Vassiliev Theory

by P. Vogel[1]

Abstract.

There exists a natural filtration on the module freely generated by knots (or links). This filtration is called the Vassiliev filtration and has many nice properties. In particular every quotient of this filtration is finite dimensional. A knot invariant which vanishes on some module of this filtration is called a Vassiliev invariant. Almost every knot invariant defined in algebraic term can be describe in term of Vassiliev invariants. Unfortunately the structure of all such invariants is completely unknown. The Kontsevich integral is, in some sense, the universal Vassiliev invariant. It takes values in a module $\mathcal{A}$ of 3-valent diagrams. So a good way to construct a knot invariant is to compose the Kontsevich integral with a linear homomorphism defined on $\mathcal{A}$.

Every Lie algebra equipped with a nonsingular bilinear symmetric invariant form produces a linear homomorphism on $\mathcal{A}$ and therefore a knot invariant. If the Lie algebra belongs to the A series, the induced knot invariant is the HOMFLY polynomial. If the Lie algebra belongs to the B-C-D series, one gets the Kauffman polynomial. The Kauffman bracket is obtained by the Lie algebra sl_2.

The structure of $\mathcal{A}$ is more or less unknown. This module is actually a polynomial algebra, but the number d_n of generators in degree n is known only for $n < 13$. A Lie algebra L induces an algebra homomorphism from the Hopf algebra $\mathcal{A}$ to the center of the enveloping algebra of L. So, in some sense, there a universal algebra $\mathcal{L}$ over an algebra Λ such that $\mathcal{A}$ is the center of the enveloping algebra of $\mathcal{L}$. This Lie algebra is defined as a category satisfying some conditions. There is many conjectures about this universal Lie algebra and the coefficient algebra Λ. The decomposition of $\mathcal{L}^{\otimes p}$ in simple modules is given for $p \leq 3$.

1. VASSILIEV INVARIANTS

1.1 Knots and links invariants

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A link is a compact 1-dimensional smooth submanifold of $\mathbf{R}^3$. A connected link is called a knot. A link may be oriented or not. Every knot is the image of a embedding f from the circle S^1 into $\mathbf{R}^3$. For a link, the situation is similar but the embedding is defined on a disjoint union of finitely many copies of the circle.

1.2 Definitions. A link L is called banded if L is equipped with a vector field V from L to $\mathbf{R}^3$ such that $V(x)$ is transverse to L for every point $x \in L$.

A link L is called framed if it is oriented and banded.

Two links L_0 and L_1 are isotopic if there exists an isotopy h_t of the ambient space $\mathbf{R}^3$ such that h_0 is the identity and L_1 is the link $h_1(L_0)$. If the links are oriented we suppose also that L_1 has the same orientation as $h_1(L_0)$. If (L_0, V_0) and (L_1, V_1) are banded links, there are called isotopic if L_0 and L_1 are isotopic via an isotopy h_t in such a way that V_1 is homotopic to the vector field $h_1(V_0)$ by an homotopy which is always transverse to L_1. So we have four isotopy relations corresponding to the four classes of links: non oriented, oriented, banded or framed.

An invariant of knots (or links) is a function from the set of knots (or links) to some module which is invariant under isotopy. It's also possible to define an invariant of oriented knots (or links), or an invariant of banded knots (or links) or an invariant of framed knots (or links).

Every link can be described by its projection on the plane if it is generic. Such a projection is called a diagram of a link. A diagram of a link is a finite graph D contained in the plane such that every vertex is of order 4. Moreover near every vertex x two edges arriving at x correspond to the over branch and the two other ones correspond to the under branch. The edges corresponding to the over branch are represented by a connected path.

Let L a link represented by a diagram D. If L is oriented, the orientation is represented by an orientation of D. If it is banded it is possible to choose the diagram D in such a way that the transverse vector field is normal to the plan $\mathbf{R}^2$ with positive last coordinate.

Using this convention every diagram defines a banded link and every oriented diagram defines a framed link and these links are well defined up to isotopy.

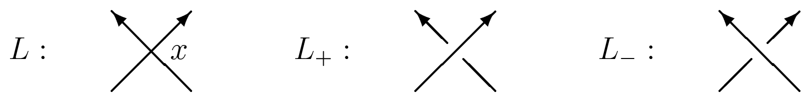
Suppose that L_0 and L_1 are two links related by a family L_t, $0 \leq t \leq 1$, of geometric objects. If every L_t is a link which depends smoothly on t (that is the union $L_{[0,1]}$ of all $L_t \times \{t\}$ is a submanifold of $\mathbf{R}^3 \times [0,1]$) the links L_0 and L_1 are isotopic. But it is possible to consider singular deformation when L_t becomes singular, for some particular values of t. The simplest example of such a singular deformation is when a branch of L crosses another one. When this crossing happens the link becomes a singular link in the following sense:

Definition. A singular link L is the image of an immersion f from a 1-dimensional compact manifold Γ to $\mathbf{R}^3$ such that f has only finitely many multiple points and every multiple point is double and transverse, together with local orientations in Γ near each singular point of f.

A singular link L is oriented if the source Γ of the immersion f is oriented and the local orientations are induced by the orientation of Γ. It is banded if L is equipped with a transverse vector field V such that, for every double point x of L, $V(x)$ is transverse to the plan which is tangent at x to the two branches of L containing x.

If D is a diagram of a link and P a subset of the set of vertices of D, one can associate to (D, P) a singular link L where the double points correspond to the points in P. With the same way as before, the diagram induces a well defined banded structure on L. If D is oriented, L is naturally framed.

Let L be a singular link and x be a double point in L. One can modify L a little bit near x and obtain a new singular link L' with one double point less. But it is possible to do that in two different ways, and one gets two new links L_+ and L_-.



Since L is supposed to be oriented near x, there is no ambiguity between L_+ and L_-.

If L is banded, the two desingularized links L_+ and L_- are still banded.

1.3 Lemma. *Let I be an invariant of oriented knots. Then I extends uniquely to an invariant defined on the set of all singular oriented knots and satisfying the following property:*

If K is a singular oriented knot and K_+ and K_- are the two knots obtained by desingularization near a double point in K, one has:

$$I(K) = I(K_+) - I(K_-)$$

The extension of I may be defined in the following way:

Let K be a singular oriented knot. Denote by X the set of double points in K and by $\mathcal{F}$ the set of functions from X to $\{\pm 1\}$. If α is a function in $\mathcal{F}$ one can desingularize K near every every double point in K by using the positive or the negative move near a point x if $\alpha(x) = 1$ or -1. So for every $\alpha \in \mathcal{F}$ one gets a knot K_α. Then one sets:

$$I(K) = \sum_{\alpha \in \mathcal{F}} \varepsilon(\alpha) I(K_\alpha)$$

where $\varepsilon(\alpha)$ is the product of all numbers $\alpha(x)$, $x \in X$.

1.4 Definition. Let I be an invariant of knots. One said that I is a Vassiliev invariant of degree $\leq n$ if I vanishes on every oriented singular knot with at least $n+1$ double points.

Remark. If I is an invariant of oriented links, or an invariant of knots (or links) or banded knots or links or framed knots or links, it is possible to extend I to the corresponding set of singular knots or links and one can define a Vassiliev invariant of knots (or links), or banded knots (or links) or framed knots (or links).

Example. Let L be a singular oriented link with only one double point x. One can modify L near x in three different ways:

These three links have no double point. The Conway polynomial ∇ is the only polynomial invariant of oriented links which is equal to 1 for the trivial knot and satisfies the following skein relation:

$$\nabla(L_+) - \nabla(L_-) = t\nabla(L_0)$$

For every oriented link L, $\nabla(L)$ is a polynomial in the ring $\mathbf{Z}[t]$.

1.5 Proposition. *The n^{th} coefficient of the polynomial ∇ is a Vassiliev invariant of degree n.*

Proof: The skein relation shows that $\nabla(L)$ is divisible by t^n if L is a singular link with at least n double points and the n^{th} coefficient a_n of the polynomial ∇ is an integral invariant of oriented links which vanishes on every singular links with at least $n+1$ double points. The result follows. □

If E is a module, denote by $\mathcal{I}(E)$ the set of invariants of knots with values in E. For every integer $n \geq 0$, denote by $V_n(E)$ the set of Vassiliev invariants of degree $\leq n$.

1.6 Proposition. *Let R be a ring. Then the R-modules $V_n(R)$ form an increasing family of finitely generated R-submodules of $\mathcal{I}(R)$:*

$$V_0(R) \subset V_1(R) \subset V_2(R) \subset \ldots \subset \mathcal{I}(R)$$

Moreover one has: $V_p(R)V_q(R) \subset V_{p+q}(R)$ for every $p, q \geq 0$.

Remark. This proposition is also true for the module of invariants of knots or banded knots or framed knots. For links it is also true but only for invariants of links with a fixed number of components.

In order to prove such a result one needs to study more precisely the set of singular links (or knots).

2. THE ALGEBRA OF KNOTS

Let $\mathcal{K}$ be the set of isotopy classes of oriented knots and $\mathbf{Z}[\mathcal{K}]$ be the free $\mathbf{Z}$-module generated by $\mathcal{K}$. The connected sum operation induces on $\mathcal{K}$ a structure of commutative monoid. With this structure $\mathbf{Z}[\mathcal{K}]$ becomes a commutative algebra. Moreover the map $K \mapsto K \otimes K$ induces a comultiplication Δ from $\mathbf{Z}[\mathcal{K}]$ to $\mathbf{Z}[\mathcal{K}] \otimes \mathbf{Z}[\mathcal{K}]$.

With these structures $\mathbf{Z}[\mathcal{K}]$ is a commutative and cocommutative Hopf algebra (but without antipode map).

The inclusion $\mathcal{K} \subset \mathbf{Z}[\mathcal{K}]$ is obviously an invariant of oriented knots. Therefore it extends to singular knots and every singular link may be see as an element of $\mathbf{Z}[\mathcal{K}]$. So, using previous notations, one has the following, for every singular oriented knot K:

$$K = \sum_{\alpha \in \mathcal{F}} \varepsilon(\alpha) I(K_\alpha)$$

Let's denote by I_n the submodule of $\mathbf{Z}[\mathcal{K}]$ generated by singular knots with at least n double points. With this notation an invariant I of oriented knots is a Vassiliev invariant of degree $\leq n$ if it vanishes on I_{n+1} and the module $V_n(R)$ is isomorphic to the R-module $\mathrm{Hom}(\mathbf{Z}[\mathcal{K}]/I_{n+1}, R)$.

2.1 Proposition. *The submodules I_n form a filtration of $\mathbf{Z}[\mathcal{K}]$ which is compatible with the Hopf algebra structure. The graded associated algebra $GA = \oplus I_n/I_{n+1}$ is a connected graded Hopf algebra.*

2.2 Proposition. *The Hopf algebra GA is finitely generated in each degree and $GA \otimes \mathbf{Q}$ is a polynomial algebra.*

Proof: We have to check the following conditions:

$$\mathbf{Z}[\mathcal{K}] = I_0 \supset I_1 \supset I_2 \supset \dots$$

$$\forall p, q \qquad I_p I_q \subset I_{p+q}$$

$$\forall n \qquad \Delta(I_n) \subset \bigoplus_{p+q=n} I_p \otimes I_q$$

The first property is obvious. The connected sum operation extends linearly to connected sum for singular links and the second property follows. The third one is a consequence of the following formula:

$$\Delta(K) = \sum K_\alpha \otimes K_{\alpha-1}$$

where K is any singular oriented knot, and α run in the set of all functions from the set X of double points of K to $\{0, 1\}$. If β is a function from X to $\{-1, 0, 1\}$, K_β denote the singular knot obtained from K by a positive (resp. negative) modification near every double point x where β is equal to 1 (resp. -1).

Then GA is a graded Hopf algebra which is connected, commutative and cocommutative. Therefore GA is rationally a symmetric algebra over the module of primitive elements (i.e. elements x satisfying $\Delta(x) = 1 \otimes x + x \otimes 1$). If $\{x_i\}$ is a homogeneous basis of the module of primitive elements, $GA \otimes \mathbf{Q}$ is the polynomial algebra $\mathbf{Q}[\{x_i\}]$.

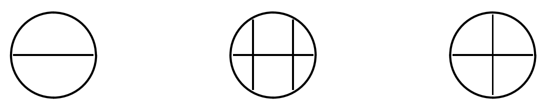
The last thing to do is to prove that I_0/I_n is finitely generated for every n. To do that, it is enough to show that I_n/I_{n+1} is finitely generated for every $n \geq 0$. This point will be proven in the next section.

Remark. If one consider the case of unoriented knots, the connected sum operation is no longer defined. We still have a Vassiliev filtration but the corresponding graded module is only a cocommutative coalgebra. The same hold for other classes of knots or links. Nevertheless, in the case of links (unoriented, oriented, banded or framed), one can use the disjoint union operation. In these cases the graded associated modules are Hopf algebras.

3. CHORD DIAGRAMS AND 3-VALENT DIAGRAMS

Definition. A *chord diagram* is a collection of disjoint pairs of points in the standard circle S^1. This structure is defined up to isotopy in the circle. Such a pair $\{a, b\}$ is called a chord. It is represented by the chord $[a, b]$ in the plan. In other words a chord diagram is a picture in the plan represented by the standard circle and finitely many chords with disjoint boundaries.

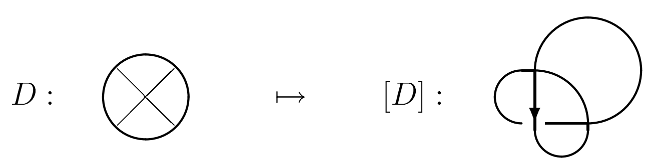
Notice that the set $\mathcal{D}_n^c$ of chord diagrams with exactly n chords is finite.



Let D be a chord diagram represented by n chords $c_i = [a_i, b_i]$, $i = 1, \dots, n$ in the circle S^1. Then there exists a unique immersion f (unique up to regular homotopy) such that the image of f is a singular knots having exactly n double points: the points $f(a_i) = f(b_i)$, $i = 1, \dots, n$.

If f and g are two such immersions, they are regularly homotopic. Then there exist p such immersions h_j such that $f = h_1$, $g = h_p$ and h_{j+1} is obtained from h_j by making a crossing change somewhere. Let K_j be the image of h_j. Then K_j and K_{j+1} are obtained from a singular knot K'_j with $n + 1$ double points by the positive and negative moves near some double point of K'_j. Therefore the difference $K_{j+1} - K_j$ belongs to I_{n+1} and the two singular knots corresponding to f and g induce the same element in I_n/I_{n+1}.

Hence every chord diagram $D \in \mathcal{D}_n^c$ induces a well defined element $[D]$ in $I_n/I_{n+1} = GA_n$.



Since I_n is generated by singular knots with exactly n double points, this correspondence induces a surjective homomorphism from $\mathbf{Z}[\mathcal{D}_n^c]$ onto I_n/I_{n+1}. Therefore the Hopf algebra GA is finitely generated in each degree.

Remark. For unoriented knots the situation is different. In order to consider a singular knot as an element of $\mathbf{Z}[\mathcal{K}]$, one need to consider local orientations near each double point. So a chord diagram in this context is a manifold Γ diffeomorphic to a circle equipped with chords and local orientations near each endpoint of a chord.

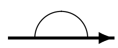
Moreover if one changes a local orientation in some place, the induced element in I_n/I_{n+1} is multiply by -1.

For banded knots there is another problem. If a chord diagram D is given, it is not possible to define naturally a banded structure on the singular knot constructed from D. The singular knot can be represented by a diagram Δ contained in the plan. This diagram is oriented and has n double points, p positive crossings and q negative crossings. If one positive crossing is replaced by a negative crossing, the difference of the two corresponding singular knots belongs to I_{n+1}. Therefore the class of the singular knot depends only on the class of $p-q$ mod 2.

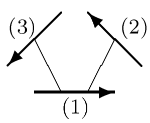
Actually there is two functions $D \mapsto [D]_0$ and $D \mapsto [D_1]$. The first one corresponds to the case $p-q$ even and the other one to the case $p-q$ odd. In the case of banded knots (or framed knots) one has a surjection from $\mathbf{Z}[\mathcal{D}_n^c] \oplus \mathbf{Z}[\mathcal{D}_n^c]$ onto I_n/I_{n+1}.

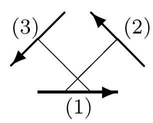
3.2 Proposition. *The morphism $D \mapsto [D]$ satisfy the following relations:*

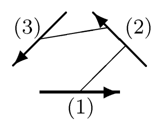
— If D contains an isolated chord, then: $[D] = 0$.

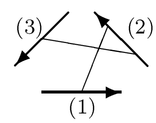
(1T) $D =$  $\Longrightarrow \quad [D] = 0$

— Let D_1, D_2, D_3 and D_4 be four chord diagrams which differ only in three parts of the circle and two chords and which have the following form near these chords:

D_1 : 

D_2 : 

D_3 : 

D_4 : 

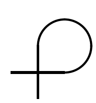
then one has the following:

(4T) $$[D_1] - [D_2] = [D_3] - [D_4]$$

Remark. The relation (1T) is called the 1T (one term) relation. It holds for oriented or unoriented knots but not for framed knots or banded knots. The relation (4T) is called the 4T (four terms) relation. It holds for every class of knots or links.

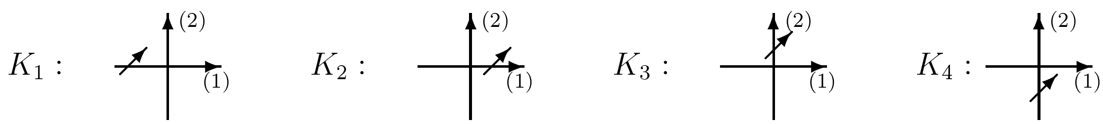
In the case of banded or framed knots, the 4T relation holds for both functions $[?]_0$ and $[?]_1$.

Proof: If D is a chord diagram with one isolated chord, the corresponding singular knot K may be chosen to be represented by the following diagram:

K : 

Therefore the knots K_+ and K_- are the same and $[K]$ is the zero element. In the case of banded or framed knots K_+ and K_- don't have the same banded structure or the same framing and the property 1T doesn't work.

Let K_1 be a singular knot corresponding to the diagram D_1. The diagrams D_2, D_3 and D_4 may be represented by singular knots K_2, K_3 and K_4 differing from K_1 only near parts (1), (2) and (3). Near this region the knots K_i look like the following:



In order to compute the class of each K_i in I_n, it is possible to modify K_i near the part (3). Denote by P the plan generated by parts (1) and (2) of the knot. This plan is cut into four pieces: the upper-right piece P_1, the upper-left piece P_2, the under-left piece P_3 and the under-right piece P_4.

If (3) doesn't cross parts (1) and (2), it crosses P in some P_i and defines a singular knot K_i'. With this conventions one has:

$$K_1 = K_3' - K_2' \qquad K_2 = K_4' - K_1' \qquad K_3 = K_1' - K_2' \qquad K_4 = K_4' - K_3'$$

and the 4T relation follows. □

This notion of chord diagram is generalizable in the following way: Let Γ be a compact one-dimensional manifold. A n-chord diagram on Γ is a finite collection of n disjoint pairs of points (or chords) in the interior of Γ and a orientation of Γ near each point in this collection. The set of n-chord diagram on Γ will be denoted by $\mathcal{D}_n^c(\Gamma)$.

If Γ has no boundary, a n-chord diagram D on Γ induces a singular link $[D]$ which is oriented near each double point. Moreover $[D]$ is well defined modulo the submodule I_{n+1} generated by singular links with $n+1$ double points. If Γ has a boundary the same construction works but one has to consider embeddings of $(\Gamma, \partial\Gamma)$ in the pair $(B^3, \partial B^3)$ instead of links.

In all this cases the 4T relation holds.

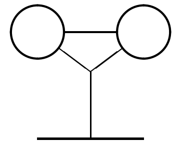
3.3 Definition. Let Γ be a curve (i.e. a compact one-dimensional manifold). A Γ-diagram is a triple (K, f, α) where:

— K is a finite graph and f is a homeomorphism from Γ to a sub-graph of K such that every point in $f(\partial\Gamma)$ is a univalent vertex of K and every other vertex of K has valency 3

— α is a function which associates to every 3-valent vertex x in K a cyclic ordering $\alpha(x)$ between the three edges of K starting from x.

Usually a Γ-diagram will be represented by a graph K immersed in the plane and containing Γ. The cyclic orderings are given by the orientation of the plane.

Such a diagram is also called 3-valent diagram in the literature.



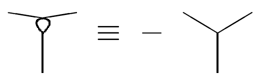
A chord diagram K is actually a particular S^1-diagram. The graph is the union of the circle and the chords. The vertices of K are the endpoints of the chords. If x is such a vertex, on has a natural cyclic ordering: first the chord, after the arc of the circle arriving at x and then the arc of the circle starting from x.

In the same way a n-cord diagram on a curve Γ is a particular Γ-diagram and we have a map from $\mathcal{D}_n^c(\Gamma)$ to the set of Γ-diagrams.

3.4 The module $\mathcal{A}(\Gamma)$. Let Γ be a compact one-dimensional manifold. Let k be a field of characteristic zero. The module $\mathcal{A}(\Gamma)$ is the k-module given by the following presentation:

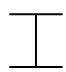
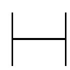
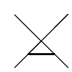
The generators are the Γ-diagrams. The relations are the following:

— If a Γ-diagram K is obtained from a Γ-diagram K' by changing a cyclic ordering in one place, then: $K \equiv -K'$.

(AS) 

— the IHX relation (also called Jacobi relation):

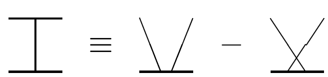
If three Γ-diagrams K, K' and K'' differ only near an edge in the following way:

K :  K' :  K'' : 

one has:

(IHX) $$K \equiv K' - K''$$

In the IHX relation, the edge is not necessary outside of Γ. If not the relation takes the following form:

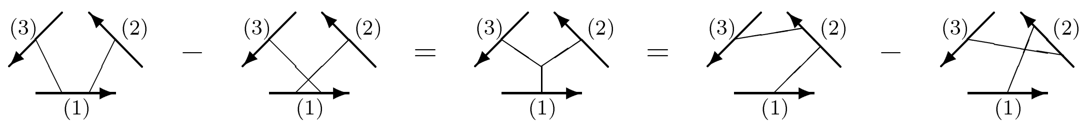
(STU) 

This relation is called the (STU) relation.

Remark. The module $\mathcal{A}(\Gamma)$ is a graded module. The degree of a Γ-diagram K is half the number of three-valent vertices of K.

3.5 Proposition. *Let Γ be a curve. Let $\mathcal{A}_c(\Gamma)$ be the submodule of $\mathcal{A}(\Gamma)$ spanned by Γ-diagrams K such that each component of K meets Γ. Then the natural map from $\mathcal{D}_n^c(\Gamma)$ to the set of Γ-diagrams induces an isomorphism from the quotient module $K[\mathcal{D}_n^c(\Gamma)]/(4T)$ to the degree n part $\mathcal{A}_c(\Gamma)_n$ of the module $\mathcal{A}_c(\Gamma)$.*

This result is proven by Bar-Natan [BN] in the case $\Gamma = S^1$. The general case can be done exactly in the same way. The fact that the 4T relation holds in $\mathcal{A}_c(\Gamma)$ is easy to check:



3.6 Algebraic properties of $\mathcal{A}$.

3.7 Proposition. *An inclusion i from a curve Γ to a curve Γ' induces a homomorphism i_* of graded modules from $\mathcal{A}(\Gamma)$ to $\mathcal{A}(\Gamma')$.*

If Γ and Γ' are two curves the disjoint union operation induces a homomorphism from $\mathcal{A}(\Gamma) \otimes \mathcal{A}(\Gamma')$ to $\mathcal{A}(\Gamma \coprod \Gamma')$.

If f is a continuous map from a curve Γ to a curve Γ' sending boundary to boundary, f induces a well define homomorphism f^* from $\mathcal{A}(\Gamma')$ to $\mathcal{A}(\Gamma)$. Moreover if f and g are two homotopic maps from $(\Gamma, \partial\Gamma)$ to $(\Gamma', \partial\Gamma')$, the homomorphisms f^* and g^* are equal.

Sketch of proof: The first homomorphism send a Γ-diagram K to the union $K \cup \Gamma'$. It is easy to see that AS and IHX relations are satisfied and this homomorphism is well defined.

The second homomorphism send $K \otimes K'$ to the disjoint union $K \coprod K'$.

The last homomorphism is more complicated to define. First suppose that f is smooth and has only finitely many critical points and distinct corresponding critical values. Let K be a Γ'-diagram. Let $\{x_i\}$ be the set of vertices of K contained in Γ' and H be the closure in K of the complement $K - \Gamma'$. Suppose also that every x_i is a regular value of f. The space H is a finite graph and the set of univalent vertices of H is exactly the set $\{x_i\}$. The diagram K is the union (over $\{x_i\}$) of Γ' and H. Moreover, for each vertex x_i, the cyclic ordering near x_i induces an orientation ω_i of a neighborough of x_i in Γ. Let S be the set of functions s from $\{x_i\}$ to Γ such that $s(x_i) \in f^{-1}(x_i)$ for every x_i. This set S is finite because f is a finite covering over a neighborough of $\{x_i\}$.

For every $s \in S$ the union of Γ and H, where each $x_i \in H$ is identify with the point $s(x_i) \in \Gamma$ is a finite graph K_s. Since f is étale near every point $s(x_i)$, the local orientations ω_i induces local orientations ω_i' near each point $s(x_i)$. The graph K_s equipped with these local orientations induces a Γ-diagram still denoted by K_s. Then one defines $f^*(K)$ as the sum in $\mathcal{A}(\Gamma)$ of all K_s.

Suppose now that f is any continuous map from $(\Gamma, \partial\Gamma)$ to $(\Gamma', \partial\Gamma')$. This function is homotopic to a smooth function f' such that every point x_i is a regular value of f'. Then we set: $f^*(K) = f'^*(K)$. If f is homotopic to another function f'' satisfying the same property, f' is smoothly homotopic to f'' by a homotopy h_t such that every h_t is smooth and only finitely many h_t has some x_i as critical value. One mays also suppose that each of these critical functions h_t has only one x_i as critical value corresponding to only one non degenerate critical point. The element $h_t^*(K)$ is well defined for every non critical function h_t. It is easy to see that this function is locally constant in t and the AS relation show that each jump of this function is trivial. Therefore this function is constant and $f'^*(K)$ is equal to $f''^*(K)$. Consequently $f'^*(K)$ depends only on K and the homotopy class of f.

In order to prove that f^* is compatible with AS and IHX relation it is enough to consider the case where f is smooth and has no critical value where the geometry of the diagrams is modified by the relation. In this case f^* sends an AS relation to a sum of AS relation and an IHX relation to a sum of IHX relations. □

Corollary. *For every curve* Γ, $\mathcal{A}_c(\Gamma)$ *is a graded cocommutative coalgebra.*

Proof: Let f be the first projection from $\Gamma \times \{0,1\}$ to Γ. The homomorphism f^* sends $\mathcal{A}_c(\Gamma)$ to $\mathcal{A}_c(\Gamma \times \{0,1\})$. Consider the module $\mathcal{A}_c(\Gamma \times \{0,1\})$ quotiented by the submodule M spanned by all diagrams K for which some component meets $\Gamma \times \{0\}$ and $\Gamma \times \{1\}$. We have a natural map:

$$\mathcal{A}_c(\Gamma \times \{0\}) \otimes \mathcal{A}_c(\Gamma \times \{1\}) \longrightarrow \mathcal{A}_c(\Gamma \times \{0,1\})/M$$

which is obviously an isomorphism. Then the homomorphism f^* composed with the quotient map induces a homomorphism Δ from $\mathcal{A}_c(\Gamma)$ to $\mathcal{A}_c(\Gamma) \otimes \mathcal{A}_c(\Gamma)$. It is not difficult to see that $\mathcal{A}_c(\Gamma)$ equipped with this homomorphism Δ is a graded cocommutative coalgebra. □

3.8 Proposition. *The graded modules* $\mathcal{A}_c([0,1])$ *and* $\mathcal{A}_c(S^1)$ *are isomorphic graded commutative and cocommutative Hopf algebras.*

Proof: Let f be an embedding from $[0,1]$ to the circle S^1 which is compatible with the orientations. This injection induces a homomorphism φ from $\mathcal{A}_c([0,1])$ to $\mathcal{A}_c(S^1)$. This homomorphism depends only on the isotopy class of f and then it is well defined. It is easy to see that φ is an epimorphism of coalgebra. The fact that φ is an isomorphism is proven in [BN1]. The algebra structure on $\mathcal{A}_c([0,1])$ comes from the embedding from $[0,1] \coprod [0,1]$ to $[0,1]$ which is an increasing bijection from the first copy of $[0,1]$ to $[0,1/3]$ and an increasing bijection from the second copy of $[0,1]$ to $[2/3,1]$.

The fact that the product is commutative can be see in the circle. If u and v are represented by $[0,1]$-diagrams K and K', uv is represented by the diagram L obtained by placing K' after K on the interval. But in the circle, one can move K' around the circle and uv is equal to vu in $\mathcal{A}_c(S^1)$. Since the inclusion map from $\mathcal{A}_c([0,1])$ to $\mathcal{A}_c(S^1)$ is bijective, one has: $uv = vu$. □

3.9 Proposition. *Let* $n > 0$ *be an integer. Then* $\mathcal{P}_n = \mathcal{A}_c([0,1] \times \{1,\ldots,n\})$ *is a cocommutative Hopf algebra.*

Proof: The coalgebra structure of $\mathcal{P}_n$ is already defined. The product is defined as follows: Denote by X the set $\{1,\ldots,n\}$ and consider a map i from $[0,1] \times \{0,1\}$ to

$[0,1]$ which is increasing with respect to the parameter in $[0,1]$ and send $[0,1]\times\{0\}$ to $[0,1/2)$ and $[0,1]\times\{1\}$ to $(1/2,1]$. Then the product map is the composite:

$$\mathcal{A}_c([0,1]\times X)\otimes\mathcal{A}_c([0,1]\times X)\to\mathcal{A}_c([0,1]\times\{0,1\}\times X)\xrightarrow{i_*}\mathcal{A}_c([0,1]\times X)$$

The antipode map is constructed by induction on the degree. □

Remark. The Hopf algebra $\mathcal{P}_1$ is isomorphic to $\mathcal{A}_c([0,1])=\mathcal{A}_c(S^1)$ and is commutative. For $n>1$ $\mathcal{P}_n$ is never commutative.

4. THE CATEGORY OF DIAGRAMS

A good way to understand knots or links is to cut such a link by horizontal planes. So one gets one-dimensional submanifolfs of $\mathbf{R}^2\times[0,1]$ with boundary in $\mathbf{R}^2\times\{0,1\}$. These objects are called tangles. The tangles form a category $\mathcal{T}$. An object in the category $\mathcal{T}$ is a finite subset X in the plane $\mathbf{R}^2$. A morphism from X to Y is an isotopy class of tangle in $\mathbf{R}^2\times[0,1]$ with boundary $X\times\{0\}\cup Y\times\{1\}$.

One may also consider the category of oriented tangles, banded tangles or framed tangles. In view on the Kontsevich integral it will be also convenient to consider categories of parenthesized tangles (or non associative tangles). In this category the morphisms are the same but the objects are more complicated. They are points written in a non-associative way in $\mathbf{R}\subset\mathbf{R}^2$.

The same consideration holds for diagrams.

Definition. Let Γ be a curve (i.e. a compact one-dimensional manifold). Let X be a finite set. A (Γ,X)-diagram is a triple (K,f,g,α) where:

— K is a finite graph and g is an injection from X to the set of univalent vertices of K

— f is a homeomorphism from Γ to a sub-graph of K such that the set of univalent vertices of K is the disjoint union of $g(X)$ and $f(\partial\Gamma)$

— every vertex in K is univalent or three-valent

— α is a function which associates to every 3-valent vertex x in K a cyclic ordering $\alpha(x)$ between the three edges of K starting from x.

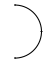
A $(\Gamma,\emptyset)$-diagram is nothing else but a Γ-diagram. Usually a (Γ,X)-diagram will be represented by a graph immersed in the plan and containing Γ and X, and the cyclic orderings are induced by the orientation of the plan.

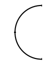
The AS and IHX relations make sense for (Γ,X)-diagrams. So one can define the quotient:

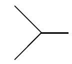
The graded module $\mathcal{A}(\Gamma,X)$. The module $\mathcal{A}(\Gamma,X)$ is the module freely generated by all (Γ,X)-diagram and quotiented by the AS and IHX relations. The degree of an element of this module which is represented by a diagram K is half the number of 3-valent vertices of K. This degree is half an integer. Twice this degree is congruent to the order of X modulo 2.

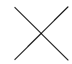
4.1 The category of diagrams $\mathcal{D}$. The objects of $\mathcal{D}$ are the finite sets $[n] = \{1, \ldots, n\}$, $n \geq 0$. A morphism from an object X and an object Y is an element of the module $\mathcal{A}(\emptyset, X \coprod Y)$. The composite of a morphism from X to Y and a morphism from Y to Z is obtained by taking the union of diagrams over Y.

4.2 Proposition. *The category $\mathcal{D}$ is a monoidal linear category. As a monoidal category it is generated by one object (the object $[1]$) and four morphisms:*

— *the morphism d_1 from $[2]$ to $[0]$ represented by* 

— *the morphism d_2 from $[0]$ to $[2]$ represented by* 

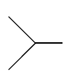
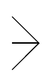
— *the morphism d_3 from $[2]$ to $[1]$ represented by* 

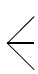
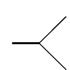
— *the morphism d_4 from $[2]$ to $[2]$ represented by* 

Proof: The monoidal structure is given by the disjoint union. The tensor product of two objects $[p]$ and $[q]$ is the object $[p+q]$, and the first p points in $[p+q]$ correspond in the standard way to the points in $[p]$ and the last q points in $[p+q]$ to the points in $[q]$. The tensor product of two morphisms u and v represented by two diagrams K and K' is the morphism represented by $K \coprod K'$.

It is easy to see that the category $\mathcal{D}$ is a monoidal category and the monoidal structure is strictly associative. Moreover for every objects X and Y the set $\mathrm{Hom}_{\mathcal{D}}(X, Y) = \mathcal{A}(\emptyset, X \coprod Y)$ is a module and the composition and the tensor product are both bilinear.

The object $[p]$ is the tensor product of p copies of $[1]$. A morphism between $[p]$ and $[q]$ is a linear combination of morphisms corresponding to diagrams. Consider now such a diagram K. This diagram is a finite graph containing $[p] \coprod [q]$. There exist a subdivision K' of K and a function f from K to $[0,1] \times \mathbf{R}$, which is affine in every edge of K' and sends every point $i \in [p]$ to $(0, i)$ and every point $j \in [q]$ to $(1, j)$. Let X be the set of double points of f and Y be the image under f of the set of vertices of K'. If f is chosen to be generic enough, the image of $X \cup Y$ under the first projection consists of distinct points. Then one can cut $[0,1] \times \mathbf{R}$ into pieces $U_i = [a_i, a_{i+1}] \times \mathbf{R}$, with: $0 = a_0 < a_1 < \ldots < a_n = 1$, and $X \cup Y$ meets U_i in at most one point. Then the morphism induced by K is the composite of n elementary morphisms corresponding to the diagram $K_i = K \cap U_i$. By construction each such morphism is on the form $\mathrm{Id} \otimes u \otimes \mathrm{Id}$ where u is the morphism d_4 or a morphism corresponding to a connected graph H with at most one 3-valent vertex. If H has no 3-valent vertex u is 1 or d_1 or d_2. If H has one 3-valent vertex, we have four possibilities for H, but each of these may be express as composite of morphisms d_1, d_2 and d_3. Precisely the four possibilities are:

 $= d_3$  $= d_1 \circ (1 \otimes d_3)$

 $= (1 \otimes d_3 \otimes 1) \circ (1 \otimes 1 \otimes d_2) \circ d_2$  $= (d_3 \otimes 1) \circ (1 \otimes d_2)$

Therefore the four morphisms d_i generate the full category. □

4.3 Remark. Actually it is possible to describe the monoidal category $\mathcal{D}$ by generators and relations. The generators are the object [1] and morphisms d_1, d_2, d_3 and d_4. The relations are the following:

— $d_3 \circ d_4 = -d_3$

— $d_3 \circ (d_3 \otimes 1) \circ \big(1 \otimes 1 \otimes 1 + (d_4 \otimes 1) \circ (1 \otimes d_4) + (1 \otimes d_4) \circ (d_4 \otimes 1)\big) = 0$

— $d_1 \circ d_4 = d_1$

— $d_1 \circ (d_3 \otimes 1) = d_1 \circ (1 \otimes d_3)$

— $d_4 \circ d_4 = 1 \otimes 1$

— $1 = (d_1 \otimes 1) \circ (1 \otimes d_2) = (1 \otimes d_1) \circ (d_2 \otimes 1)$

— $(d_4 \otimes 1) \circ (1 \otimes d_4) \circ (d_4 \otimes 1) = (1 \otimes d_4) \circ (d_4 \otimes 1) \circ (1 \otimes d_4)$

— $(1 \otimes d_3) \circ (d_4 \otimes 1) \circ (1 \otimes d_4) = d_4 \circ (d_3 \otimes 1)$

— $(1 \otimes d_1) \circ (d_4 \otimes 1) = (d_1 \otimes 1) \circ (1 \otimes d_4)$

4.4 The category of diagrams $\mathcal{D}(E_0, E_1)$. Consider two sets E_0 and E_1. Consider a Γ-diagram K where Γ is a curve. Suppose that only some components of Γ are oriented. Suppose that each oriented component of Γ is "colored" by an element of E_1 and each unoriented component by an element of E_0. If one cuts such a diagram into pieces one gets diagrams with univalent vertices. This univalent vertices are of different type:

– points outside of Γ

– points in a unoriented component of Γ. Such a point is colored by an element of E_0.

– points in an oriented component of Γ. Such a point is colored by an element of E_1 and the orientation of Γ defines a sign on it.

The points of the first type are called standard points. The points of the second type are called unoriented point. They are colored by E_0. The points in the last type are called oriented points. They are colored by E_1 and equipped with a sign.

So we are able to define a category corresponding to this situation.

An object of the category $\mathcal{D}(E_0, E_1)$ is a 5-tuple $(X, X_0, X_1, \alpha, \beta)$, where X, X_0 and X_1 are disjoint finite sets, α is a function from $X_0 \coprod X_1$ to $E_0 \coprod E_1$ sending X_0 into E_0 and X_1 into E_1, and β is a function from E_1 to $\{\pm 1\}$.

A morphism from $(X, X_0, X_1, \alpha, \beta)$ to $(X', X'_0, X'_1, \alpha', \beta')$ is a triple (Γ, f, u), where Γ is a compact partially oriented one-dimensional manifold with boundary $\partial\Gamma = X_0 \coprod X_1 \coprod X'_0 \coprod X'_1$, f is a function from the set of components of Γ to E and u is an element of $\mathcal{A}(\Gamma, X \coprod X')$, such that:

– the oriented components of Γ are sent by f into E_1

– the unoriented components of Γ are sent by f into E_0

– the restriction of f on the boundary of Γ is the function $\alpha \coprod \alpha'$

– the sign induced by the partial orientation of Γ on its boundary is the function $\beta' \coprod -\beta$.

It is easy to see that all these data define a category. The composition is given by gluing. We have also a monoidal structure obtained by disjoint union.

Remark. There is also a completed version of this category, where a morphism is a triple (Γ, f, u) satisfying the same condition as above except that u lies now in the completion $\mathcal{A}(\Gamma, X \coprod X')\widehat{\ }$ of $\mathcal{A}(\Gamma, X \coprod X')$ (completion with respect to the degree).

Remark.
The most important cases are the unoriented category $\mathcal{D}(un) = \mathcal{D}([1], \emptyset)$ and the oriented category $\mathcal{D}(or) = \mathcal{D}(\emptyset, [1])$ and also their completed versions $\mathcal{D}(un)\widehat{\ }$ and $\mathcal{D}(or)\widehat{\ }$.

5. THE KONTSEVICH INTEGRAL

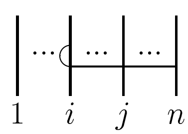
The Kontsevich integral was originally constructed to associate to every knot $\alpha : S^1 \to \mathbf{R}^3$ an element $Z(\alpha)$ lying in a a quotient of the module $\mathcal{A}(S^1)\widehat{\ }$. This construction was generalized to every link and every tangle.

5.1 The case of braids.

Let Γ be a braid with n strands. This braid is nothing else but the image of embedding $\alpha : [0,1] \times \{1, \ldots, n\} \to \mathbf{R}^2 \times [0,1] = \mathbf{C} \times [0,1]$ such that $\alpha(t, i)$ is on the form $(\beta_i(t), t)$ for every $i = 1, \ldots, n$ and every $t \in [0,1]$. Let $\omega(t)$ be the degree 1 form defined by:

$$\omega(t) = \sum_{1 \le i < j \le n} h_{ij} \frac{d\beta_i(t) - d\beta_j(t)}{\beta_i(t) - \beta_j(t)}$$

where h_{ij} is the diagram obtained by adding one edge a to $[0,1] \times \{1, \ldots, n\}$ attached on points $(1/2, i)$ and $(1/2, j)$, with equivalent cyclic orderings (equivalent with respect to the involution $(t, i) \leftrightarrow (t, j)$). The form $\omega(t)$ has its coefficients in the algebra $\mathcal{P}_n$ (defined over $\mathbf{C}$).

$h_{ij} =$ 

The Kontsevich integral of u is defined be the following series:

$$Z(u) = \alpha_* \sum_{n \ge 0} \int_{\Delta_n} \omega(t_1) \wedge \cdots \wedge \omega(t_n)$$

where Δ_n is the simplex $0 \le t_1 \le \cdots \le t_n \le 1$. This integral belongs to $\mathcal{A}_c(\Gamma)\widehat{\ }$, where Γ is considered as an abstract curve.

5.2 Proposition. *If two braids are isotopic, they have the same Kontsevich integral. Moreover if Γ_1 and Γ_2 are composable, one has a product $\mathcal{A}_c(\Gamma_1)\widehat{\ } \otimes \mathcal{A}_c(\Gamma_2)\widehat{\ } \to \mathcal{A}_c(\Gamma_1\Gamma_2)\widehat{\ }$ defined by union of diagrams and one has:*

$$Z(\Gamma_1\Gamma_2) = Z(\Gamma_1)Z(\Gamma_2)$$

Sketch of proof: If one consider a braid as a path in the configuration space X_n of all n-tuples of n distinct points in $\mathbf{C}$, $\omega(t)$ appears as the inverse image under the path of a form ω defined over X_n. This form (with coefficients in $\mathcal{P}_n$) is actually a flat connexion called the Knizhnik-Zamolodchikov connexion. More precisely, one has $d\omega = 0$ and $\omega \wedge \omega = 0$. The first equality is trivial and the second one is an easy consequence of the IHX relation.

With this point of view, $Z(\Gamma)$ is the monodromy of this connexion and, because the connexion is flat, $Z(\Gamma)$ depends only on the homotopy class of Γ.

The fact that Z is compatible with the product is a trivial consequence of the Fubini formula. □

Actually the braids form a category: the category of braids B. An object in this category is a finite subset in $\mathbf{C}$ and a morphism from a set $X \subset \mathbf{C}$ to a set $Y \subset \mathbf{C}$ is an isotopy class of braids $\Gamma \subset \mathbf{C} \times [0,1]$ meeting transversally the boundary of $\mathbf{C} \times [0,1]$ in $X \times \{0\} \cup Y \times \{1\}$. The properties implies that Z is a functor from the category of braids B to the category $\mathcal{D}(un)\widehat{}$.

5.3 The case of tangles.

The category of braids B is a subcategory of the category of tangles. The functor cannot be extended to this category but it is the case if one consider tangles equipped with vector fields.

5.4 Definitions. A banded tangle (or b-tangle) is a 1-dimensional compact submanifold $\Gamma \subset \mathbf{C} \times [0,1]$ equipped with a vector field V such that:

— Γ meets transversally $\mathbf{C} \times \{0,1\}$ on its boundary

— V is transverse to Γ, i.e. for every $x \in \Gamma$, $V(x)$ is a vector in $\mathbf{C} \times \mathbf{R}$ which is not tangent to Γ

— for every $x \in \partial\Gamma$, $V(x)$ is the vector $(i, 0)$.

Two b-tangles (Γ, V) and (Γ', V') are called isotopic if there exists an isotopy h_t on $\mathbf{C} \times [0,1]$ such that:

— h_t is the identity on $\mathbf{C} \times \partial[0,1]$ (for every $t \in [0,1]$)

— h_0 is the identity

— $h_1(\Gamma) = \Gamma'$

— V' is homotopic to $dh_1 \circ V$ by a homotopy which is constant on $\partial\Gamma'$ and always transverse to Γ'.

A b-link is a b-tangle without boundary. It is contained in $\mathbf{C} \times (0,1)$.

Usually a b-tangle will be represented by a diagram in the plane contained in $\mathbf{R} \times [0,1]$ equipped with the constant vector field $(i, 0)$.

5.5 Proposition. *The b-tangles form a category $\mathcal{T}$ containing the category of braids B.*

Proof: The objects of $\mathcal{T}$ are the finite subset of $\mathbf{C}$. A morphism in $\mathcal{T}$ from $X \subset \mathbf{C}$ to $Y \subset \mathbf{C}$ is an isotopy class of b-tangle Γ meeting the boundary of $\mathbf{C} \times [0,1]$ in $X \times \{0\} \cup Y \times \{1\}$. The composition is obtained by gluing. The inclusion functor

$B \to \mathcal{T}$ send a braid Γ to the b-tangle (Γ, V) where V is the constant vector field $(i, 0)$.

5.6 Proposition. *The functor Z extends to the category of b-tangles $\mathcal{T}$.*

This extension is unique under certain conditions. See [L] for a proof. There is two bad things about this functor: it is very difficult to compute and Z is not a monoidal functor. The target category of Z is a monoidal category (via disjoint union) but not the source category $\mathcal{T}$.

The category of q-tangles.

Consider the free non associative monoid F generated by one object . called point. The unique monoid homomorphism from F to $(\mathbf{Z}, +)$ sending the point to 1 is called the degree map. Every element $u \in F$ of degree > 1 can be written is a unique way: $u = vw$. Moreover F has a unique antiinvolution $u \mapsto u^*$ which sent the point to itself. The elements of F are called q-objects.

A q-object is nothing else but a finite set equipped with a parenthetization. For example there is 5 q-objects in degree 4:

$$u = (..)(..) \qquad v = .(.(..)) \qquad v^* = ((..).). \qquad w = (.(..)). \qquad w^* = .((..).)$$

To every q-object X one can associate to it the subset $\widehat{X} \subset \mathbf{C}$ given by the integers $1, \ldots, n$ corresponding to points in X.

One define now the category of q-tangles $\mathcal{T}'$ by the following:

The objects of $\mathcal{T}'$ are the q-objects. A morphism in $\mathcal{T}'$ from a q-object X to a q-object Y is an isotopy class of q-tangles corresponding to a morphism in the category $\mathcal{T}$ from $\widehat{X}$ to $\widehat{Y}$.

The category $\mathcal{T}'$ is a monoidal category. The monoidal structure on the level of objects is the product in F. On the level of morphisms, the monoidal structure is obtained by putting the second tangle on the right hand size of the first one.

There is another algebraic operation on the category $\mathcal{T}'$: the doubling.

Let Γ be a q-tangle. It is a morphism in the category $\mathcal{T}'$ from a q-object X to a q-object Y. Let Γ_0 be a component of Γ with non empty boundary. Suppose that Γ joins a point in $\mathbf{C} \times \{0\}$ to a point in $\mathbf{C} \times \{1\}$. These two points correspond to a point x in X and a point y in Y. Let u be a q-object of degree n. Then one can replace Γ by n parallel copies of Γ sitting in a band normal to the vector field which is a thickening of Γ_0. One can also replace x in X by u and y in Y by u in order to obtain new q-objects X' and Y'. So one gets a new q-tangle Γ' joining X' to Y'.

If the boundary of Γ_0 is contained in $\mathbf{C} \times \{0\}$ or in $\mathbf{C} \times \{1\}$, x and y are both in X or both in Y. In these cases one has to replace x by u and y by u^*. In all cases one gets a new q-tangle Γ' joining X' to Y'. This q-tangle Γ' is called a doubling of Γ.

The category $\mathcal{T}'$ is generated, as a monoidal category, by one object (the point) and the following morphisms:

$X^+ =$  $X^- =$  $u =$  $v =$  $W =$ 

the last morphism is the morphism from a q-object $w_1(w_2w_3)$ to $(w_1w_2)w_3$ represented by the trivial tangle. It is obtained by doubling from the associator morphism from .(..) to (..). .

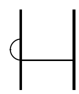
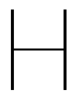
5.7 Theorem [Dr]. *There exist a functor Z from the category of q-tangles $\mathcal{T}'$ to the category of diagrams $\widehat{\mathcal{D}(un)}$ such that:*

— Z is a functor of monoidal categories

— for every q-tangle T with underlying curve Γ, $Z(T)$ is on the form (Γ, u) where u is a group-like element in $\widehat{\mathcal{A}_c(\Gamma)}$

— if T' is obtained from a braid T by doubling, if Γ and Γ' are the underlying curves and if $Z(T) = (\Gamma, u)$, then one has: $Z(T') = (\Gamma', f^*(u))$, where f is the canonical projection from Γ' to Γ.

— if $U_\pm$ is the positive or negative half-twist, one has: $Z(U_\pm) = \exp(\pm h/2)\sigma$, where σ is the transposition in $\mathfrak{S}_2$ and h is the following element in $\mathcal{P}_2$:

$h =$  $= -$ 

This functor Z is not unique, but for every banded link L, $Z'(L)$ is unique. More precisely, if Z_1 and Z_2 are two functor satisfying the properties of theorem 5.7, $Z_1(L) = Z_2(L)$ for every banded link L. Actually Z_2 is obtained from Z_1 by a gauge transformation in the following sense: for every q-object X there is a group-like element $F(X) \in \mathcal{A}_c(X \times [0,1]) \subset \mathrm{End}_{\widehat{\mathcal{D}(un)}}(X)$ such that, for every morphism T from a q-object X to a q-object Y one has: $Z_2(T) = F(Y)Z_1(T)F(X)^{-1}$.

Another interesting results is the following: every functor Z satisfying the properties of theorem 5.7 is gauge conjugate to a rational functor [LM].

The functor Z is characterized by the following elements:

$$\alpha = Z(u) \qquad \beta = Z(v) \qquad \Phi = Z(w)$$

where w is the associator from .(..) to (..). represented by the trivial b-tangle.

$u =$  $v =$  $w =$ 

The element Φ, also called associator, lies in the algebra $\widehat{\mathcal{P}_3}$. Via the projection from u and v to $[O,1]$ sending left boundaries to 0 and right boundaries to 1, one may consider α and β as element in the algebra $\widehat{\mathcal{P}_1} = \widehat{\mathcal{A}_c([0,1])}$.

The associator Φ is not unique. It has to satisfy many properties. Up to now there is only one associator which explicitly known. It come from the Kontsevich integral by taking a limit of this integral when sources and target goes to certain limit configurations.

Remark. The functor Z extends to the category $\widetilde{\mathcal{T}}$ of framed q-tangles. An object of $\widetilde{\mathcal{T}}$ is a q-object equipped with signs. A morphism is a framed tangle T, i.e. an oriented banded tangle. If this tangle is a morphism from X to Y, the sign of a point x in X is the opposite of the sign coming from the orientation of T and the sign of a point y in Y is the sign coming from the orientation of T. If we don't forget the orientation Z induces a functor from $\widetilde{\mathcal{T}}$ to the category $\mathcal{D}(or)\widehat{\ }$. For simplicity all these functors will be called the Kontsevich integral and denoted by Z.

5.8 Theorem. *Let Θ be the chord diagram with only one chord. Then the Kontsevich integral Z induces an isomorphism from the completion of $\mathbf{Q}[\mathcal{K}]$ with respect to the Vassiliev filtration to $\mathcal{A}_c(S^1)\widehat{\ }/(\Theta)$.*

Let $\mathcal{K}'$ be the monoid of framed knots. Then the Kontsevich integral Z induces an isomorphism from the completion of $\mathbf{Q}[\mathcal{K}']$ with respect to the Vassiliev filtration to the product of two copies of $\mathcal{A}_c(S^1)\widehat{\ }$.

Proof: The Kontsevich integral sends a framed knot K to an element $Z(K)$ in the completion $\mathcal{A}_c(S^1)\widehat{\ }$ of $\mathcal{A}_c(S^1)$. The element $Z(K)$ is an infinite series:

$$Z(K) = \sum_{0}^{\infty} Z_n(K)$$

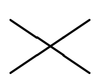
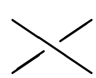
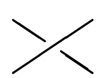
where $Z_n(K)$ belongs to $\mathcal{A}_c(S^1)_n$. Moreover the constant term $Z_0(K)$ is equal to 1 for every knot K.

Up to gauge transformation we may as well suppose that Z has rational coefficients.

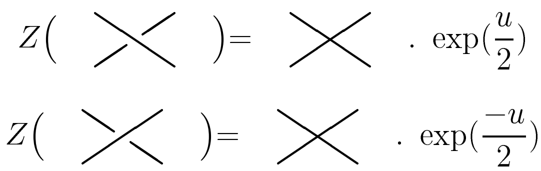
Denote by $\widetilde{\mathcal{T}}$ the category of framed q-tangles. If we replace the morphisms of this category by the sets of formal linear combinations of framed tangles, we get a new category $\mathbf{Q}[\widetilde{\mathcal{T}}]$. By linearity Z induces a functor still denoted by Z from the category $\mathbf{Q}[\widetilde{\mathcal{T}}]$ to the oriented category $\mathcal{D}(or)\widehat{\ }$.

A framed singular knot K is represented by an oriented diagram Δ. Let D be the chord diagram associated to K. The diagram Δ has n double points, p positive crossings and q negative crossings. Let a be the class of $p-q$ mod 2. Then in $\mathbf{Q}[\widetilde{\mathcal{T}}]$ one has: $K = [D]_a$. The knot K may be seen as an endomorphism of the emptyset in the category $\mathbf{Q}[\widetilde{\mathcal{T}}]$. Cutting D into pieces produces a decomposition of K as a composite (in $\mathbf{Q}[\widetilde{\mathcal{T}}]$) of elementary morphisms K_i. Some of these K_i are standard tangles, but n of these morphisms have the form $\mathrm{Id}\otimes T_i \otimes \mathrm{Id}$, where T_i is a morphism between two points and two points (with some sign) represented by a singular tangle with one double point. In this case the singular tangle represents the difference between a positive and a negative half-twist.

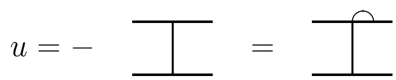
If K_i is standard, $Z(K_i)$ has constant term 1. If K_i has a double point, $Z(K_i)$ is on the form $\mathrm{Id} \otimes Z(T_i) \otimes \mathrm{Id}$ and we have:

$Z(T_i) = Z\Big($  $\Big) = \varepsilon Z\Big($  $\Big) - \varepsilon Z\Big($  $\Big)$

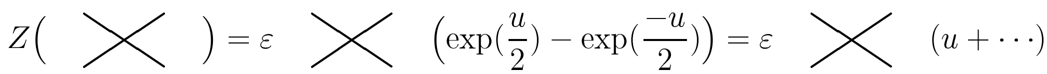
where ε depends on the local orientations near the double points in T_i. On the other hand the images under Z of the half twists are given by the following exponentials:



where u is the following diagram:



Then we have:



Therefore the degree one term of $Z(K_i)$ is exactly the chord diagram corresponding to K_i. Hence the Kontsevich integral $Z(K)$ is trivial in degree $< n$ and its degree n term is represented by the chord diagram $[D]_a$ associate to the diagram Δ.

Let I'_n be the n^{th} filtration of $\mathbf{Q}[\mathcal{K}']$ in the Vassiliev filtration. The degree n term of the Kontsevich integral Z induces a homomorphism Z_n from the quotient I'_n/I'_{n+1} to $\mathcal{A}_c(S^1)_n$ and, for every chord diagram $D \in \mathcal{D}^c_n$ and every $a \in \{0,1\}$, $Z_n([D]_a)$ is the class of D in the module $\mathcal{A}_c(S^1)_n$.

Consider the ring $R = \mathbf{Q}[\theta]/(\theta^2 - 1)$. Every framed knot K has a self linking number $a = \lambda(K,K)$. If K is represented by a diagram Δ this number a is the difference between the number of positive crossings of Δ and the number of negative crossings. Consider the following homomorphism φ from $\mathbf{Q}[\mathcal{K}']$ to $R \otimes \mathcal{A}_c(S^1)\widehat{}$: $K \mapsto \varphi(K) = \theta^{\lambda(K,K)} \otimes Z(K)$.

This homomorphism induces a function $\varphi_n : I'_n/I'_{n+1} \to R \otimes \mathcal{A}_c(S^1)_n$. We have the following diagram:

$$\mathbf{Q}[\mathcal{D}^c_n]/(4T) \oplus \mathbf{Q}[\mathcal{D}^c_n]/(4T) \stackrel{\psi}{\longrightarrow} I'_n/I'_{n+1} \stackrel{\varphi_n}{\longrightarrow} R \otimes \mathcal{A}_c(S^1)_n$$

where ψ is the map $D \oplus D' \mapsto [D]_0 + [D]_1$.

We have shown that ψ is surjective. On the other hand the composite $\varphi_n \circ \psi$ is the map: $D \oplus D' \mapsto f(D) + \theta f(D')$ where f is the isomorphism from $\mathbf{Q}[\mathcal{D}^c_n]/(4T)$ to $\mathcal{A}_c(S^1)_n$. Hence ψ and φ_n are isomorphisms.

In the case of oriented knots, we have only one function $D \mapsto [D]$ which satisfies also the 1T relation. On the other hand the Kontsevich integral is well defined, but only in the quotient $\mathcal{A}_c(S^1)\widehat{}\,/(\Theta)$. Since the natural map f from $\mathbf{Q}[\mathcal{D}^c_n]/(4T)$ to $\mathcal{A}_c(S^1)_n$ is an isomorphism, it induces an isomorphism f_1 from $\mathbf{Q}[\mathcal{D}^c_n]/(4T, 1T)$ to $\mathcal{A}^r_c(S^1)_n = \big(\mathcal{A}_c(S^1)/(\Theta)\big)_n$. So we have the following diagram:

$$\mathbf{Q}[\mathcal{D}^c_n]/(4T, 1T) \stackrel{\psi_1}{\longrightarrow} I_n/I_{n+1} \stackrel{Z_n}{\longrightarrow} \mathcal{A}^r_c(S^1)_n$$

where ψ_1 is the map $D \mapsto [D]$. Since ψ_1 is surjective and the composite $Z_n \circ \psi_1 = f_1$ bijective, ψ_1 and Z_n are isomorphisms. $\square$

6. WEIGHT FUNCTIONS

The Kontsevich integral of a link or a tangle lies in a completion of a module of diagrams $\mathcal{A}(\Gamma, X)$. In order to have an invariant of links or tangles it's enough to construct linear homomorphisms from a module of diagrams to some module.

A *weight function* on (Γ, X) is a homomorphism from the module $\mathcal{A}(\Gamma, X)$ to some module E. Such a function associates to every (Γ, X)-diagram K an element $\varphi(K) \in E$ in such a way that φ satisfy AS and IHX relations.

The standard way to construct a weight function into a ring or a module is obtained with the help of a Lie algebra or a Lie superalgebra.

Definitions. Let k be a characteristic zero field. A Lie algebra over k is a pair $L = (L, [\ ,\ ])$ where L is a k-vector space and $[\ ,\ ]$ a bilinear homomorphism from $L \otimes L$ to L (called the bracket or the Lie bracket) such that:

— the Lie bracket is antisymmetric: $[x, y] = -[y, x]$ for every x, y in L

— the Lie bracket satisfies the Jacobi identity: $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for every x, y, z in L.

Let L be a Lie algebra. A L representation (or a L-module) is a vector space E together with a bilinear map $x \otimes e \mapsto xe$ from $L \otimes E$ to E such that;

— for every x, y in L and every e in E one has: $x(ye) - y(xe) = [x, y]e$

Let Mod_L be the category of finite dimensional L-modules. The objects of this category are the finite dimensional L-modules and the morphisms are the linear maps compatible with the L-action. The category Mod_L is a k-linear category, but it is also a monoidal category. If E and E' are L-modules the vector space $E \otimes E'$ is also a L-module by the action: $x(e \otimes e') = xe \otimes e' + e \otimes xe'$. The dual E^* of a L-module is also a L-module by the rule: $(xf)(e) = -f(xe)$.

The simplest example of L-module is the module L itself with the action: $x(y) = [x, y]$. This module is called the adjoint representation.

In order to obtain weight function we need to consider Lie algebras equipped with a bilinear form.

Let E be a finite dimensional k-vector space equipped with a non singular bilinear form $b : E \otimes E \to k$. With such a form there is a well defined associated element $\omega = \sum e_i \otimes e'_i$ in $E \otimes E$ satisfying:

$$\forall e \in E \qquad e = \sum b(e, e_i)e'_i = \sum e_i b(e'_i, e)$$

This element is called the Casimir element of E. If the form b is symmetric the Casimir element is symmetric too.

If E isn't equipped with a bilinear form, one still have a Casimir element $\omega = \sum e_i \otimes e'_i \in E \otimes E^*$. It satisfies the following:

$$\forall e \in E \qquad e = \sum e_i\ e'i(e)$$

Definition. A quadratic Lie algebra is a triple $L = (L, [\ ,\ ], <\ ,\ >)$ where $(L, [\ ,\ ]$ is a finite dimensional Lie algebra over k and $<\ ,\ >$ is a non singular symmetric bilinear form on L satisfying the following:

$$\forall x, y, z \in L \qquad < [x, y], z >=< x, [y, z] >$$

This condition is equivalent to said that b is a L-linear map from $L \otimes L$ to k, where L is considered as the adjoint representation and k the trivial module (the module k equipped with the trivial action).

Supermodules and Lie superalgebras. A supermodule is a $\mathbf{Z}/2$-graded module $E = E_0 \oplus E_1$. The degree of an element in E_i is $i \in \mathbf{Z}/2$. If E and E' are supermodules their tensor product is also a supermodule: $(E \otimes E')_0 = E_0 \otimes E'_0 \oplus E_1 \otimes E'_1$ and $(E \otimes E')_1 = E_0 \otimes E'_1 \oplus E_1 \otimes E'_0$. In the classical case one has a symmetry T from $E \otimes E'$ to $E' \otimes E$ sending $e \otimes e'$ to $e' \otimes e$. In the super case T is replaced by the supersymmetry T sending $e \otimes e'$ to $\varepsilon e' \otimes e$ where ε is the sign $(-1)^{pq}$, with $p = \partial^\circ e$ and $q = \partial^\circ e'$. In this context, a supersymmetric bilinear form on a super module E is a symmetric bilinear form on E_0 together with an antisymmetric bilinear form on E_1.

As in the classical case there is a Casimir element associated to every supermodule equipped (or not) with a non singular bilinear form.

Definition. A Lie superalgebra is a pair $(L, [\ ,\ ])$ where L is a k-supermodule and $[\ ,\ ]$ is a morphism from $L \otimes L$ to L which is antisymmetric (in the super sense) and satisfies the super Jacobi identity:

$$\forall x \in L_p, y \in L_q, z \in L_r \qquad [[x, y], z] + (-1)^{p(q+r)}[[y, z], x] + (-1)^{r(p+q)}[[z, x], y] = 0$$

A L-supermodule is a supermodule E equipped with an action $L \otimes E \to E$ such that:

$$\forall x \in L_p, \forall y \in L_q, \quad \forall e \in E \qquad x(ye) - (-1)^{pq} y(xe) = [x, y]e$$

The category of L-supermodules is still a monoidal k-linear category.

A quadratic Lie superalgebra is a triple $(L, [\ ,\ ], <\ ,\ >)$ where $(L, [\ ,\ ])$ is a finite dimensional Lie superalgebra and $<\ ,\ >$ is a non singular supersymmetric bilinear form on L which is invariant (i.e. the map from $L \otimes L$ to k is L-linear, where L is the adjoint representation and k the trivial L-module).

Let L be a quadratic Lie (super)algebra. Let Γ be an oriented curve. A L-*coloring* of Γ is a map f which associates to each component of Γ a finite dimensional L-(super)module. With such a coloring each point x in $\partial\Gamma$ has an associated module $E_0(x)$. On the other hand each point x in $\partial\Gamma$ is equipped with a sign $\varepsilon(x)$. If Γ start from x, the sign is negative. It is positive in the other case. We'll said that the color of x is the dual module $E(x) = E_0(x)$ if $\varepsilon(x) = 1$ and the module $E(x) = E_0(x)^*$ if $\varepsilon(x) = -1$.

Suppose now that Γ is only partially oriented. Then we define a L-coloring of Γ as a map f which associates to each oriented component of Γ a finite dimensional L-module and to each unoriented component a finite dimensional L-module E equipped with a non singular (super)symmetric invariant bilinear form: $E \otimes E \to k$. Let x be a point in $\partial\Gamma$. If x lies in the oriented part of Γ, its sign $\varepsilon(x)$ and its color $E(x)$ are defined. In the other case x has only a color: the module $E(x)$ which is the coloring of the component of Γ containing x.

6.1 Proposition. *Let L be a quadratic Lie (super)algebra. Let Γ be a partially oriented curve and $X = \{x_1, \ldots, x_n\}$ be a finite set. Let $\{y_1, \ldots, y_p\}$ be the set of points in $\partial\Gamma$. Let f be a L-coloring of Γ and $E_1, \ldots, E_p$ be the corresponding colors of $y_1, \ldots, y_p$.*

Then these data induce a well defined homomorphism $\Phi(L, f)$ from $\mathcal{A}(\Gamma, X)$ to the module $L^{\otimes n} \otimes E_1 \otimes \cdots \otimes E_p$.

Proof: Let K be a (Γ, X)-diagram. Let K' be a subdivision of K such that K' has only standard edges (i.e. no circle). Let A be the set of vertices of K' and A_3 (resp. A_2, A_1) be the set of 3-valent (resp. 2-valent, univalent) vertices. Let B be the set of edges in K' and $\widetilde{B}$ be the set of edges in K' equipped with an orientation. The orientation changing is an involution $\alpha \mapsto -\alpha$ in $\widetilde{B}$. The end-point mapping is a map ∂_+ from $\widetilde{B}$ to A and the starting point mapping is a map ∂_- from $\widetilde{B}$ to A. For every $\alpha \in B$ one has: $\partial_+(-\alpha) = \partial_-(\alpha)$.

Let α be an oriented edge in $\widetilde{B}$. If α is not contained in Γ, we set: $E(\alpha) = L$. If α is contained in a oriented component Γ_0 of Γ, there is two possibilities: if the orientation of α is compatible with the orientation of Γ_0 we define $E(\alpha)$ as the color of Γ_0. If the orientations don't agree we define $E(\alpha)$ as the dual of the color of Γ_0. If α is contained in an unoriented component Γ_0, $E(\alpha)$ is defined as the color of Γ_0.

Let a be an (unoriented) edge in K'. Let α and $-\alpha$ be the corresponding oriented edges. If a is not contained in Γ denote by $\omega(a)$ the Casimir element $\Omega \in L \otimes L = E(\alpha) \otimes E(-\alpha)$. Suppose that a is contained in an oriented component Γ_0 which is colored by E. Up to taking another choice for α we may as well suppose that the orientations of α and Γ_0 are compatible. In this case we set $\omega(a)$ to be the Casimir element in $E \otimes E^*$. Suppose a is contained in a unoriented component Γ colored by E. On E we have a non singular (super)symmetric invariant bilinear form and we define $\omega(a)$ as the Casimir element associated to it.

So for every $a \in B$, the element $\omega(a)$ is an element in $E(\alpha) \otimes E(-\alpha)$. Notice that $\omega(a)$ doesn't depend on the choice of α, because a Casimir element is (super)symmetric.

Take a numbering $\{\alpha_0, \ldots \alpha_{2q-1}\}$ of $\widetilde{B}$ such that $\alpha_{2j+1} = -\alpha_{2j}$ for every $j < q$, and denote by a_j the unoriented edge corresponding to α_{2j} (and α_{2j+1}). Let $\mathcal{E}$ be the following module: $\mathcal{E} = E(\alpha_0) \otimes E(\alpha_1) \cdots \otimes E(\alpha_{2q-1})$. The tensor product $\Omega = \omega(a_1) \otimes \cdots \otimes \omega(a_{q-1})$ belongs to the module $\mathcal{E}$.

Consider another numbering $\{\beta_0, \ldots, \beta_{2q-1}\}$ of $\widetilde{B}$ satisfying the following properties:

– for every 3-valent vertex x in K the three oriented edges arriving to x are $\beta_j, \beta_{j+1}, \beta_{j+2}$ for some j (in the right cyclic ordering)

– for every 2-valent vertex x in K' the two oriented edges arriving to x are β_j, β_{j+1} for some j

– the oriented edges arriving to some vertex in X or in $\partial\Gamma$ appear in the ordering corresponding to the given ordering of $X \cup \partial\Gamma = \{x_1, \ldots, x_n, y_1, \ldots, y_p\}$.

This new numbering is obtained by a permutation σ of the set $\widetilde{B}$. The group of permutations of $\widetilde{B}$ acts on $\mathcal{E}$. In the super case this action permutes a pure tensor and multiply it by a sign. In particular a transposition acts as the supertransposition: $x \otimes y \mapsto (-1)^{ij} y \otimes x$ where x and y are of degree i and j.

Denote by Ω' the image of Ω under σ.

Let x be a 2-valent vertex of K' and α_j and α_{j+1} the oriented edges arriving to x. In any case one has a canonical bilinear form b_x defined on $E(\alpha_j) \otimes E(\alpha_{j+1})$. This form is the given form on $L \otimes L$, the evaluation map from some $E^* \otimes E$ to k or a given form on $E \otimes E$ where E is a color corresponding to a unoriented component of Γ.

Let x be a 3-valent vertex of K'. This vertex is a vertex of K. Let α_j, α_{j+1} and α_{j+2} be the oriented edges arriving to x. If x is not contained in Γ we have a trilinear form b_x on $E(\alpha_j) \otimes E(\alpha_{j+1}) \otimes E(\alpha_{j+2}) = L^{\otimes 3}$: the form $u \otimes v \otimes w \mapsto < u, [v, w] >=< [u, v], w >$. If x belongs to the curve Γ, the module $E(\alpha_j) \otimes E(\alpha_{j+1}) \otimes E(\alpha_{j+2})$ is, up to a (unique) cyclic permutation a module on the form $E^* \otimes L \otimes E$ and the form b_x is defined (up to this permutation) as the form: $e^* \otimes u \otimes e \mapsto e^*(ue)$.

Now we are able to define the element $\Phi(L, f)(K)$ as the image of Ω under the tensor product of all forms b_x. The fact that $\Phi(L, f)(K)$ doesn't depend on any choice follows from the construction. The morphism $\Phi(L, f)$ is compatible with the AS relation because the form b_x is completely antisymmetric (if x is not contained in Γ). The compatibility of the IHX relation comes from the Jacobi identity and the STU relation from the algebraic property of an action of L on a L-module. □

Let L be a quadratic Lie (super)algebra. The category Mod_L is a monoidal category. Denote by L the adjoint representation.

The Casimir element may be seen as a homomorphism from $k = L^{\otimes 0}$ to $L \otimes L = L^{\otimes 2}$. The form <?,?> is a homomorphism from $L^{\otimes 2}$ to $L^{\otimes 0}$ and the bracket is a homomorphism from $L^{\otimes 2}$ to $L = L^{\otimes 1}$. The symmetry (or the supersymmetry) T is a homomorphism from $L^{\otimes 2}$ to itself.

6.2 Theorem. *Let L be a quadratic Lie (super)algebra. Then there exists a unique functor Φ_L of monoidal categories from the category of diagrams $\mathcal{D}$ to the category Mod_L sending $[1]$ to the adjoint representation L and morphisms $d_1 - d_4$ to the invariant form, the Casimir element, the Lie bracket and the symmetry T respectively.*

Proof: The morphisms d_i are defined in 4.2. The unicity of such a functor comes from the fact that $\mathcal{D}$ is generated by $[1]$ and the d_i's morphisms. In order to construct Φ_L it is enough to prove that Φ_L is compatible with all the relations satisfied by the d_i's.

The first relation means that the bracket is antisymmetric. The second one which correspond to the IHX relation is send by Φ_L to the Jacobi relation. The other relations are easy to check.

A direct way to construct Φ_L is the following: Set $\Phi_L([n]) = L^{\otimes n}$. Let u be a morphism from an object $[p]$ to an object $[q]$. This morphism belongs to the module $\mathcal{A}(\emptyset, [p] \coprod [q])$. Then $\Phi(L, -)(u)$ is an element of the module $L^{\otimes p} \otimes L^{\otimes q}$. But L is canonically isomorphic to its dual (as a L-module). Then $\Phi(L, -)(u)$ may be seen as an element of the module $L^{*\otimes p} \otimes L^{\otimes q}$ or an element $\Phi_L(u)$ of $\mathrm{Hom}(L^{\otimes p}, L^{\otimes q})$.

So Φ_L is well defined. The fact Φ_L satisfy the desired properties is a formal consequence of the construction. □

Remark. This theorem gives a very efficient way to compute the image under $\Phi(L, -)$ of a (Γ, X)-diagram when Γ is empty. Just take a bijection between X and some $[n]$, and the diagram becomes a morphism from $[0]$ to $[n]$ and can be decomposed in a composite of the morphisms d_i's. Then the theorem give the desired computation.

This theorem may be generalized in the case of colored curves.

6.3 Theorem. *Let L be a quadratic Lie (super)algebra. Let E_0 and E_1 be two sets. Let f_0 and f_1 be two coloring functions sending every $x \in E_1$ to a finite dimensional L-module $f_1(x)$ and every $x \in E_0$ to a finite dimensional L-module $f_0(x)$ equipped with a non singular bilinear (super)symmetric invariant form. Then there exist a unique monoidal functor $\Phi(L, E_0, E_1, f_0, f_1)$ from the category $\mathcal{D}(E_0, E_1)$ to the category Mod_L satisfying the following properties:*

— An object of $\mathcal{D}(E_0, E_1)$ reduced to a standard point is sent to L. An oriented point with color $e \in E_1$ and sign ε is sent to the module $f_1(e)$ if $\varepsilon = 1$ or to its dual in the other case. An unoriented point with color $e \in E_0$ is sent to the module $f_0(e)$

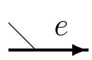
— The functor send morphisms d_1, d_2, d_3 and d_4 to the bilinear form $<\ ,\ >$, the Casimir element, the bracket and the (super)symmetry

— If X is an object with two points x and y and K is a diagram composed with two edges joining x to y and y to x, then the image of this morphism under the functor is the (super)symmetry

— If X is an object with only two points x and y colored by E_0 or E_1 and u is a morphism from X to $\emptyset$ represented by a diagram K with only one edge, the image of u is the evaluation map if K is oriented and the given bilinear form in K if not

— If X is an object with only two points x and y colored by E_0 or E_1 and u is a morphism from $\emptyset$ to X represented by a diagram K with only one edge, the image of u is the corresponding Casimir element

— If u is the morphism corresponding to the following diagram, its image is the action map:



$$\Longrightarrow \qquad (x \otimes u \mapsto xu)$$

where x is in L and u in the color module associated to e (or its dual if the curve is oriented in the other way).

Sketch of proof: It is easy to see that the objects and morphisms described in the theorem generate the monoidal category $\mathcal{D}(E_0, E_1)$ and the functor is unique.

The construction of the functor is exactly the same as above. The functor is defined on the objects. On morphisms the definition uses the functions $\Phi(L, f)$

constructed in proposition 6.1.

7. INVARIANTS OF LINKS

Let L be a quadratic Lie (super)algebra over a field k and E be a finite dimensional L-module. Let K be a framed link. The Kontsevich integral $Z(L)$ of K may be seen as a morphism from $\emptyset$ to itself in the oriented category $\widehat{\mathcal{D}(or)}$. This integral has an expansion $Z(K) = \sum Z_n(K)$ and each $Z_n(K)$ is an morphism of degree n in the category $\mathcal{D}(or)$. Let Φ be the functor $\Phi(L, \emptyset, [1], -, E)$, where E is the coloring $1 \mapsto E$, we have a series of numbers: $a_n(K) = \Phi(Z_n(K))$.

In order to force the series $\sum a_n(K)$ to be convergent, we'll consider a formal modification of the functor Φ.

Let X and Y be two objects in the category $\mathcal{D}(E_0, E_1)$ and u be a morphism from X to Y. Suppose that X and Y have p and q elements and that u is represented by a diagram with n 3-valent vertices. Set: $\partial^\circ u = (n + q - p)/2$. This new degree is an integer and, with this degree, the category $\mathcal{D}(E_0, E_1)$ becomes a graded monoidal category.

Let $\Phi = \Phi(L, E_0, E_1, f_0, f_1)$ be a functor from $\mathcal{D}(E_0, E_1)$ to Mod_L corresponding to some colorings. Then we have a new graded functor $\widetilde{\Phi}$ from $\widehat{\mathcal{D}(E_0, E_1)}$ to $k[[t]][t^{-1}] \otimes \mathrm{Mod}_L$ defined by: $\widetilde{\Phi}(\sum u_n) = \sum \Phi(u_n) t^{\partial^\circ u_n}$.

With these conventions, we have: $\widetilde{\Phi}(Z(K)) = \sum a_n(K) t^n \in k[[t]][t^{-1}]$, and $\widetilde{\Phi} \circ Z$ is an invariant or framed links.

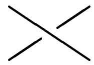
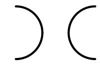
If we want to have an invariant of banded links, we have to consider other data. If K is a banded link, $Z(K)$ is a morphism from $\emptyset$ to itself in the unoriented category $\widehat{\mathcal{D}(un)}$. In order to have a weight function in this case, we have to take a L-module E equipped with a non singular (super)symmetric bilinear invariant form. The same construction as before applied to the functor $\Phi = \Phi(L, [1], \emptyset, E, -)$ gives rise to an invariant $\widetilde{\Phi} \circ Z$ of banded links.

The Kauffman bracket. Consider the simplest Lie algebra: $L = sl_2$ of all 2×2 matrices with zero trace. This Lie algebra is equipped with a form: $\alpha \otimes \beta \mapsto <\alpha, \beta> = tr(\alpha\beta)$. With this form, L becomes a quadratic Lie algebra. Consider the standard representation E of dimension 2. An isomorphism $\bigwedge^2 E \simeq k$ induces an antisymmetric bilinear invariant form b from $E \otimes E$ to k. Consider now the following L-supermodule E': the degree 0 part of E' is trivial and the degree 1 part of E' is the module E. Then the (super)dimension of E' is -2 and the form b induces a supersymmetric form b' on E'. Let Φ be the functor $\Phi(L, [1], \emptyset, E', -)$. The construction above gives rise to a functor $\widetilde{\Phi}$ and an invariant of banded links.

7.1 Theorem. *Let $K \mapsto <K>$ be the invariant of banded links induced by the sl_2 equipped with the standard representation consider as a supermodule of superdimension -2. Set: $A = -\exp(t/4)$. Then this invariant satisfies the following properties:*

— *for every banded link K, $<K>$ belongs to $k[[t]]$*

— *$< >$ is multiplicative with respect to the disjoint union operation*
— *$< \emptyset >= 1$ and the invariant of the trivial banded knot is $-A^2 - A^{-2}$*
— *the invariant $< >$ satisfy the following skein relation:*

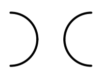
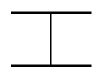
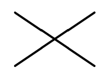
$<$  $>= A <$  $> +A^{-1} <$  $>$

Proof. Let $\{e_1, e_2\}$ be a basis of the standard representation E. The Casimir element $\Omega \in L \otimes L$ is the following:

$$\Omega = \sum_{ij} e_{ij} \otimes e_{ji} - \frac{1}{2}\mathrm{Id} \otimes \mathrm{Id}$$

where e_{ij} is the elementary matrix with 1 in the (i, j) place. Consider the supermodule $E' = 0 \oplus E$. We have the following decomposition: $E' \otimes E' \simeq k \oplus L$, where k is the trivial L-module.

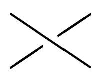
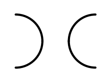
We have particular endomorphisms of $E' \otimes E'$: the identity, and the images under $\widetilde{\Phi}$ of the following diagrams:

$h =$  $u = -$  $T =$ 

An easy computation gives the following:

$$\mathrm{Id} = (1, 1) \qquad \widetilde{\Phi}(h) = (-2, 0) \qquad \widetilde{\Phi}(u) = (-3t/2, t/2) \qquad \widetilde{\Phi}(T) = (1, -1)$$

Let K_+, K_0 and K_∞ be the following tangles:

$K_+ =$  $K_0 =$ $K_\infty =$ 

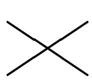
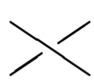
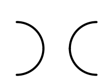
For every tangle K denote its invariant $\widetilde{\Phi}(Z(K)$ by $< K >$. With this notation we have the following:

$$< K_+ >= \widetilde{\Phi}(T \circ \exp(u/2)) = (1, -1)(\exp(-3t/4, t/4) = (\exp(-3t/4), -\exp(t/4))$$
$$< K_0 >= \widetilde{\Phi}(\mathrm{Id}) = (1, 1)$$
$$< K_\infty >= \widetilde{\Phi}(K_\infty) = (a, 0)$$

for some $a \in k[[t]]$. Therefore there exists an element $b \in k[[t]]$ such that:

$$< K_+ >= -\exp(t/4) < K_0 > +b < K_\infty >$$

Consider a singular banded link L with only one double point x. We can modify L near x in oder to get three banded links:

$L :$  $L_+ :$  $L_0 :$ $L_\infty :$ 

In the category of non associative tangles we can express this links on the following form:

$$L_+ = u \circ ((\mathrm{Id} \otimes K_+) \otimes \mathrm{Id}) \circ v$$

$$L_0 = u \circ ((\mathrm{Id} \otimes K_0) \otimes \mathrm{Id}) \circ v$$

$$L_\infty = u \circ ((\mathrm{Id} \otimes K_\infty) \otimes \mathrm{Id}) \circ v$$

and we have:

$$< L_+ >=< u > \circ(\mathrm{Id}\otimes < K_+ > \otimes\mathrm{Id})\circ < v >$$

$$= -\exp(t/4) < u > \circ(\mathrm{Id}\otimes < K_0 > \otimes\mathrm{Id})\circ < v >$$

$$+b < u > \circ(\mathrm{Id}\otimes < K_\infty > \otimes\mathrm{Id})\circ < v >= -\exp(t/4) < L_0 > +b < L_\infty >$$

Applying this method to the negative crossing (with $-u$ instead of u), we get also the following:

$$< L_- >= -\exp(-t/4) < L_0 > +b' < L_\infty >$$

for some $b' \in k[[t]]$. By applying a $90°$ rotation of the picture we get:

$$< L_+ >= -\exp(-t/4) < L_\infty > +b' < L_0 >$$

and b is equal to $-\exp(-t/4)$. So the invariant $K \mapsto < K >$ satisfies the desired skein relation. The other relations are easy to check. □

Remark. This invariant is actually the Kauffman bracket. It can be computed using the skein relation. It is easy to prove by induction that $< K >$ is an element of $\mathbf{Z}[A, A^{-1}]$.

The Kauffman polynomial. Consider now a n-dimensional vector space E equipped with a non singular symmetric form b. Let $L = o(E)$ be the Lie algebra of antisymmetric endomorphisms of E. The trace of the product induces a form $< \, , \, >$ on L and L is a quadratic Lie algebra. The module E is a L-module and functors Φ and $\widetilde{\Phi}$ are defined. So we get an invariant of banded links.

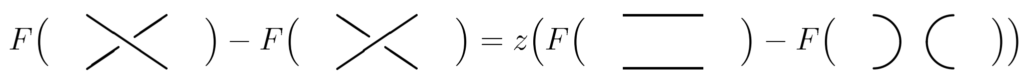
7.2 Theorem. *Let $K \mapsto F(K)$ be the invariant of banded links induced by the quadratic Lie algebra $o(E)$ equipped with the standard representation E. Set: $\alpha = \exp((n-1)t/4)$ and $z = 2sh(t/4)$. Then this invariant satisfies the following properties:*

— *for every banded link K, $F(K)$ belongs to $k[[t]]$*
— *F is multiplicative with respect to the disjoint union operation*
— *$F(\emptyset) = 1$ and the invariant of the trivial banded knot δ is:*

$$F(\delta) = 1 + \frac{\alpha - \alpha^{-1}}{z}$$

— *if K' is obtained from a banded link K by a positive twist, one has: $F(K') = \alpha F(K)$*

— *the invariant F satisfies the following skein relation:*



Proof: The module $\bigwedge^2 E$ is isomorphic to L by the rule:

$$e \wedge e' \mapsto \Big(u \mapsto b(u,x)y - b(y,u)x\Big)$$

The module S^2E contains the trivial module generated by the Casimir of E as a direct summand. So we get the following decomposition:

$$E \otimes E \simeq k \oplus L \oplus F$$

The situation is similar as above except that we have three modules instead of two. With the same notations, we have:

$$\mathrm{Id} = (1,1,1) \qquad \widetilde{\Phi}(h) = (n,0,0) \qquad \widetilde{\Phi}(u) = ((1-n)t/2, -t/2, t/2)$$

$$\widetilde{\Phi}(T) = (1,-1,1)$$

If we extend F to all non associative banded tangle, we have, with the same notation as above (and K_- as the inverse of K_+):

$$F(K_+) = (\exp((1-n)t/4), -\exp(-t/4), \exp(t/4))$$

$$F(K_-) = (\exp((n-1)t/4), -\exp(t/4), \exp(-t/4))$$

$$F(K_0) = (1,1,1)$$

$$F(K_\infty) = (a,0,0)$$

for some $a \in k[[t]]$. Then there exists an element $b \in k[[t]]$ such that:

$$F(K_+) - F(K_-) = 2\mathrm{sh}(t/4)F(K_0) - bF(K_\infty)$$

and the same argument as before shows that F verifies the following skein relation for every banded link L:

$$F(L_+) - F(L_-) = 2\mathrm{sh}(t/4)F(L_0) - bF(L_\infty)$$

By applying a 90° rotation of the picture we get:

$$F(L_+) - F(L_-) = bF(L_0) - 2\mathrm{sh}(t/4)F(L_\infty)$$

and therefore:

$$b = 2\mathrm{sh}(t/4) \qquad \Longrightarrow \qquad a = 1 + \frac{\mathrm{sh}((n-1)t/4)}{\mathrm{sh}(t/4)}$$

On the other hand we have:

$$F(K_\infty K_\infty) = (a^2, 0, 0) = F(\delta)F(K_\infty)$$

where δ is the trivial banded knot. Then we have: $F(\delta) = a$. Let L be a link and L' and L'' be the link obtained from L by applying a positive or negative twist. By construction there exists a series α in $[[t]]$, with constant term 1, such that:

$$F(L') = \alpha F(L) \qquad\qquad F(L'') = \alpha^{-1}F(L)$$

and the skein relation shows the following:

$$\alpha - \alpha^{-1} = 2\mathrm{sh}(t/4)(F(\delta) - 1)$$

which implies:

$$\alpha = \exp((n-1)t/4)$$

□

Remark. This invariant is called the Kauffman polynomial. For every banded link L, $F(L)$ belongs the the algebra $\mathbf{Z}[z, z^{-1}, \alpha, \alpha^{-1}]$.

The HOMFLY polynomial. Consider now the Lie algebra $L = sl_n$ of $n \times n$-matrices with zero trace. This Lie algebra is quadratic by taking the trace of the product as bilinear form. The standard representation E is n-dimensional. The module E is a L-module and functors Φ and $\widetilde{\Phi}$ are defined on the category of non associative framed tangles. So we get an invariant of framed links.

7.3 Theorem. *Let $K \mapsto P(K)$ be the invariant of framed links induced by the quadratic Lie algebra $sl_n = sl(E)$ equipped with the standard representation E. Set: $\alpha = \exp(t/(2n))$, $\beta = \exp(nt/2)$ and $z = \exp(t/2) - \exp(-t/2)$. Then this invariant satisfies the following properties:*

— for every framed link K, $P(K)$ belongs to $k[[t]]$
— P is multiplicative with respect to the disjoint union
— $P(\emptyset) = 1$ and the invariant of the trivial banded knot δ is:

$$F(\delta) = \frac{\beta - \beta^{-1}}{z}$$

— if K' is obtained from a banded link K by a positive twist, one has:

$$P(K') = \beta\alpha^{-1}P(K)$$

— If K_+, K_- and K_0 are obtained from a singular framed link by the three standard modifications, on has:

$$\alpha P(K_+) - \alpha^{-1}P(K_-) = zP(K_0)$$

Proof: Let $\{e_i\}$ be a basis of the standard representation E. The Casimir element $\Omega \in L \otimes L$ is the following:

$$\Omega = \sum_{ij} e_{ij} \otimes e_{ji} - \frac{1}{n}\mathrm{Id} \otimes \mathrm{Id}$$

where e_{ij} is the elementary matrix with 1 in the (i, j) place.

With the same notations as above, we have three endomorphisms of $E \otimes E$: The identity, the symmetry T, and the endomorphism u. An easy computation shows the following:

$$u = tT - \frac{t}{n}\mathrm{Id}$$

We have the decomposition:

$$E \otimes E \simeq S^2E \oplus \bigwedge{}^2 E$$

With respect to this decomposition, we have:

$$\mathrm{Id} = (1, 1) \qquad T = (1, -1) \qquad u = (t - t/n, -t - t/n)$$

If K_+ and K_- are the framed tangle corresponding to positive and negative half twists, we have:

$$P(K_+) = T \circ \exp(u/2) = (\exp(\frac{t}{2} - \frac{t}{2n}), -\exp(-\frac{t}{2} - \frac{t}{2n}))$$

and that implies:

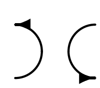
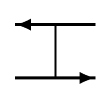
$$\alpha P(K_+) - \alpha^{-1} P(K_-) = zP(K_0)$$

With the same argument as above, this formula becomes true for every framed link, and the skein relation is proven.

On the other hand, we have the following decomposition:

$$E^* \otimes E \simeq k \oplus L$$

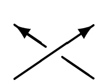
We have also three endomorphisms: the identity and the image under $\widetilde{\Phi}$ of the following diagrams:

h :  u : 

In this decomposition, we have:

$$\mathrm{Id} = (1, 1) \qquad \widetilde{\Phi}(h) = (n, 0) \qquad \widetilde{\Phi}(u) = (nt - t/n, -t/n)$$

Let K_+ be the following tangle:



we have:

$$F(K_+) = T \circ \exp(u/2) = T \circ (\exp(\frac{nt}{2} - \frac{t}{2n}), \exp(-\frac{t}{2n}))$$

Denote by K_- the tangle K_+, but with a negative crossing. We have:

$$\alpha F(K_+) - \alpha^{-1} F(K_-) = (\exp(nt/2) - \exp(-nt/2)) T \circ (1, 0)$$

On the other hand the skein relation is the following:

$$\alpha F(K_+) - \alpha^{-1} F(K_-) = (\exp(t/2) - \exp(-t/)) F(K_0)$$

and: $F(K_0) = F(\delta) T \circ (1, 0)$. Thus we get the desired formula for $F(\delta)$.

Let L be a framed link and L' and L'' be the link obtained from L by applying a positive or negative twist. By construction there exists a series γ in $[[t]]$, with constant term 1, such that:

$$P(L') = \gamma P(L) \qquad\qquad F(L'') = \gamma^{-1} P(L)$$

If one applies the skein relation to L_+ one gets:

$$\alpha\gamma - \alpha^{-1}\gamma^{-1} = z P(\delta) = \beta - \beta^{-1}$$

Since the constant term of γ is 1 the only possibility is:

$$\alpha\gamma = \beta \qquad \Longrightarrow \qquad \gamma = \beta\alpha^{-1}$$

and that finishes the proof. □

Remark. This polynomial invariant can be computed using the skein relation. It belongs to the subring of $k[[t]]$ generated by α, α^{-1}, β, β^{-1}, z and $(\beta - \beta^{-1})/z$. But the three variables α, β and z are algebraically independent in the following sense: there is no polynomial Q such that $Q(\alpha, \beta, z)$ vanishes in $k[[t]]$ for every value of n.

Then the polynomial invariant P belongs to a ring contained in the polynomial algebra $\mathbf{Z}[\alpha, \alpha^{-1}, \beta, \beta^{-1}, z, z^{-1}]$. If we want to have an invariant of oriented link, it's enough to set: $\beta = \alpha$. This polynomial is the HOMFLY polynomial. It satisfies all properties of the theorem, except that $\alpha = \beta$ and z are formal variables.

Remark. If one consider a k-supermodule E with non zero superdimension n as a module over the Lie superalgebra $L = sl(E)$, one gets exactly the same polynomial invariant as before in theorem 7.3. If one takes a k-supermodule of superdimension n, equipped with a non singular symmetric form and the Lie superalgebra $L = osp(E)$, one gets the same polynomial invariant as before in theorem 7.2. Roughly speaking, the A series give the HOMFLY polynomial and the B-C-D series give the Kauffman polynomial.

8. THE ALGEBRA Λ

8.1 Construction of Λ

Let Γ be a curve (i.e. a one-dimensional compact manifold), and X be a finite set. Denote by $\mathcal{A}(\Gamma, X)$ the $\mathbf{Q}$-module generated by all (Γ, X)-diagrams and divided by the AS and IHX-relations. If Γ is empty, the module $\mathcal{A}(\Gamma, X)$ will be simply denoted by $\mathcal{D}(X)$. Denote also by $\mathcal{D}_c(X)$ the submodule of $\mathcal{D}(X)$ generated by connected $(\emptyset, X)$-diagrams, and by $\mathcal{D}_s(X)$ the submodule of $\mathcal{D}(X)$ generated by connected non-empty $(\emptyset, X)$-diagrams having at least one 3-valent vertex. It is easy to see the following:

8.2 Proposition. *Let X be a finite set. Let $\pi(X)$ be the set of partitions of X. Then there is a canonical isomorphism:*

$$\mathcal{D}(X) = \mathcal{D}(\emptyset) \otimes \Big(\bigoplus_{\pi \in \pi(X)} \bigotimes_{Y \in \pi} \mathcal{D}_c(Y) \Big)$$

If X has 0 or 2 elements, one has:

$$\mathcal{D}_c(X) \simeq \mathbf{Q} \oplus \mathcal{D}_s(X)$$

If X has one element, $\mathcal{D}_c(X)$ and $\mathcal{D}_s(X)$ are trivial modules. If X has at least 3 elements the two modules $\mathcal{D}_c(X)$ and $\mathcal{D}_s(X)$ are equal.

Proof: The first formula is a consequence of the fact that a $(\emptyset, X)$-diagram K may be written in a unique way as a disjoint union: $K = H \cup \big(\cup_i K_i\big)$, where H has no univalent vertex, and K_i are connected and non-empty. The sets $X \cap K_i$ form a partition of X, and the formula follows.

On the other hand every non-empty connected $(\emptyset, X)$-diagram has a 3-valent vertex except the circle if X is empty or the interval $[0, 1]$ if X has 2 elements. The fact that $\mathcal{D}_c(X) = \mathcal{D}_s(X) = 0$ when X has only one element, is an easy exercise (see [V2] for a proof). □

If X is a set, the symmetric group $\mathfrak{S}(X)$ acts on modules $\mathcal{D}(X)$, $\mathcal{D}_c(X)$ and $\mathcal{D}_s(X)$. In particular for every $n > 0$ the module $F(n) = \mathcal{D}_s([n])$ is a $\mathfrak{S}_n$-module.

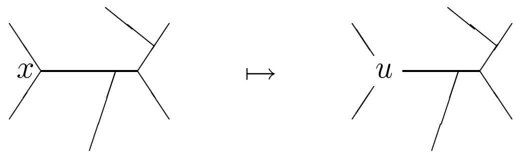
8.3 Definition. Let Λ be the submodule of $F(3) = \mathcal{D}_s([3])$ consisting of all elements $u \in F(3)$ satisfying the following:

$$\forall \sigma \in \mathfrak{S}_3, \qquad \sigma(u) = \varepsilon(\sigma) u$$

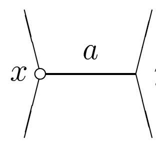
where ε is the signature homomorphism. The degree of an element $u \in \Lambda$ represented by a diagram K is $(n-4)/2$, where n is the number of vertices of K. This degree is also the rank of $\mathrm{H}_1(K)$. With this degree, Λ is a graded $\mathbf{Q}$-module.

8.4 Proposition. *The module Λ is actually a graded $\mathbf{Q}$-algebra. Moreover, for every set X, $\mathcal{D}_s(X)$ is equipped with a natural Λ-algebra structure.*

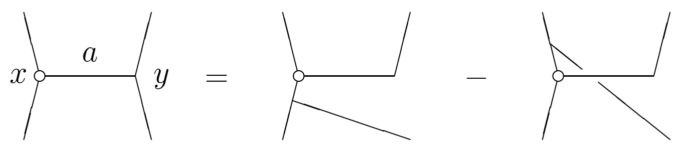
Proof: Let X be a finite set. Let K be a $(\emptyset, X)$-diagram. Suppose that K is connected and has some 3-valent vertex x. Let u be an element of Λ represented by a $(\emptyset, [3])$-diagram H. Because of the numbering of the set of edges arriving to x, one can insert H in K near x and one gets a new diagram $K(x, H)$. Since H is completely antisymmetric with respect with the $\mathfrak{S}_3$-action, the class of $K(x, H)$ in $\mathcal{D}(X)$ doesn't depend on the choice of the numbering. Moreover it depends only on K, x and u and will be denoted by $K(x, u)$.



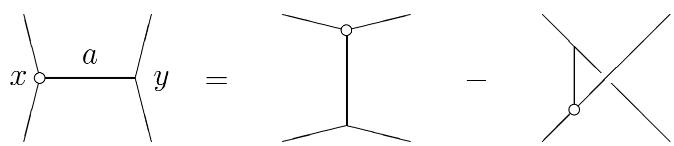
Consider an edge a in K with vertices x and y. Consider the following part of $K(x, u)$, where the small circle represents H inserted near x:



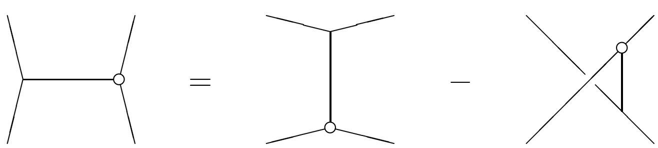
Because of the next lemma the bottom right edge may cross H and we have in $\mathcal{D}_s(X)$:



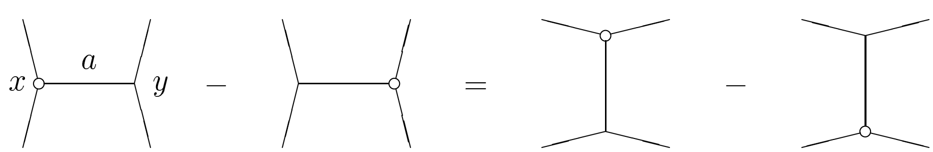
Or equivalently:



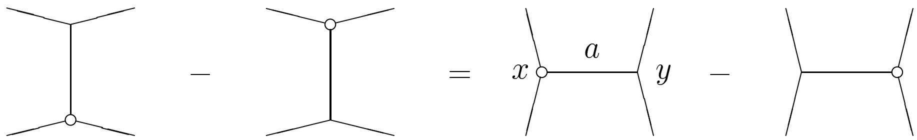
In the same way we have:



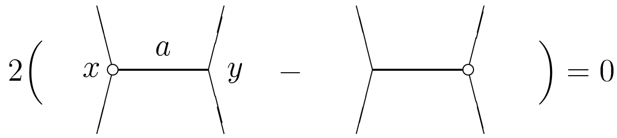
which implies:



and, by applying a rotation of the picture, we have:



and then:

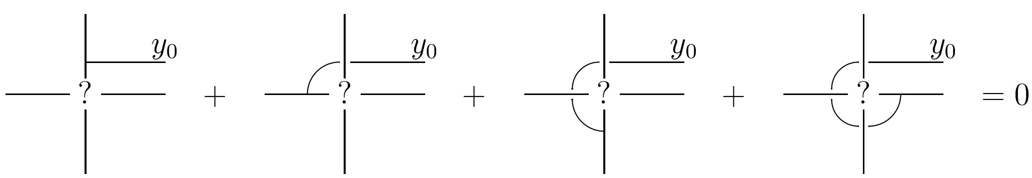


Therefore inserting H near x or y gives the same element in $\mathcal{D}_s(X)$ and the element $K(x,u)$ doesn't depend on the choice of the vertex x. Then $K(x,u)$ depends only on K and the class u of H in Λ. It is easy to see that the map $K \mapsto K(x,u)$ is compatible with the AS relation. But this transformation is also compatible with the IHX relation because such a relation corresponds to an edge a in K and the transformation may be done near a vertex outside a. If K has only two vertices this proof doesn't work but a direct computation shows also the compatibility with the IHX relation.

Hence this transformation induces a well defined homomorphism from $\Lambda \otimes \mathcal{D}_s(X)$ to $\mathcal{D}_s(X)$. In particular this homomorphism induces a morphism from $\Lambda \otimes \Lambda$ to Λ and Λ becomes an algebra. It is easy to see that the previous morphism from $\Lambda \otimes \mathcal{D}_s(X)$ to $\mathcal{D}_s(X)$ induces on $\mathcal{D}_s(X)$ a structure of Λ-module. So the last thing to do is to prove the following lemma:

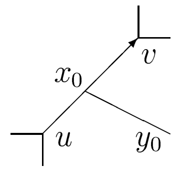
8.5 Lemma. *Let X be a finite set and Y be the set X with one extra point y_0 added. Let K be a connected $(\emptyset, X)$-diagram. For every $x \in X$ denote by K_x the $(\emptyset, Y)$-diagram obtained by adding to K an extra edge from y_0 to a point in K near x, the cyclic ordering near the new vertex being given by taking the edge coming from y_0 first, the edge coming from x after and the last edge at the end.*

Then the element $\sum_x K_x$ is trivial in the module $F(Y)$.



Proof: For every oriented edge a of K from a vertex u to a vertex v, we can connect y_0 to K by adding an extra edge from y_0 to a new vertex x_0 in a and we get a $(\emptyset, Y)$-diagram K_a where the cyclic order between edges arriving at x_0 is

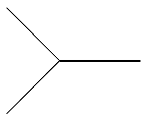
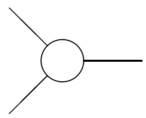
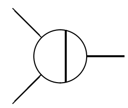
(x_0u, x_0y_0, x_0v).

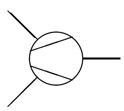
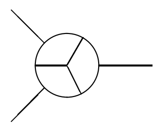


It is clear that the expression $K_a + K_b$ is trivial if b is the edge a with the opposite orientation. Moreover if a, b and c are the three edges starting from a 3-valent vertex of K, the sum $K_a + K_b + K_c$ is also trivial. Therefore the sum $\sum K_a$ for all oriented edge a of K is trivial and is equal to the sum $\sum K_a$ for all oriented edge a starting from a vertex in X. That proves the lemma. □

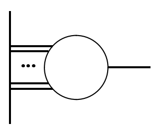
Remark. The algebra is commutative. In [V2] the algebra Λ is constructed with integral coefficients and it is shown that $12ab = 12ba$ for every a and b in Λ. In this situation Λ is defined over the rationals and is commutative.

In degree less to 4, the module Λ is generated (over $\mathbf{Q}$) by the following diagrams:

$1 =$  $t =$  $t^2 =$ 

$t^3 =$  $x_3 =$ 

Let $n > 0$ be an integer. We have the following element in $F(3)$:

$x_n =$ 

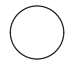
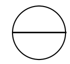
having n horizontal edges on the left hand side of the picture. It is proven in [V2] that x_n lies in Λ for every $n > 0$. We have:

$$x_1 = 2t \qquad \text{and} \qquad x_2 = t^2$$

Moreover the even $x_n's$ can be express in term of the odd $x_n's$.

8.6 Proposition. *The algebra* $End_{\mathcal{D}}([0])$ *of endomorphisms of the emptyset* $[0]$ *in the category* $\mathcal{D}$ *is isomorphic to the tensor product of the polynomial algebra* $\mathbf{Q}[\delta]$ *and the symmetric algebra* $S(E)$ *of the free* Λ*-module* E *generated by the* Θ*-diagram.*

Proof: It is clear that $\mathrm{End}_{\mathcal{D}}([0])$ is the symmetric algebra of the module $\mathcal{D}_c([0])$ of connected non-empty diagrams. The module $\mathcal{D}_c([0])$ is the direct sum of the $\mathbf{Q}$-module generated by the circle δ and the module $\mathcal{D}_s([0])$. But this last module is equipped with a Λ-module structure. Actually $\mathcal{D}_s([0])$ is the free Λ-module generated by the Θ-diagram with two vertices and three edges joining them. The result follows. □

$\delta =$  $\Theta =$ 

Only partial things are known about the structure of Λ. Every simple quadratic Lie (super)algebra L produces an algebra homomorphism from Λ to the coefficient ring of L (see next section). By this way one gets 8 algebra homomorphisms from Λ to different polynomial algebras. Another point which is known is the following: the elements x_1, x_3, x_5, ... are not algebraically independent. A family of relations including a special relation in degree 10 considered in [V2] was discovered by Kneissler [K]. In order to explain these relations, one has to consider the following algebras:

Let α, β and γ be formal variables of degree 1. Let R be the algebra of symmetric polynomials in α, β and γ. This algebra R is a subalgebra of $\mathbf{Q}[\alpha,\beta,\gamma]$. If $t=\alpha+\beta+\gamma$, $s=\alpha\beta+\beta\gamma+\gamma\alpha$ and $p=\alpha\beta\gamma$ are the elementary symmetric polynomials, R is the algebra $\mathbf{Q}[t,s,p]$. Set: $\omega=(t+\alpha)(t+\beta)(t+\gamma)=p+st+2t^3$ and define R_0 to be the subalgebra $\mathbf{Q}[t]\oplus\omega R$ of R.

On the other hand consider the elements $x'_n\in R$, $n\geq 0$ defined by the following:

— $x'_0=0 \qquad x'_1=2t \qquad x'_2=t^2$

— $\forall n\geq 0 \qquad x'_{n+3}=tx'_{n+2}-sx'_{n+1}+px'_n+\dfrac{st^{n+1}}{2}-\dfrac{pt^n}{2}-p(2t)^n$

It is an easy exercise to check that all these elements belong to the subalgebra R_0.

With these algebras the result of Kneissler may be express in the following way:

8.7 Theorem. *There exists a unique homomorphism φ of graded algebras from R_0 to Λ satisfying the following conditions:*

— φ sends t to t

— for every $n>0$, φ sends x'_n to x_n.

Related with this result we can formulate different conjectures:

8.8 Conjecture. *The morphism φ is injective.*

8.9 Conjecture. *The morphism φ is bijective.*

Presently φ is known to be bijective in degree <11 and injective in degree <16.

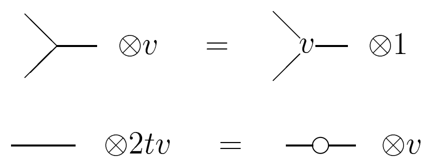
9. THE CATEGORY $\mathcal{D}'$

The algebra Λ acts on many modules of diagrams but not on all. In particular the set of morphisms $\mathrm{Hom}_{\mathcal{D}}([p],[q])$ in $\mathcal{D}$ between two object $[p]$ and $[q]$ are not Λ-modules.

The category $\mathcal{D}'$ is the category $\mathcal{D}$ where an action of Λ is forced. More precisely the objects of the category $\mathcal{D}'$ are the sets $[p]$, $n \geq 0$ and if $[p]$ and $[q]$ are two objects in $\mathcal{D}$ (or $\mathcal{D}'$), the module $\mathrm{Hom}_{\mathcal{D}'}([p],[q])$ of morphisms in $\mathcal{D}'$ from $[p]$ to $[q]$ is defined to be the quotient of $\mathrm{Hom}_{\mathcal{D}}([p],[q]) \otimes \Lambda$ by the following relations:

— Let u is an element of $\mathrm{Hom}_{\mathcal{D}}([p],[q])$ represented by a diagram K, and x be a 3-valent vertex in K. Let v be an element in Λ. Then $u \otimes v$ is equivalent to $u' \otimes 1$ where u' is obtained from u by inserting v near x.

— Let u is an element of $\mathrm{Hom}_{\mathcal{D}}([p],[q])$ represented by a non empty diagram K and v be an element in Λ. Then $u \otimes 2tv$ is equivalent to $u' \otimes v$ where u' is obtained from u by inserting a circle in some edge in K.



9.1 Proposition. *The category $\mathcal{D}'$ is a linear monoidal category over the polynomial algebra $\Lambda[\delta]$. Moreover the canonical functor from $\mathcal{D}$ to $\mathcal{D}'$ induces, for every $[p]$ and $[q]$ a morphism from $\mathrm{Hom}_{\mathcal{D}}([p],[q])$ to $\mathrm{Hom}_{\mathcal{D}'}([p],[q])$ which is injective on the submodule of $\mathrm{Hom}_{\mathcal{D}}([p],[q])$ generated by connected diagrams.*

Proof: If we apply the construction above to the functors $\mathcal{D}$, $\mathcal{D}_c$ and $\mathcal{D}_s$ considered in section 1.1, we get new functors $\mathcal{D}'$, $\mathcal{D}'_c$ and $\mathcal{D}'_s$ from finite sets to Λ-modules. By construction, $\mathcal{D}'_s(X)$ is isomorphic to $\mathcal{D}_s(X)$ for every finite set X. If X is finite, one has: $\mathcal{D}_c(X) = \mathbf{Q}^n \oplus \mathcal{D}_s(X)$, where $n = 1$ if the order of X is 0 or 2 and $n = 0$ otherwise. Therefore we get:

$$\mathcal{D}'_c(X) = \Lambda^n \oplus \mathcal{D}_s(X)$$

and $\mathcal{D}_c(X)$ is contained in $\mathcal{D}'_c(X)$.

The module $\mathcal{D}(\emptyset)$ is actually an algebra by the disjoint union of diagrams. More precisely $\mathcal{D}(\emptyset)$ is the symmetric algebra of the graded module $\mathcal{D}_c(\emptyset)$:

$$\mathcal{D}(\emptyset) = S(\mathcal{D}_c(\emptyset)) = \mathbf{Q}[\delta] \otimes S(\mathcal{D}_s(\emptyset)) = \mathbf{Q}[\delta] \otimes S(\Lambda\Theta) = \mathbf{Q}[\delta] \otimes S(2t\Lambda\delta)$$

The module $\mathcal{D}'(\emptyset)$ is also an algebra, but over Λ:

$$\mathcal{D}'(\emptyset) = \Lambda[\delta]$$

For an arbitrary finite set X, we have:

$$\mathcal{D}(X) = \mathcal{D}(\emptyset) \otimes \Big(\bigoplus_{\pi \in \pi(X)} \bigotimes_{Y \in \pi} \mathcal{D}_c(Y) \Big)$$

$$\mathcal{D}'(X) = \Lambda[\delta] \otimes_\Lambda \Big(\bigoplus_{\pi \in \pi(X)} \bigotimes_{Y \in \pi} \mathcal{D}'_c(Y) \Big)$$

where the last tensor product is over Λ.

By applying this results to the morphisms of the categories $\mathcal{D}$ and $\mathcal{D}'$, one gets the result. The action of Λ on modules of homomorphisms is obtained by construction. The multiplication by δ is the disjoint union with a circle.

9.2 Proposition. *Let L be a quadratic Lie (super)algebra over a coefficient field k. Suppose L is simple with a non trivial bracket. Then there exists a unique algebra homomorphism χ_L from $\Lambda[\delta]$ to k and a unique functor Φ_L of monoidal categories from $\mathcal{D}'$ to the category Mod_L of L-modules such that:*

— $\chi_L(\delta)$ is the (super)dimension of L

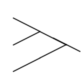
— Φ_L sends $[1]$ to the adjoint representation L and the following diagrams:

to the scalar product, the Casimir element, the Lie bracket and the (super)symmetry respectively

— For every morphism $f \in \mathcal{D}'$ and every $v \in \Lambda[\delta]$ one has:

$$\Phi_L(vf) = \chi_L(v)\Phi_L(f)$$

Proof: In section 6 a functor Φ from the category $\mathcal{D}$ to Mod$\mathcal{L}$ is constructed. It satisfies all the properties above except the properties relative to Λ and δ. Let v be an element in Λ considered as a morphism in $\mathcal{D}$ from $[2]$ to $[1]$. Denote by φ the homomorphism from $L \otimes L$ to L induced under Φ by v. Consider the following morphism D from $[3]$ to $[1]$:



The image of D is the morphism:

$$x \otimes y \otimes z \in L^{\otimes 3} \mapsto [[x,y],z]$$

If we multiply D by v in the two different ways we obtain, for every x, y and z in L:

$$\varphi([x,y],z) = [\varphi(x,y),z]$$

Therefore $\varphi(x,y)$ depends only on the bracket $[x,y]$ and there exists an endomorphism f of L such that: $\varphi(x,y) = f([x,y])$. Since L is supposed to be simple, f is the multiplication by an element $a = \chi_L(v) \in k$ which depends only on v. It is easy to check that χ_L is a homomorphism of algebras and satisfies the following:

$$\Phi(vD) = \chi_L(v)\Phi(D)$$

for every $v \in \Lambda$ and every diagram D for which vD is defined.

On the other hand Φ transform the circle considered as a morphism from [0] to itself to the multiplication by the dimension d of L. Therefore the functor Φ factorizes through the category $\mathcal{D}'$ by a functor Φ_L satisfying all the desired properties. $\square$

The algebra homomorphism χ_L is described in [V2] for every simple quadratic Lie (super)algebra. We get the following:

— If L is the Lie superalgebra $sl(E)$ where E is a super k-module of superdimension n, the character χ_L restricted to R_0 is obtained by sending α, β and γ to n, 2 and -2.

— If L is the Lie superalgebra $osp(E)$ where E is a super k-module of superdimension n equipped with a non singular supersymmetric bilinear form, the character χ_L restricted to R_0 is obtained by sending α, β and γ to $n-4$, 4 and -2.

— If L is a Lie superalgebra of type D(2, 1,?), the character χ_L restricted to R_0 is obtained by sending α, β and γ to arbitrary elements in the coefficient field with the only condition: $\alpha + \beta + \gamma = 0$.

— If L is an exceptional Lie algebra of type E6, E7, E8, F4 or G2, the character χ_L restricted to R_0 is obtained by sending (α, β, γ) to $(3, -1, 4)$, $(4, -1, 6)$, $(6, -1, 10)$, $(5, -2, 6)$ and $(5, -3, 4)$ respectively.

There are few other examples of quadratic simple Lie superalgebras. The character corresponding to $psl(n, n)$ may be defined in term of sl characters. Lie superalgebras G(3) and F(4) induce the same character as sl_2 and sl_3 (see [P]). The Hamiltonian algebras induce the augmentation character.

The characters χ_L where L is of sl type fit together in one graded algebra homomorphism χ_{sl} from Λ to $R/(sl)$ where (sl) is the ideal of R generated by the polynomial $P_{sl} = \prod(\alpha + \beta) = p - st$. The osp-type characters fit together in one graded algebra homomorphism χ_{osp} from Λ to $R/(osp)$ where (osp) is the ideal of R generated by the polynomial $P_{osp} = \prod(\alpha + 2\beta) = 8s^2t^2 + 4s^3 + 4pt^3 - 18pst + 27p^2$. In the same way the $D(2, 1, \alpha)$-type characters induce a graded algebra homomorphism χ_{sup} from Λ to $R/(sup)$ where (sup) is the ideal of R generated by the polynomial $P_{sup} = t$. The exceptional Lie algebras induce graded algebra homomorphisms χ_i from Λ to $R/(exc_i)$ where (exc_i) is the ideal of R generated by $P_{exc} = \prod(3\alpha - 2t) = 4t^3 - 18st + 27p$ and the polynomial P_i equal to $36s-5t^2$, $81s-14t^2$, $225s-44t^2$, $81s-8t^2$ or $36s+7t^2$ if the Lie algebra is E6, E7, E8, F4 or G2. A last interesting character is obtained by the Lie algebra sl_2. It can be seen as a graded algebra homomorphism from Λ to $R/(sl_2)$ where (sl_2) is the ideal generated by the polynomial $P_{sl_2} = \prod(t+\alpha) = \omega = p+st+2t^3$.

All these characters are compatible in the following sense:

9.3 Theorem [P]. *Let I be the intersection in R of the ideals (sl), (osp), (sup), (exc_i) and (sl_2). Then all the characters above induce a graded algebra homomorphism χ*

from Λ *to* R_0/I. *Moreover the composite* $\chi \circ \varphi$ *from* R_0 *to* R_0/I *is the quotient homomorphism.*

Remark. Since the first non trivial element in I is the product $P_{sl}P_{osp}P_{sup}P_{exc}P_{sl_2}$ which is a polynomial of degree 16, the first element in R_0 which may be killed in Λ is this polynomial in degree 16.

10. THE UNIVERSAL LIE ALGEBRA

Pseudo quadratic Lie algebra.

Let L be a quadratic Lie (super)algebra over a commutative ring k. Let Mod_L be the category of L-modules. This category is monoidal and k-linear. The adjoint representation still denoted by L is a particular module in this category. On the other hand the scalar product $f_1 = <?,?>$, the Casimir element $f_2 = \Omega$, the Lie bracket $f_3 = [?,?]$ and the (super)symmetry $f_4 = T$ are homomorphisms in Mod_L from $L^{\otimes 2}$ to $L^{\otimes 0}$, from $L^{\otimes 0}$ to $L^{\otimes 2}$, from $L^{\otimes 2}$ to $L^{\otimes 1}$ and from $L^{\otimes 2}$ to $L^{\otimes 2}$ respectively.

Moreover we have the following properties:

— $f_3 \circ f_4 = -f_3$

— $f_3 \circ (f_3 \otimes 1) \circ \Big(1 \otimes 1 \otimes 1 + (f_4 \otimes 1) \circ (1 \otimes f_4) + (1 \otimes f_4) \circ (f_4 \otimes 1)\Big) = 0$

— $f_1 \circ f_4 = f_1$

— $f_1 \circ (f_3 \otimes 1) = f_1 \circ (1 \otimes f_3)$

— $f_4 \circ f_4 = 1 \otimes 1$

— $1 = (f_1 \otimes 1) \circ (1 \otimes f_2) = (1 \otimes f_1) \circ (f_2 \otimes 1)$

— $(f_4 \otimes 1) \circ (1 \otimes f_4) \circ (f_4 \otimes 1) = (1 \otimes f_4) \circ (f_4 \otimes 1) \circ (1 \otimes f_4)$

— $(1 \otimes f_3) \circ (f_4 \otimes 1) \circ (1 \otimes f_4) = f_4 \circ (f_3 \otimes 1)$

— $(1 \otimes f_1) \circ (f_4 \otimes 1) = (f_1 \otimes 1) \circ (1 \otimes f_4)$

The category Mod_L is not strictly associative. But the full subcategory Mod'_L of Mod_L generated by the tensor products of L contains the morphisms f_i and is strictly associative.

Definition. Let k be a commutative ring. A pseudo quadratic Lie algebra L over k is a monoidal k-linear category $\mathcal{L}$ equipped with an object L and four morphisms f_1, f_2, f_3 and f_4 such that:

— the objects of $\mathcal{L}$ are the objects $L^{\otimes n}$, $n \geq 0$

— f_1 is a morphism from $L^{\otimes 2}$ to $L^{\otimes 0}$

— f_2 is a morphism from $L^{\otimes 0}$ to $L^{\otimes 2}$

— f_3 is a morphism from $L^{\otimes 2}$ to $L^{\otimes 1}$

— f_4 is a morphism from $L^{\otimes 2}$ to $L^{\otimes 2}$

— the morphisms f_i satisfy the nine properties above.

For simplicity the unit object $L^{\otimes 0}$ will be also denoted by k.

Definition. Let k and k' be commutative rings. Let $L = (L, f_1, f_2, f_3, f_4)$ and $L' = (L', f'_1, f'_2, f'_3, f'_4)$ be two pseudo quadratic Lie algebras over k and k'. A morphism from L to L' is a ring homomorphism χ from k to k' together with a functor of monoidal categories Φ from L to L' sending L to L' and morphisms f_i to f'_i and such that Φ is linear over χ on the modules of homomorphisms.

Remarks. Let L be a quadratic Lie (super)algebra. Then the category Mod'_L satisfy all the properties of a pseudo quadratic Lie algebra. In this sense, a quadratic Lie (super)algebra is a particular pseudo quadratic Lie algebra.

Because of this canonical example the morphism f_1 is called the scalar product, f_2 the casimir element, f_3 the Lie bracket and f_4 the symmetry.

The categories of diagrams $\mathcal{D}$ and $\mathcal{D}'$ are particular examples of pseudo quadratic Lie algebras. The first one is over $\mathbf{Q}$ and the second one over $\Lambda[\delta]$.

10.1 Theorem. *Let L be a pseudo quadratic Lie algebra over a $\mathbf{Q}$-algebra k. Then there exists a unique morphism Φ from $\mathcal{D}$ to L.*

Sketch of proof: The functor is obviously defined on the objects. On the coefficients ring it's the unique ring homomorphism from $\mathbf{Q}$ to k. To define Φ on the modules of morphisms, it is enough to defined $\Phi(D)$ where D is a diagram which represents a morphism from an object $[p]$ to another object $[q]$. Consider $[p]$ included in the standard way in $\mathbf{R}\times\{0\}$ and $[q]$ in $\mathbf{R}\times\{1\}$. Let f be a PL map from D to $\mathbf{R}\times[0,1]$ which extends the previous inclusions. If f is chosen to be generic enough, its image doesn't contain any vertical segment and has only finitely many double points. We may also suppose that two vertices or double points are not in a common vertical line. Then, by cutting $f(D)$ by vertical lines, one obtains a decomposition of D as a composite of morphisms of the form $\mathrm{Id}\otimes d_i\otimes\mathrm{Id}$. By using the same expression but with f_i instead of d_i one gets an morphism $\Phi(D)$ from $L^{\otimes p}$ to $L^{\otimes p}$.

Suppose now that g is another generic PL map from D to $\mathbf{R}\times[0,1]$ which satisfies the same condition as above. Then one construct a homotopy h_t between f and g which as generic as possible. For such a homotopy, h_t is generic except for finitely many values of t. For a generic t the corresponding morphism $\Phi(D)_t$ is defined. This function is locally constant and has maybe some jump on non generic t. The non generic values of t correspond to the case where some edge becomes vertical, or a double point (or a vertex) crosses some edge, or two double points (or vertices) have a common first coordinate. One have to check all these cases, but each of these corresponds to some formula satisfied by the f_i's and the function $t \mapsto \Phi(D)_t$ has no jump. That implies that $\Phi(D)$ doesn't depend on the choice of f. The fact that Φ is compatible with AS and IHX relations is easy to check.

So the functor is defined and the theorem is proven. □

Definition. Let L be a pseudo quadratic Lie algebra over a commutative ring k. Then L is called reduced if the algebra of endomorphisms of $L^{\otimes 0}$ is the module $k\mathrm{Id}$. It is called simple if $\mathrm{End}(L)$ is also the module $k\mathrm{Id}$.

10.2 Theorem. *Let L be a simple pseudo quadratic Lie algebra over a $\mathbf{Q}$-algebra k. Suppose that the following diagram is cartesian:*

$$\begin{array}{ccc} Hom(L,L) & \longrightarrow & Hom(L,L^{\otimes 2}) \\ \downarrow & & \downarrow \\ Hom(L^{\otimes 2},L) & \longrightarrow & Hom(L^{\otimes 2},L^{\otimes 2}) \end{array}$$

where the horizontal morphisms are the composition from the left with the cobracket (the dual of the bracket), and the vertical morphisms are the composition from the right with the bracket.

Then there exists a unique morphism Φ from $\mathcal{D}'$ to L.

Remark. Actually this condition is always satisfied if L is a simple quadratic Lie (super)algebra over a field with a non zero bracket.

Proof: Because of the last theorem, there is a unique functor Φ_0 from $\mathcal{D}$ to L which a morphism of pseudo quadratic Lie algebra. We have to prove that Φ_0 factorizes uniquely through the category $\mathcal{D}'$. Let d_3 be the bracket in the category $\mathcal{D}$ and d'_3 be the cobracket. These morphisms are represented by the following diagrams:

Let v be an element in Λ. The morphisms $\Phi_0(vd_3)$ and $\Phi_0(vd'_3)$ lye in $\mathrm{Hom}(L^{\otimes 2},L)$ and $\mathrm{Hom}(L,L^{\otimes 2})$ respectively. Moreover they induce the same morphism from $L^{\otimes 2}$ to itself. Because of the property of L, there exists a unique morphism f from L to L inducing $\Phi_0(vd_3)$ and $\Phi_0(vd'_3)$. On the other hand, L is supposed to be simple and there exists a unique element $a \in k$ such that f is the morphism $a\mathrm{Id}$. This element a depends only on v. Denote it by $\chi(v)$. It is easy to see that χ is actually an algebra homomorphism from Λ to k.

On the other hand the circle δ induces under Φ_0 the scalar form applied to the Casimir element. This endomorphism of $L^{\otimes 0}$ is the multiplication by an element $d \in k$. So we have a well defined algebra homomorphism from $\Lambda[\delta]$ to k. This homomorphism, still denoted by χ, is the previous χ on Λ and send δ to d.

Now it is easy to see that the functor Φ_0 factorizes in a unique way through $\mathcal{D}'$ and the functor Φ is constructed. □

10.3 Direct summand and dimension.

Let L be a pseudo quadratic Lie algebra. Suppose L is reduced (i.e every endomorphism of the unit object $L^{\otimes 0}$ is scalar). It is possible to construct forms b_X and Casimir elements Ω_X for every object X in L. If $X = L^{\otimes n}$ is an object in L, denote by X^* the object $L^{\otimes n}$ where the components are written in the opposite order. So we have: $(X \otimes Y)^* = Y^* \otimes X^*$ for every objects X et Y in L. The form b_X is a morphism from $X^* \otimes X$ to $L^{\otimes 0} = k$ and Ω_X is an morphism from k to $X \otimes X^*$.

If X is the object L itself b_X is the scalar form and Ω_X the Casimir element. For general objects we construct b_X and Ω_X by induction:

$$b_{X\otimes Y} = b_X \circ (\mathrm{Id}_X \otimes b_Y \otimes \mathrm{Id}_X)$$

$$\Omega_{X\otimes Y} = (\mathrm{Id}_X \otimes \Omega_Y \otimes \mathrm{Id}_X) \circ \Omega_X$$

The form b_X is a morphism from $X \otimes X$ to $L^{\otimes 0}$ and Ω_X is a morphism from $L^{\otimes 0}$ to $X \otimes X$.

If f is an endomorphism from an object X to itself, one defines its trace by:

$$\tau(f) = b_X \circ (f \otimes \mathrm{Id}_X) \circ \Omega_X$$

This morphism is an endomorphism of the unit object. Since L is supposed to be reduced, this morphism is represented by a number. So the trace $\tau(f)$ of an endomorphism f is an element of the coefficient ring k.

It's an easy exercise to show that τ has the formal properties of a trace. More precisely we have:

10.4 Proposition. *The trace homomorphisms are linear. If f is a morphism from an object X to an object Y and f is a morphism from Y to X, on has: $\tau(f \circ g) = \tau(g \circ f)$.*

If f is an endomorphism of an object X and g is an endomorphism of an object Y, on has: $\tau(f \otimes g) = \tau(f)\tau(g)$.

Let π be a projector, that is an endomorphism of an object X such that: $\pi \circ \pi = \pi$. It is possible to consider π as a projection onto a direct summand X_π. This new object lies in a new category. Formally X_π is the projector π itself and, if π and π' are two projectors in $\mathrm{End}(X)$ and $\mathrm{End}(Y)$ respectively, the set of morphisms $\mathrm{Hom}(X_\pi, X_{\pi'})$ is defined by: $\pi'\mathrm{Hom}(X, X')\pi$. So we have a bigger category which is still a monoidal linear category. In this new category the object X decomposes into a direct sum of two objects: the object X_π and $X_{1-\pi}$. The dimension of the object X_π is simply the trace of the projector π.

In order to simplify the terminology, these new objects are called modules, or L-modules. So every L-module in this category has a dimension. This dimension is an element of the coefficient ring k.

Definition. Let L be a pseudo quadratic Lie algebra over an integral domain k. Let M be a L-module. One said that M is simple (resp. absolutely simple) if $\mathrm{End}(M)$ is a commutative integral domain containing $k\,\mathrm{Id}$ (resp. is contained in a localization of $k\,\mathrm{Id}$)

Examples. If L is a quadratic Lie superalgebra, the trace is the supertrace: the trace of the even part minus the trace of the odd part. The dimension of a module is the superdimension: the dimension of the even component minus the dimension of the odd component.

In the category $\mathcal{D}'$, the dimension of $L = [1]$ is simply the element $\delta \in \Lambda[\delta]$.

10.5 Theorem. *Let $\Lambda[\delta] \to A$ be an algebra homomorphism sending t, ω and $p\omega$ to invertible elements in A. Let $L = [1]$ be the generator module in the category $\mathcal{D}'_A = \mathcal{D}' \otimes A$. Let $\bigwedge^2 L$ be the second exterior power of L. Then $\bigwedge^2 L$ decomposes into a direct sum of three modules X_1, X_2 and E. Moreover the dimensions of these modules are:*

$$dimX_1 = \delta$$

$$dimX_2 = -\frac{\omega^2}{p\omega}\delta$$

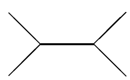
$$dimE = \delta\Big(\frac{\delta - 3}{2} + \frac{\omega^2}{p\omega}\Big)$$

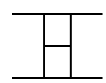
Proof: The elements t, $\omega = p+st+2t^3$ and $p\omega$ are elements of the algebra $R_0 = \mathbf{Q}[t] \oplus \omega R \subset R = \mathbf{Q}[t,s,p]$ and R_0 is sending into Λ by a canonical algebra homomorphism (see 8.7). Then the elements t, ω and $p\omega$ belong to Λ and become invertible in A. The category $\mathcal{D}'_A$ is obtained from $\mathcal{D}'$ by tensoring every module of morphisms by A. This category is still a pseudo quadratic Lie algebra but over A.

Let π be the projector $\pi = (\mathrm{Id} - T)/2$. It's an endomorphism of $L^{\otimes 2} = [2]$ and the corresponding module is $\bigwedge^2 L$. The trace of Id is δ^2, because it corresponds to two circles and the trace of T is δ. So we have:

$$\dim \bigwedge{}^2 L = \frac{\delta(\delta - 1)}{2}$$

Consider the endomorphisms U and V of $[2]$ corresponding to the following diagrams:





It is easy to see the following:

$$\pi U = U\pi = U \qquad\qquad U^2 = 2tU$$

Since t is invertible there exists a projector π' such that: $U = 2t\pi$. On the other hand Kneissler [K] has shown the following formula:

$$\omega(\pi V \pi)^2 = -\frac{3p\omega}{2}\pi V \pi + \Big(4t^3 - \frac{3\omega}{2}\Big)\Big(\frac{\omega t^2}{2} - \frac{3s\omega}{4}\Big)U$$

So there exists a projector π'' in $\mathcal{D}'_A$ such that:

$$\pi V \pi = (4t^3 - \frac{3\omega}{2})\pi' - \frac{3p}{2}\pi''$$

where the element p is defined in A as the quotient $\frac{p\omega}{\omega}$. Let X_1, X_2 and E be the images of π', π'' and $\pi - \pi''$. It is easy to compute the traces of U, V and $\pi V \pi$:

$$\tau(U) = 2t\delta \qquad\qquad \tau(V) = 8t^3\delta \qquad\qquad \tau(\pi V \pi) = \tau(\pi V) = 4t^3\delta$$

The result follows. □

Remark. Let L be a simple quadratic Lie (super)algebra over a ring k. Suppose that χ send t, ω and $p\omega$ to invertible elements in k. Suppose also that the dimension d of L is invertible in k. Then the functor Φ extends to a functor defined on $\mathcal{D}'_A$. One can check, case by case, that Φ send E to the zero module, and therefore the dimension d is given by: $d = 3 - \chi(\frac{2\omega}{p})$.

Actually the module E seems to be very poor. We don't have presently any counterexample to the following conjecture:

Conjecture. For every morphism u from [2] to [2], represented by a connected diagram, the induced morphism from E to E is trivial.

With regard to this conjecture, one mays expect to kill E in a suitable quotient of $\mathcal{D}'$ without loosing any important information. More precisely one has the following conjecture:

10.6 Conjecture. There exists a simple pseudo quadratic Lie algebra $\mathcal{L}$ over a ring Λ' and a morphism Φ from $\mathcal{D}'$ to $\mathcal{L}$ such that:

— the algebra Λ' is an integral domain contained in a localization of Λ

— if $\mathrm{Hom}_c([p],[q])$ is the module of homomorphisms in $\mathcal{D}'$ from an object $[p]$ to an object $[q]$ represented by connected diagrams, the functor Φ is injective on $\mathrm{Hom}_c([p],[q])$

— the module $\bigwedge^2 \mathcal{L}$ decomposes in a direct sum of two modules $X_1 \simeq \mathcal{L}$ and X_2, such that X_2 is absolutely simple (i.e. $\mathrm{End}(X_2)$ is contained in a localization of Λ)

Remark. If L is a simple quadratic Lie (super)algebra over a field k, the second exterior power $\bigwedge^2 L$ decomposes in a direct sum: $X_1 \oplus X_2$, where X_1 is isomorphic to L via the bracket, and X_2 is the kernel of the bracket. In many cases, X_1 and X_2 are simple. In the *sl* case, the module X_2 is not simple, but, in the subcategory of $\mathrm{Mod}(L)$ generated by L, the scalar product, the Casimir element, the bracket and the symmetry, it is simple. If L is the Lie superalgebra $\mathrm{D}(2,1,\alpha)$, X_2 is not simple, but the endomorphism ring of $\bigwedge^2 L$ is two-dimensional.

10.7 Theorem. *Suppose the conjecture 10.6 is true. Then there exist an extension Λ'' of Λ' and a decomposition in $\mathcal{L} \otimes \Lambda''$:*

$$\bigwedge{}^2\mathcal{L} = X_1 \oplus X_2$$

$$S^2\mathcal{L} = X_0 \oplus Y_2(\alpha) \oplus Y_2(\beta) \oplus Y_2(\gamma)$$

such that X_0, X_1, X_2, $Y_2(\alpha)$, $Y_2(\beta)$ and $Y_2(\gamma)$ are absolutely simple. Moreover there exists three elements α, β, γ in Λ'' such that: $t = \alpha+\beta+\gamma$, $s = \alpha\beta+\beta\gamma+\gamma\alpha$, $p = \alpha\beta\gamma$, and half the casimir operator acts on X_0, X_1, X_2, $Y_2(\alpha)$, $Y_2(\beta)$ and $Y_2(\gamma)$ by multiplication by 0, t, $2t$, $2t-\alpha$, $2t-\beta$ and $2t-\gamma$ respectively.

The dimension of these modules are the following:

$$dim X_0 = 1$$

$$dimX_1 = dimL = -\frac{(2t-\alpha)(2t-\beta)(2t-\gamma)}{\alpha\beta\gamma}$$
$$dimX_2 = \frac{(2t-\alpha)(2t-\beta)(2t-\gamma)(t+\alpha)(t+\beta)(t+\gamma)}{\alpha^2\beta^2\gamma^2}$$
$$dimY_2(\alpha) = -\frac{t(2t-\beta)(2t-\gamma)(t+\beta)(t+\gamma)(3\alpha-2t)}{\alpha^2\beta\gamma(\alpha-\beta)(\alpha-\gamma)}$$

A Galois group $\mathfrak{S}_3$ acts by permuting the elements α, β and γ and the modules $Y_2(\alpha)$, $Y_2(\beta)$ and $Y_2(\gamma)$.

Sketch of proof. By assumption Λ is contained in an integral domain. Consider the algebra homomorphism $\varphi : R_0 \to \Lambda$. Suppose that φ is not injective. Let $P \in \mathbf{Q}[t,s,p]$ be an polynomial killed by φ, with $P \neq 0$. This polynomial has the following form: $P = t^n Q(t,s,p)$ with $Q(0,s,p) \neq 0$. Because t is not zero and Λ has no zero divisor, the polynomial Q is killed by φ. Consider the character from Λ to $\mathbf{Q}[s,p]$ associated with the Lie algebras of type $\mathrm{D}(2,1,\alpha)$. This character send Q to $Q(0,s,p)$ and Q cannot be zero.

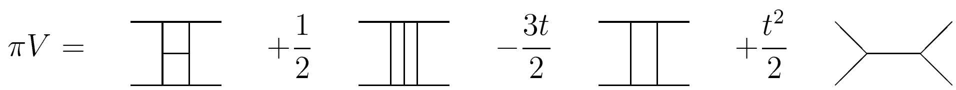
Thus R_0 is contained in $\Lambda \subset \Lambda'$. Up to inverting some elements in Λ and taking some algebraic extension, we may as well suppose that Λ'' contains $\mathbf{Q}[\alpha,\beta,\gamma]$ and every non zero homogeneous element in $\mathbf{Q}[\alpha,\beta,\gamma]$ are invertible in Λ'.

Consider the diagram U and V defined in the proof of theorem 3.4. Since the functor is injective on every module of connected diagrams, there is no relations between U and V in $\mathcal{L}$, and projectors π' and π'' are non zero. Therefore π' generates X_1 and π'' generates X_2 and the projector π is the sum: $\pi' + \pi''$.

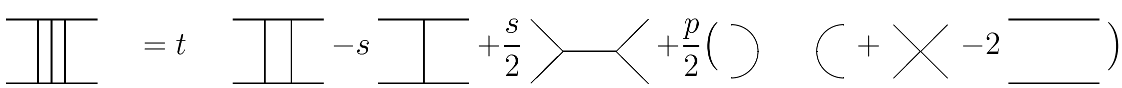
This relation may be written in the following way:

$$\pi V = (\frac{t^2}{2} - \frac{3s}{4})U - \frac{3p}{2}\pi$$

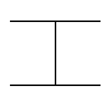
On the other hand it is easy to show the following:



If one substracts to this expression twice the same expression rotated by an half turn, one gets the following relation:



and that implies that the morphism ψ represented by



satisfies the following on the module $S^2\mathcal{L}$ divided by the image of the Casimir:

$$\psi^3 = t\psi^2 - s\psi + p$$

Hence the action of ψ on $S^2\mathcal{L}$ has three eigenspaces X_0, $Y_2(\alpha)$, $Y_2(\beta)$ and $Y_2(\gamma)$ corresponding to the eigenvalues $2t$, α, β and γ. On X_1 and X_2, ψ acts by multiplication by t and 0.

On the other hand the action oh half the Casimir on $\mathcal{L}\otimes\mathcal{L}$ is $2t-\psi$. So one gets the desired action.

The module $\mathrm{End}(\mathcal{L}^{\otimes 2}) = \mathrm{Hom}([2],[2])$ is isomorphic to the module $\mathrm{Hom}([0],[4])$ and the group $\mathfrak{S}_4$ acts on it. So we have the following decomposition:

$$\mathrm{Hom}([2],[2]) = E_+\otimes(4)\oplus E_-\otimes(1111)\oplus F_+\otimes(31)\oplus F_-\otimes(211)\oplus G\otimes(22)$$

where (4), (1111), ... are simple $\mathfrak{S}_4$-modules corresponding to Young diagrams. The assumption about the structure of $\bigwedge^2\mathcal{L}$ implies that G is isomorphic to the module $\mathrm{End}(X_1)\oplus\mathrm{End}(X_2)$ and that E_- and F_- are trivial modules. Since ψ has different eigenvalues in $S^2\mathcal{L}$ and $\bigwedge^2\mathcal{L}$, there is no homomorphism from $S^2\mathcal{L}$ to $\bigwedge^2\mathcal{L}$. Thus F_+ is zero too.

At the end we prove that E_+ is two dimensional and the dimension of the module of endomorphisms of $\mathcal{L}^{\otimes 2}$ is 6 (over some extension of Λ). The simplicity of the modules follows.

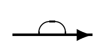
The computation of dimensions follows directly from:

$$\forall n>0 \qquad \tau(\psi^n) = 2tx_{n-1}\delta$$

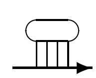
□

For the decomposition of $\mathcal{L}^{\otimes 3}$ the technique is much more complicated but we find a complete decomposition. In order to have absolutely simple modules we need to consider an algebraic extension of the ring. The first extension was necessary in order to have the modules $Y_2(\alpha)$, $Y_2(\beta)$ and $Y_2(\gamma)$. This extension is the Galois extension of the polynomial X^3-tX^2+sX-p and the Galois group is $\mathfrak{S}_3$. This group permutes α, β and γ and permutes some modules. In order to have a complete decomposition of $\mathcal{L}^{\otimes 3}$, one needs another Galois extension with Galois group G still isomorphic to $\mathfrak{S}_3$.

In order to detect some module, one need another operator. The first operator ψ was strongly related to the Casimir operator. Denote by π half this operator. Actually every element in the algebra $\mathcal{A}(S^1)$ induces an operator on every module. The Casimir operator 2π is obtained from the diagram:



Consider the operator π' represented by the diagram:



an set: $\sigma = \pi' - (8t^3-3\omega)\pi$. This element acts on every module and in particular on every direct summand of $\mathcal{L}^{\otimes n}$. On an absolutely simple module it acts by a scalar.

In order to describe the decomposition of $\mathcal{L}^{\otimes 3}$, we will use the standard action of $\mathfrak{S}_3$ on $\mathcal{L}^{\otimes 3}$. For every Young diagram a there is a corresponding module $(a)\mathcal{L}$. In this case we have the following modules $(3)\mathcal{L}$, $(21)\mathcal{L}$ and $(111)\mathcal{L}$. The first one is the symmetric power $S^3\mathcal{L}$ and the last one is the exterior power $\bigwedge^3 \mathcal{L}$.

Now we need to define another cubic extension of the ring. Consider the following elements in Λ':

$$p = \alpha\beta\gamma \quad q = t(\alpha\beta + \beta\gamma + \gamma\alpha) \quad r = t^3$$

Set also:

$$a = -\frac{9p}{8} + \frac{q}{4} - \frac{r}{2}$$

$$b = -\frac{27p^2}{32} + \frac{15pq}{16} - \frac{pr}{8} - \frac{q^2}{2}$$

$$c = \frac{p^2}{64}(27p - 18q + 4r)$$

Now define Λ'' as the Galois extension corresponding to the polynomial: $\Pi = X^3 - aX^2 + bX - c$. In this extension Π has three roots λ, μ and ν and the Galois group G is isomorphic to $\mathfrak{S}_3$. The Galois group of the complete extension is isomorphic to $\mathfrak{S}_3 \times \mathfrak{S}_3$.

10.8 Theorem. *Suppose the conjecture is true. Then in some localization of Λ'', $\mathcal{L}^{\otimes 3}$ has the following decomposition in absolutely simple modules:*

— $(3)\mathcal{L} = S^3\mathcal{L} = 2X_1 \oplus X_2 \oplus B(\alpha) \oplus B(\beta) \oplus B(\gamma) \oplus Y_3(\alpha) \oplus Y_3(\beta) \oplus Y_3(\gamma)$

— $(21)\mathcal{L} = 2X_1 \oplus 2X_2 \oplus Y_2(\alpha) \oplus Y_2(\beta) \oplus Y_2(\gamma) \oplus B(\alpha) \oplus B(\beta) \oplus B(\gamma) \oplus C(\alpha) \oplus C(\beta) \oplus C(\gamma)$

—$(111)\mathcal{L} = X_0 \oplus X_2 \oplus Y_2(\alpha) \oplus Y_2(\beta) \oplus Y_2(\gamma) \oplus X_3(\lambda) \oplus X_3(\mu) \oplus X_3(\nu)$

— *one has also the following decomposition:*

$X_1 \otimes Y_2(\alpha) = X_1 \oplus X_2 \oplus Y_2(\alpha) \oplus Y_3(\alpha) \oplus B(\beta) \oplus B(\gamma) \oplus C(\alpha)$

$X_1 \otimes X_2 = X_1 \oplus 2X_2 \oplus Y_2(\alpha) \oplus Y_2(\beta) \oplus Y_2(\gamma) \oplus B(\beta) \oplus B(\gamma) \oplus C(\alpha) \oplus C(\beta) \oplus C(\gamma) \oplus X_3(\lambda) \oplus X_3(\mu) \oplus X_3(\nu)$

— *the Galois group $\mathfrak{S}_3$ permutes α, β and γ and G permutes λ, μ, ν.*

— *$\mathfrak{S}_3$ permutes $X(\alpha)$, $X(\beta)$ and $X(\gamma)$ for $X = Y_2$, Y_3, B or C and G permutes $X_3(\lambda)$, $X_3(\mu)$ and $X_3(\nu)$*

— *the actions of π and σ are the following:*

X_0 :	$\pi = 0$	$\sigma = 0$
X_1 :	$\pi = t$	$\sigma = 0$
X_2 :	$\pi = 2t$	$\sigma = -18pt$
$Y_2(\alpha)$:	$\pi = 2t - \alpha$	$\sigma = 2\alpha(\alpha - t)(\alpha - 2t)(3\alpha - t)$
$X_3(\lambda)$:	$\pi = 3t$	$\sigma = -6t(9p + 4\lambda)$
$Y_3(\alpha)$:	$\pi = 3t - 3\alpha$	$\sigma = 6\alpha(\alpha - t)(3\alpha - t)(3\alpha - 2t)$
$B(\alpha)$:	$\pi = 2t + \alpha$	$\sigma = 2\alpha(\alpha + t)((\beta + \gamma)(2\beta + 2\gamma - \alpha) - 12\beta\gamma)$
$C(\alpha)$:	$\pi = 3t - 3\alpha/2$	$\sigma = 3\alpha(\alpha - 2t)(t^2 - 9s/2 + 9\beta\gamma)$

— *the dimensions are the following:*

$$dimY_3(\alpha)=\frac{1}{3}\frac{t(t+\beta)(t+\gamma)(2t-\alpha)(2t-\beta)(2t-\gamma)(5\alpha-2t)(2\beta+\gamma)(2\gamma+\beta)}{\alpha^3\beta\gamma(2\alpha-\beta)(2\alpha-\gamma)(\alpha-\beta)(\alpha-\gamma)}$$

$$dimB(\alpha)=-\frac{t(t+\beta)(t+\gamma)(2t-\alpha)(2t-\beta)(2t-\gamma)(2t-3\beta)(2t-3\gamma)(2\alpha+\beta)(2\alpha+\gamma)}{\alpha^2\beta^2\gamma^2(\alpha-\beta)(\alpha-\gamma)(2\beta-\gamma)(2\gamma-\beta)}$$

$$dimC(\alpha)=-\frac{32}{3}\frac{t(t+\alpha)(t+\beta)(t+\gamma)(2t-\beta)(2t-\gamma)(\beta+\gamma)(\beta+2\gamma)(\gamma+2\beta)}{\alpha^3\beta\gamma(\alpha-2\beta)(\alpha-2\gamma)(\alpha-\beta)(\alpha-\gamma)}$$

$$dimX_3(\lambda)=\frac{d(27p-18q+4r)}{12\lambda(\lambda-\mu)(\lambda-\nu)}\Big(\frac{1}{16}(q+2r)(7p+2q+4r)-\lambda(\mu+\nu)+\lambda(-\frac{3}{4}p+\frac{3}{2}q+r)\Big)$$

where $d=dimX_1$ is the dimension of $\mathcal{L}$.

Remark. The computation is rather difficult. The program maple is very useful for that. Actually this decomposition holds for every simple quadratic Lie (super)algebra. But sometime, some of these modules are zero. Another possibility is that the sum of two modules is zero.

In the sl case, we have $\alpha=t$. The polynomial Π has roots $\lambda=-p/4$ and $\mu=p/2$ and: $C(\alpha)=X_3(\nu)=0$.

In the osp case, we have $\beta+2\gamma=0$. We have: $\lambda=3p/4+t\gamma^2$ and $\mu=-3p/2-2t\gamma^2$ and: $Y_3(\alpha)=C(\alpha)=B(\gamma)=X_3(\nu)=0$.

In the exceptional cases we have $3\alpha=2t$ and $\lambda=0$ and:

$$B(\beta)=B(\gamma)=Y_2(\alpha)=C(\alpha)\oplus X_2=Y_3(\alpha)\oplus X_1=X_3(\mu)=X_3(\nu)=0$$

If $\alpha+t=0$, we get the sl_2 case and we have: $\lambda=-2t^3-9p/4$ and:

$$X_2=Y_2(\beta)=Y_2(\gamma)=Y_3(\beta)=Y_3(\gamma)=B(\beta)=B(\gamma)=C(\alpha)=C(\beta)=C(\gamma)=0$$

$$X_3(\mu)=X_3(\nu)=B(\alpha)+X_1=Y_2(\alpha)+X_3(\lambda)=0$$

In the D(2, 1,?) case, we have $t=0$ and $\lambda=3p/4$, $\mu=-3p/2$ and all the modules are zero dimensional except X_0, X_1, X_2, $X_3(\lambda)$ and $X_3(\mu)$.

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