th.25.3) *Let g be submultiplicative, measurable, and bounded on every compact set in $\mathbb{R}^d$. Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ with generating triplet (A, ν, γ). Then, $\{X_t\}$ has a finite g-moment for every $t > 0$ or, equivalently, for some $t > 0$, if and only if $1_{\{|x|>1\}}\nu$ has finite g-moment.*

This theorem has a wide applicability. For every $\alpha > 0$ and $0 < \beta \leq 1$, the following functions can serve as the function $g(x)$: $|x|^\alpha \vee 1$, $\exp(\alpha|x|^\beta)$, $[\log(|x| \vee e)]^\alpha$, $[\log\log(|x| \vee e^e)]^\alpha$, and also, denoting the j-th component of x by x_j, $|x_j|^\alpha \vee 1$, $(x_j \vee 1)^\alpha$, $\exp(\alpha|x_j|^\beta)$, $\exp(\alpha(x_j \vee 1)^\beta)$, and so on.

Example. Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ with generating triplet (A, ν, γ). Let ν be its Lévy measure. If $\int_{|x|>1} |x|\nu(dx) < \infty$, then $E|X_t| < \infty$ and $EX_t = t\left(\gamma + \int_{|x|>1} |x|\nu(dx)\right) = t\gamma_1$, where γ_1 is given in (A1.4). If $\int_{|x|>1} |x|\nu(dx) = \infty$, then $E|X_t| = \infty$ for $t > 0$. For $d = 1$, $E[0 \vee X_t] < \infty$ for $t > 0$ if and only if $\int_{(0,\infty)} x\nu(dx) < \infty$.

Example. Letting $g(x) = |x|^\eta$ with $\eta > 0$, we see from Theorem A1.6 that, if μ is α-stable on $\mathbb{R}^d$ with $0 < \alpha < 2$, then $\int |x|^\eta \mu(dx)$ is finite for $0 < \eta < \alpha$ and infinite for $\eta \geq \alpha$.

If g is submultiplicative and bounded on every compact set, then $g(x) \leq be^{c|x|}$ with some $b > 0$ and $c > 0$. Theorem A4.1 does not apply to functions which increase more rapidly than the exponential function.

But, in the case of $g(x) = e^{\alpha|x|\log|x|}$, the g-moment is connected to the size of the support of ν.

Theorem A4.2 (Sato)**.** ([S] Th.26.1) *Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ with generating triplet (A, ν, γ). Let*

$$c = \inf\{a > 0 \colon \nu(\{|x| > a\}) = 0\}$$

if the support of ν is bounded, and let $c = \infty$ if the support of ν is unbounded. Then $E[e^{\alpha|X_t|\log|X_t|}]$ is finite for $t > 0$ if $0 < \alpha < 1/c$; it is infinite for $t > 0$ if $\alpha > 1/c$. Here we understand $1/\infty = 0$ and $1/0 = \infty$.

The case of exponential moments is worth to be mentioned.

Theorem A4.3. ([S] Th.25.17) *Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ with generating triplet (A, ν, γ). Let*

$$C = \left\{ c \in \mathbb{R}^d \colon \int_{|x|>1} e^{\langle c,x\rangle} \nu(dx) < \infty \right\}.$$

(i) *The set C is a convex set containing the origin.*

(ii) *$Ee^{\langle c,X_t\rangle} < \infty$ for some $t > 0$ or, equivalently, for every $t > 0$, if and only if $c \in C$.*

(iii) *If $w \in \mathbb{C}^d$ is such that*[12] $\operatorname{Re} w \in C$, *then $E|e^{\langle w,X_t\rangle}| < \infty$ and*

$$E[e^{\langle w,X_t\rangle}] = e^{t\Psi(w)}$$

for $t > 0$, where $\Psi(w)$ is defined by (A1.5)*.*

Next let us consider continuity properties.

Definition. Let μ be a measure on $\mathbb{R}^d$. It is *discrete* if there is a countable set C such that $\mu(\mathbb{R}^d \setminus C) = 0$. It is *continuous* if $\mu(\{x\}) = 0$ for every $x \in \mathbb{R}^d$. It is *absolutely continuous* or *singular* if it is so with respect to the Lebesgue measure on $\mathbb{R}^d$.

Theorem A4.4. ([S] Cor.27.5) *Let μ be infinitely divisible on $\mathbb{R}^d$ with generating triplet (A, ν, γ). Then μ is discrete if and only if $A = 0$, $\nu(\mathbb{R}^d) < \infty$, and ν is discrete.*

---

[12]For $w = (w_j)_{1\le j\le d} \in \mathbb{C}^d$, $\operatorname{Re} w$ is defined to be the vector $(\operatorname{Re} w_j)_{1\le j\le d}$ in $\mathbb{R}^d$.

Theorem A4.5 (Dœblin). ([S] Th.27.4) *Let μ be infinitely divisible on $\mathbb{R}^d$ with generating triplet (A, ν, γ). Then μ is continuous if and only if $A \neq 0$ or $\nu(\mathbb{R}^d) = \infty$.*

Special classes of infinitely divisible distributions are studied.

Theorem A4.6 (Sato). ([S] Th.27.13) *Any nondegenerate selfdecomposable distribution on $\mathbb{R}^d$ is absolutely continuous.*

It is easy to see that, if $\nu(\mathbb{R}^d) = \infty$ and ν is absolutely continuous, then μ is absolutely continuous. But the Lévy measure of a nondegenerate selfdecomposable distribution on $\mathbb{R}^d$ can be singular in the case $d \geq 2$. A necessary and sufficient condition for absolute continuity of an infinitely divisible distribution is not known.

Are there continuous singular infinitely divisible distributions on $\mathbb{R}$? The answer is yes. Theorem A4.8 below gives examples.

Definition. A probability measure μ on $\mathbb{R}^d$ is *semi-selfdecomposable* if there are a real number $b > 1$ and an infinitely divisible probability measure ρ such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}(z)$. This b is called a *span* of μ.

If μ is semi-selfdecomposable, then μ is infinitely divisible and b and ρ determine μ. In this case the Lévy process corresponding to μ is called a *semi-selfdecomposable process.*

Theorem A4.7 (Wolfe). ([S] Th.27.15) *If μ is nontrivial and semi-selfdecomposable on $\mathbb{R}^d$, then μ is either absolutely continuous or continuous and singular.*

Theorem A4.8 (Watanabe[13]). ([S] Th.27.19) *Let b be an integer ≥ 2. Let $\{X_t\}$ be a nontrivial Lévy process on $\mathbb{R}$ with $A = 0$ and Lévy measure*

$$\nu = \sum_{n=-\infty}^{\infty} k_n \delta_{b^n} + \sum_{n=-\infty}^{\infty} l_n \delta_{-b^n}$$

satisfying $k_n \geq 0$, $l_n \geq 0$, $k_n \geq k_{n+1}$, $l_n \geq l_{n+1}$, $\sup_n k_n < \infty$, and $\sup_n l_n < \infty$. Then $\mathcal{L}(X_t)$ is continuous and singular for every $t > 0$.

[13][496] by Toshiro Watanabe is to appear in Probab. Theory Related Fields.

Theorem A4.9 (Watanabe[13]). ([S] Rem.27.22) *For some choice of a real number $b > 1$ we can find a semi-selfdecomposable process on $\mathbb{R}$ with span b with $A = 0$ and*

$$\nu = \sum_{n=-\infty}^{\infty} k_n \delta_{b^n}$$

such that, for some finite positive t_0, $\mathcal{L}(X_t)$ is continuous and singular for $0 < t < t_0$ and absolutely continuous for $t > t_0$.

This is a remarkable time evolution in a distributional property of a Lévy process. Other examples exhibiting this kind of time evolution are given by Rubin [384], Tucker [478], and Sato [403].

Let us consider unimodality.

Definition. ([S] Def.23.2) A measure ρ on $\mathbb{R}$ is *unimodal with mode* a if $\mu = c\delta_a + f(x)dx$ with $c \geq 0$ and $f(x)$ increasing on $(-\infty, a)$ and decreasing on (a, ∞).

Theorem A4.10 (Wolfe and Medgyessy). ([S] Th.54.1, 54.2) *Let $\{X_t\}$ be a Lévy process on $\mathbb{R}$ with generating triplet (A, ν, γ) and $\mathcal{L}(X_t) = \mu^t$. Consider the following conditions:*

(1) *For every $t > 0$, μ^t is unimodal.*
(2) *There are $t_n > 0$, $n = 1, 2, \ldots$, such that $t_n \to 0$ and μ^{t_n} is unimodal for each n.*
(3) *The Lévy measure ν is unimodal with mode 0.*

In general, (2) implies (3). In the case where $\{X_t\}$ is symmetric, the conditions (1), (2), and (3) are equivalent.

Here is a nice sufficient condition for unimodality.

Theorem A4.11 (Yamazato). ([S] Th.53.1) *Let $\{X_t\}$ be a selfdecomposable process on $\mathbb{R}$. Then, for every $t > 0$, $\mathcal{L}(X_t)$ is unimodal.*

Corollary. *Stable distributions on $\mathbb{R}$ are unimodal.*

Let us consider another class.

Theorem A4.12 (Goldie). ([S] Th.51.6) *Let μ be a probability measure on $\mathbb{R}$ expressible as*

$$\mu(B) = \int_{(0,\infty]} \mu_\theta(B)\rho(d\theta), \qquad B \in \mathcal{B}_{\mathbb{R}},$$

with a probability measure ρ on $(0,\infty]$, where μ_θ is exponential with parameter θ for $0 < \theta < \infty$ and $\mu_\infty = \delta_0$. Then μ is infinitely divisible.

The probability measure μ above is called a *mixture* of $\{\mu_\theta\}$. Mixtures of infinitely divisible distributions are not always infinitely divisible (see Exercise 4.2).

Theorem A4.13 (Watanabe[14]). ([S] Rem.54.19) *Let n be an integer ≥ 2. Let*

$$\mu(dx) = 1_{(0,\infty)}(x) \sum_{j=1}^{n} q_j a_j e^{-a_j x} dx,$$

where $q_j > 0$, $\sum_{j=1}^n q_j = 1$, and $a_1, \dots, a_n$ are distinct positive reals. Let $\{X_t\}$ be the Lévy process with $\mathcal{L}(X_1) = \mu$. Then there are t_0, t_1 with $1 \leq t_0 < t_1 < \infty$ such that

(1) *$\mathcal{L}(X_t)$ is unimodal with mode 0 for $0 \leq t \leq t_0$;*
(2) *$\mathcal{L}(X_t)$ is at most n-modal for $t_0 < t < t_1$;*
(3) *$\mathcal{L}(X_t)$ is unimodal with a positive mode for $t \geq t_1$.*

Further, if we choose $q_1, \dots, q_n, a_1, \dots, a_n$ appropriately, there is $t \in (t_0, t_1)$ such that $\mathcal{L}(X_t)$ is n-modal.

This is a nice example of evolution in unimodality and multimodality of a Lévy process. In the above we have used the word n-modal, which means, intuitively, that the density has n peaks. See [S] Def.52.1 for the precise definition of n-modality.

Exercises.

4.1. Let $\{X_t\}$ be a compound Poisson process on $\mathbb{R}$ with $c = 1$ and $\sigma(dx) = e^{-x} 1_{(0,\infty)} dx$. Show that $\mathcal{L}(X_t)$ is unimodal for $0 \leq t \leq 2$ and non-unimodal (actually, bimodal) for $t > 2$. ([S] Ex.23.4, Prop.54.12)

[14] [497] by Toshiro Watanabe has appeared in Japan. J. Math. **25** (1999), 227–256.

4.2. Let μ_a be Gaussian with mean a and variance 1 on $\mathbb{R}$. Let $\mu = (1/2)\mu_0 + (1/2)\mu_a$ with $a \neq 0$. Show that μ is not infinitely divisible. *Hint.* Use [S] Lem.7.5.

Appendix B. Corrections of the book *Lévy Processes and Infinitely Divisible Distributions*

April 30, 2000

p. 11, line 17: read e for e

p. 15, line 6: read parameter for mean

p. 84, line -3: read and (14.18) give for of (14.18) gives

p. 105, line -5: read $E[\mathrm{e}^{\mathrm{i}\langle a_j z, Z_{u_j} - Z_{u_{j-1}}\rangle}]$ for $\mathrm{e}^{\mathrm{i}\langle a_j z, Z_{u_j} - Z_{u_{j-1}}\rangle}$

p. 120, line 13: read *1.6* for *(1.6)*

p. 164, line 6: read $Z_1 \le x, \ldots, Z_{j-1} \le x$, $Z_j > x$ for $X_1 \le x, \ldots, X_{j-1} \le x$, $X_j > x$

p. 190, line -5: read $|\widehat{\mu}(z)|$ for $|\widetilde{\mu}(z)|$

p. 202, line 3: read $Y'_{Z_2(t)}$ for $Y_{Z_2(t)}$

p. 221, line 12: delete *positive*

p. 236, line -5: insert

Newman [**324**] and

before Brockett

p. 236, line -4: delete $A = A^\sharp$ and

p. 240, insert

Fix $a > 0$.

before the first sentence of Lemma 35.5.

p. 240, line 3: read $\varepsilon > 0$ for $a > 0$

p. 253, line -2: read ≥ 0 for ≤ 0

p. 256, the last two lines should be

$$K^+ = \textstyle\int_{(2,\infty)} x \big(\int_{-x}^{-1} \nu(-\infty, y)\mathrm{d}y\big)^{-1} \nu(\mathrm{d}x),$$

$$K^- = \textstyle\int_{(-\infty,-2)} |x| \big(\int_1^{|x|} \nu(y, \infty)\mathrm{d}y\big)^{-1} \nu(\mathrm{d}x).$$

p. 287, line 10: read $\{X_s \colon 0 \le s < t\}$ for $\{X_t \colon 0 \le s < t\}$

p. 288, line -5: read $\liminf_{y\to x} f_n(y)$ for $\liminf_{y\to x} f_n(x)$

p. 307, line 11: read *Proof of* for *Proofof*

p. 313, line 16: Move the first word to left.

p. 327, line -16: read $1_{\mathbb{R}^d\setminus\{x\}}$ for $1_{\mathbb{R}\setminus\{x\}}$

p. 359, line 3: read e for e

p. 381, line 13: read $\log\log(1/u)$ for $\log\log(1/s)$

p. 381, line 14: read u for s

p. 384, line 12: the displayed identity should be

$$E[\mathrm{e}^{-uL^{-1}(1)-vM(L^{-1}(1))}]$$
$$= \exp\left[-c\exp\left[\int_0^\infty t^{-1}\mathrm{d}t\int_{(0,\infty)}(\mathrm{e}^{-t}-\mathrm{e}^{-ut-vx})\mu^t(\mathrm{d}x)\right]\right]$$

p. 439, line 13: read Use the result of [**337**], p. 159, for Use E 34.3

p. 456: [113] and [114] should be interchanged. Lines 25–28 should be as follows:

[113] Erdős, P. (1942) On the law of the iterated logarithm, *Ann. Math.* **43**, 419–436. *358*

[114] Erdős, P. and Révész, P. (1997) On the radius of the largest ball left empty by a Wiener process, *Stud. Sci. Math. Hungar.* **33**, 117–125. *368*

Consequently, citation of [**113**] and [**114**] in pp. 358, 368, 469 should be changed to [**114**] and [**113**], respectively.

p. 459, [175]: read (1972) for (1973)

p. 464, lines −5, −2 and p. 465, lines 2, 6, 10: read Gauthier for Gauthie

p. 466, [322]: read reversions for reversal

p. 466, [324]: read *236*$_3$ for *236*$_2$

p. 467, [337]: read *234,439* for *234*

p. 470, [398]: read *Probability* for *Probabability*

p. 474, [462]: read walks for waks

p. 474, [469]: read stationary for ststionary

p. 478, [534]: read 653–662 for 653–664

The author thanks the readers who were kind enough to inform him of errata in the book.

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