Spectral Analysis of
Nonrelativistic Quantum Electrodynamics

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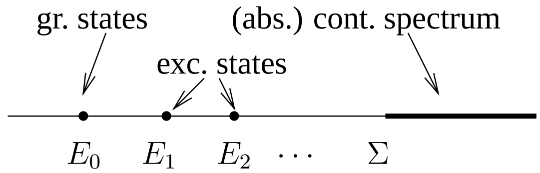
In Section I, covering my first and second lecture, I review the progress achieved during the past decade on the mathematical description of quantum mechanical matter interacting with the quantized radiation field. My main focus is the results I have obtained in collaboration with Jürg Fröhlich and Israel Michael Sigal.

The purpose of the third and fourth lecture, contained in Section II, is to sketch the basic ideas underlying the renormalization group method developed by Jürg Fröhlich, Israel Michael Sigal, and myself to analyze spectral properties of a standard model of nonrelativistic matter coupled to the quantized radiation field.

I appended an extended list of references on the subject.



I A Review on the Mathematical Progress in Nonrelativistic QED in the past Decade

I.1 Atoms and Molecules

• Atom ($M = 1$) or Molecule ($M \geq 2$) consisting of N electrons and M static nuclei.

• Hilbert space of electrons

$$\mathcal{H}_{el} \ := \ \mathcal{A}_N \bigotimes_{j=1}^N L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \, .$$

• Hamiltonian for the electron dynamics:

$$H_{el} \ = \ -\Delta_x + V(x) \, ,$$

where $\Delta_x = \sum \Delta_{x_j}$ and $V(x) = \sum_{m=1}^M \sum_{j=1}^N \frac{-Z_m}{|x_j - R_m|} + \sum_{1 \leq i < j \leq N} |x_i - x_j|^{-1}$ is the Coulomb potential of the electrons and the nuclei in the atom or molecule.

Spectral Properties of H_{el}

• $\sigma[H_{el}] \cap (-\infty, \Sigma] \ = \ \{E_0, E_1, E_2, \ldots\}$, eigenvalues of finite multiplicity (possibly accum. at ionization threshold Σ).

• $\sigma[H_{el}] \cap [\Sigma, \infty) = \sigma_{\mathrm{ess}}[H_{el}]$.

I.2 Quantized Photon Field

• Fock space $\mathcal{F} = \mathcal{F}_b[L^2(\mathbb{R}^3 \times \mathbb{Z}_2)] = \bigoplus_{n=0}^\infty \mathcal{F}^{(n)}$ is the photon Hilbert space,

• $\mathcal{F}^{(n)} \ = \ \{\psi_n \in \otimes^n L^2 \, | \, \forall \pi : \ \psi_n(k_1, \ldots, k_n)$
$= \psi_n(k_{\pi(1)}, \ldots, k_{\pi(n)})\}$ is the n-photon sector.

• $\mathcal{F}^{(0)} := \mathbb{C}\,\Omega$, where Ω is the vacuum vector.

• On $\mathcal{F}$, we have creation and annihilation operators, obeying CCR: $[a(k), a(k')] = [a^*(k), a^*(k')] = 0$,
$[a(k), a^*(k')] = \delta(k - k')$, $a(k)\Omega = 0$.

• Using creation and annihilation op's, the Hamiltonian of the free photon field is given by

$$H_f = \int dk \, \omega(k) \, a^*(k) a(k) \, .$$

• $\omega(k) = |\vec{k}| = \sqrt{\vec{k}^2 + m^2}\,|_{m=0}$ is the photon dispersion,

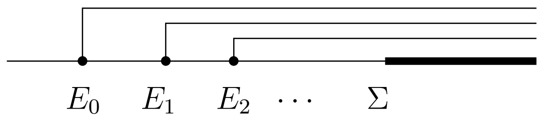
• Explicitly, $H_f = \bigoplus_{n=0}^\infty H_f^{(n)}$, $H_f \Omega := 0$, and
$\big[H_f^{(n)} \, \psi_n\big](k_1, \ldots, k_n) \ = \ \sum_{j=1}^n \omega(k_j) \, \psi_n(k_1, \ldots, k_n)$.

Spectral Properties of H_f

• $\sigma[H_f] = \mathbb{R}_{+,0}$,

• on $\mathbb{R}_+$ the spectrum is abs. cont., $\sigma_{\mathrm{ac}}[H_f] = \mathbb{R}_+$,

• Zero is a simple (the only) eigenvalue: $H_f \Omega = 0$.



I.3 Electron + Photons (non-interacting)

• Hilbert space is tensor product space, $\mathcal{H} = \mathcal{H}_{el} \otimes \mathcal{F}$,

• noninteracting Hamiltonian

$$H_0 \ := \ H_{el} \otimes \mathbf{1}_f \ + \ \mathbf{1}_{el} \otimes H_f \ ,$$

electrons and photons don't know about each other.

Spectral Properties of H_0

• Spectrum is sum of spectra,

$$\sigma[H_0] = \sigma[H_{el}] + \sigma[H_f] = [E_0 \ , \ \infty),$$

• Ground state energy $\inf \sigma[H_0] = E_0$ is eigenvalue at the bottom of the continuum,

• Excited eigenval's $E_1, E_2, \ldots$ are embedded in continuum.

I.4 Interacting Electron-Photon System

• Hilbert space is unchanged, $\mathcal{H} = \mathcal{H}_{el} \otimes \mathcal{F}$,

• interacting Hamiltonian ($g \geq 0$ coupling constant):

$$H_g \ := \ H_0 \ + \ gW \ ,$$

• W is the interaction operator, in physics $g \approx 1/500$

• To simplify the exposition, we neglect terms in the interaction quadratic in creat. and annihil. ops.

• Minimal coupling, neglecting the A^2 term: $W = W_{mc} = 2A(x) \cdot i\nabla_x$, where $A(x)$ is quantized vector potential,

• Dipole Approximation: $W = W_{dip} = E(0) \cdot x$, where $E(0)$ is quantized electric field at $x = 0$.

• Both interactions are of the form

$$W = \int dk \, G(k) \otimes a^*(k) + G(k)^* \otimes a(k)$$

where $G(k)$ is an operator on $\mathcal{H}_{el}$,

• $\|G_{\mathrm{mc}}(k) \, (-\Delta_x + 1)^{-1/2}\| \leq |k|^{-1/2}, \quad \text{for } |k| \leq 1,$

• $|G_{\mathrm{dip}}(x,k)| \leq |k|^{1/2}|x|, \quad \text{for } |k| \leq 1.$

I.5 Tasks which have been addressed

• **Models and Selfadjointness:** physically interesting H_g that defines a semibounded, s.a. Hamiltonian.

• **Binding:** To prove that H_g has a ground state, i.e., $E_0(g) := \inf \sigma(H_g)$ is an eigenvalue.

• **Resonances:** To prove that the embedded excited energies turn into resonances with corresponding metastable states of finite life-time.

• **Scattering Theory:** To study e^{itH_g}, as $t \to \pm\infty$. Ultimately, to prove asymptotic completeness of scattering of such systems.

• **Positive Temperatures:** To study the systems for non-zero temperature, $\beta < \infty$, by means of a Liovillian operator, L_g.

• **Feshbach Renormalization Map:** To develop a renormalization group that allows for a direct analysis of the spectral properties of H_g and L_g.

I.6 Models and Selfadjointness

• Let $J(k) := \|G(k)\,(-\Delta_x + \mathbf{1})^{-1/2}\|$.
• If $\int (1 + \omega(k)^{-1}) J(k)^2 dk < \infty$ then $H_g > -\mathcal{O}(g^2)$ is selfadjoint on $\mathrm{Dom}(H_0)$.
• If interaction includes terms quadratic in $a(k)$, $a^*(k)$ then same condition for selfadjointness, but $|g| \leq C$ must not be too large (otherwise, may still have selfadj. but different domain).
• The condition

$$\Lambda_{UV} := \int_{\omega(k)>1} J(k)^2 dk < \infty$$

makes the introduction of an UV cutoff necessary.
• The condition

$$\int_{\omega(k)<1} J(k)^2 \, \frac{dk}{\omega(k)} < \infty$$

is fulfilled by all physically interesting models.
• Stability of matter, $H_g \geq -C\,(N+M)$ has been proved with $C = C(\Lambda_{UV}, \underline{Z})$ in [31, 29, 18, 30, Bugliaro + Fefferman + Fröhlich + Graf 95–97].
• Best energy bounds for $\inf \sigma(H_g)$, as $\Lambda_{UV} \to \infty$, in [60, Lieb + Loss 99].

I.7 Binding

Thm:[11, BFS 98a] Assume that

$$\Lambda_2 := \int (1 + \omega(k)^{-1}) J(k)^2 dk < \infty$$

and that $g^2 \Lambda_2 \ll \Sigma - E_0$. Then $E_0(g)$ is eigenval. with eigenvec. $\Psi_0(g) \in \mathcal{H}$, obeying

$$\left\| e^{\alpha|x|} \otimes (N_f + 1)^{1/2} \Psi_0(g) \right\| \; < \; \infty \, ,$$

for some $\alpha > 0$.

• First result of this type in [32, 33, Fröhlich 73-74]

• For the spin-boson model, see [67, 69, Spohn 89, 98], proof with strat. sim. to [11, BFS 98a] in [6, Arai + Hirokawa 95];

• No smallness condition for $|g|$ in [36, Gerard 99];

• Nondeneracy of $E_0(g)$ in [11, BFS 98a] [44, Hiroshima 98].

• Thm is optimal in the sense that assertion is wrong, in general, if $\Lambda_2 = \infty$ [32, 33, Fröhlich 73-74], [67, Spohn 89], [1, Arai + Hirokawa + Hiroshima 99].

• For minimal coupling model (not neglecting the A^2 term), $\Lambda_2 = \infty$, but proof can be modified to include this special case [13, BFS 99].

• For minimal coupling model, existence has been shown for arbitrary values of g and Λ_0 (not neglecting the A^2 term) in [38, Griesemer + Lieb + Loss 00].

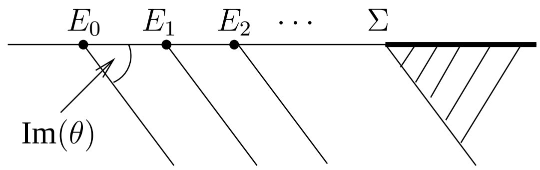


Figure 1: The spectrum of $H_0(i\vartheta)$, with $\vartheta > 0$.

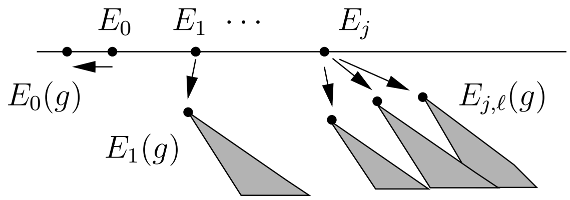


Figure 2: The spectrum of $H_g(i\vartheta)$, with $\vartheta > 0$, up to $\mathcal{O}(g^{2+\varepsilon})$-neighbourhoods, for some $\varepsilon > 0$.

I.8 Resonances

• Time-decay of
$\langle\psi|e^{-itH_g}\psi\rangle \;=\; \frac{1}{2\pi i}\int e^{-itE}\langle\psi|(H_g - E - i0)^{-1}\psi\rangle dE$
• leads to analytic continuation of $\langle\psi|(H_g - z)^{-1}\psi\rangle$ from $\mathbb{C}^+$ across $\mathbb{R}$ into $\mathbb{C}^-$;
• **Complex dilatation:** $U_\theta\psi(k) := e^{-3\theta/2}\psi(e^{-\theta}k)$ is unitary for $\theta \in \mathbb{R}$;
• $\mathcal{S} \ni \theta \mapsto H_g(\theta) := U_\theta H_g U_\theta^{-1}$ defines an analytic family of type A (in a strip $\mathcal{S}$ about $\mathbb{R}$);
• $\Rightarrow \langle\psi|(H_g - z)^{-1}\psi\rangle = \langle U_{\bar\theta}\psi|(H_g(\theta) - z)^{-1}U_\theta\psi\rangle$, for all $\theta \in \mathcal{S}$.
• If $\|U_{\bar\theta}\psi\|, \|U_\theta\psi\| < \infty$ then $z \mapsto \langle\psi|(H_g - z)^{-1}\psi\rangle$ is regular whenever $\|(H_g(\theta) - z)^{-1}\| < \infty$.
• $E_0, E_1, \ldots$ are eigenvalues of $H_0(\theta)$ at the tips of branches of continuous spectrum;
• **Thm:** [11, 13, BFS 98a, 99] Assume that

$$\Lambda_\beta := \int (1 + \omega(k)^{-\beta}) J(k)^2 dk < \infty,$$

for some $\beta > 1$ and that $|g| \ll \vartheta \ll 1$. Then, for $j \geq 1$, ex. Γ_j, c_j s.th.

$$[E_{j-1} + c_j g \;,\; E_{j+1} - c_j g] \;+\; i(-g^2\,\Gamma_j \;,\; \infty) \;\subseteq\; \rho[H_g(i\vartheta)],$$

• Thm $\Rightarrow \sigma[H_g]$ in $[E_0(g) + Cg \;,\; \Sigma - Cg]$ is a.c.
• Constr. of metast. st's in [11, 13, BFS 98a, 99], [64, Mück 00].
• Assuming $J(k) \leq \omega(k)^{-(1-\mu)/2}$, for some $\mu > 0$ (does not apply to m.c.), Thm can be strengthened by RG methods [BFS 98b]:
For each $j \geq 1$, $H_g(\theta)$ possesses complex eigenval's, $E_j(g) = E_j + \mathcal{O}(g) \in \mathbb{C}^-$. The spectrum of H_g is (locally) cont. in cuspidal domains

$$\sigma[H_g(\theta)] \;\subseteq\; E_j(g) + e^{-i\vartheta}\big\{a + ib\big|\, a \geq 0,\, |b| \leq Ca^{1+\mu/4}\big\}.$$

The eigenval's $E_j(g)$ and eigenvec's $\Psi_j(g)$ are obtained from an iterated application of the Feshbach renormalization map.

I.9 Scattering Theory

• Massive photons, $\omega_m(k) := \sqrt{\vec{k}^2 + m^2}$;

• Atom or Molecule is confined, $\lim_{|x|\to\infty} V(x) = \infty$ or $\big\|(|x|+1)^{1+\mu}\, G(k)\, (-\Delta_{\vec{x}} + \mathbf{1})^{-1/2}\big\| \leq J(k)$

• asymp. cre. + ann. op's exist [45, 46, 47, 48, Hoegh-Krohn 68-69],

$$a^{\#}_{\pm}(f) = s - \lim_{t\to\pm\infty} e^{-itH_g} a^{\#}(e^{it\omega} f) e^{itH_g}$$

• Let $\mathcal{K}_{\pm} := \{\psi |\; a_{\pm}(f)\psi = 0,\; \forall f\}$.

• **Thm:** [35, Gerard 96], [23, Deresinski + Gerard 99], [37, Griesemer 99] Scattering is asymptotically complete, i.e.,

$$\mathcal{H}_{pp}(H_g) \;=\; \mathcal{K}_+ \;=\; \mathcal{K}_-.$$

• Thm ⇒ exists unitary $J_{\pm} : \mathcal{H} \to \mathcal{H}_{pp}(H_g) \otimes \mathcal{F}$ s.th.

$$a^{\#}_{\pm}(f) = J\, a^{\#}(f)\, J^*.$$

• If $\mathcal{H}_{pp}(H_g) = \mathbb{C} \cdot \Psi_0(g)$ then

$$J_{\pm} H_g J^*_{\pm} = E_0(g) + \int dk\, \omega(k)\, a^*_{\pm}(k) a_{\pm}(k).$$

• Basic input for Thm: positive commutator estimates. Also derived in [49, 50, 51, Hübner + Spohn 94–95], [66, Skibsted 98], [15, BFS+Soffer 99].

I.10 Positive Temperatures

• At $\beta = \infty$, dynamics is generated by Hamiltonian H_g on $\mathcal{H}$.

• For $\beta < \infty$ no Gibbs state, $\mathrm{Tr}\{e^{-\beta H_g}\} = \infty$.

• ⇒ pass to descr. in which dyn. is gen. by Liouvillian

$$\begin{aligned} L_g &= L_0 + \ell[W_g] - r[W_g] \\ &= H_0 \otimes \mathbf{1} - \mathbf{1} \otimes H_0 + \ell[W_g] - r[W_g]\,, \end{aligned}$$

on $\mathcal{H} \otimes \mathcal{H}$, where, e.g.,

$$\ell[a(f)] = a\Big(\sqrt{1+\rho_\beta}\, f\Big) \otimes \mathbf{1} \;+\; \mathbf{1} \otimes a^*\Big(\sqrt{\rho_\beta}\, \overline{f}\Big)$$

and $\rho_\beta(k) := (e^{\beta\omega(k)} - 1)^{-1}$.

• Finite state atom: $H_{el} = \mathrm{diag}(E_0, E_1, \ldots, E_{N-1})$ on $\mathcal{H}_{el} = \mathbb{C}^N$ (gen. Spin-Boson model).

• Note that $\sigma[L_g] = \mathbb{R}$;

• Therm. equil. states are characterized by KMS cond.

• **Thm:** [52, 53, 54, 55, Jaksic + Pillet 95–96] If $|g| \ll \beta^{-1}$ then $\mathrm{Ker} L_g = \mathbb{C} \cdot \omega$ and $\mathbb{R} \setminus \{0\} = \sigma_{\mathrm{ac}}[L_g]$.

• Thm ⇒ ω is unique KMS state, and **Return to equilibrium** holds: $A_t := \alpha_t(A_0)$ time evol. of (loc.) observ., ρ state, normal w.r.t. ω, then

$$\lim_{t\to\infty} \rho(A_t) \;=\; \omega(A_0).$$

• Framew. devel. in [52, 53, 54, 55, Jaksic + Pillet 95–96] building upon [41, Haag + Hugenholz + Winnink 67], [8, Araki + Woods 63], [7, Araki 73].

• For Spin-Boson model uniqueness est. in [27, Fannes + Verbeure + Nachtergaele 88].

• Thm is improved in [14, BFS 00]:
no restriction $|g| \ll \beta^{-1}$ necessary.

• Use compl. deform. for spectr. analysis of L_g:
Complex translations in [52, 53, 54, 55, Jaksic + Pillet 95–96],
Complex dilatations plus RG methods in [14, BFS 00].

• Thm is improved in [24, Derezinski + Jaksic 98]:
no restriction $|g| \ll \beta^{-1}$ necessary, no restriction on finiteness of the number N of degrees of freedom necessary, either.

I.11 Renormalization Map

• Feshbach map $H \mapsto$

$$\mathcal{F}_P(H) \; := \; PHP - PH\overline{P}(\overline{P}H\overline{P})^{-1}\overline{P}HP$$

on $P\mathcal{H}$, where $P = P^2 =: \mathbf{1} - \overline{P}$;

• H is invertible on $\mathcal{H} \Leftrightarrow \mathcal{F}_P(H)$ is invertible on $P\mathcal{H}$.

• Proj. out high energy degr. of freedom with $\mathcal{F}_{P_0}$, where $P_0 = |\varphi_0\rangle\langle\varphi_0| \otimes \chi[H_f < 1]$ (assuming that $E_1 - E_0 \geq 2$).

• Effective Hamiltonian

$$|\varphi_0\rangle\langle\varphi_0| \otimes H_{(0)}[z] \; := \; \mathcal{F}_{P_0}(H_g - z)$$

on $\mathcal{H}_{\rm red} = \chi[H_f < 1]\mathcal{F}$,

• $z \in \sigma(H_g) \Leftrightarrow 0 \in \sigma(H_{(0)}[z])$.

• Normal-order effective Hamiltonian,

$$H_{(0)} \; = \; E_{(0)} + T_{(0)}[H_f] + W_{(0)},$$

where $E_{(0)} \in \mathbb{C}$, $\partial_r T_{(0)}[r] \approx 1$, $W_{(0)} = \sum_{M+N\geq 1} W^{(0)}_{M,N}$, and

$$W^{(0)}_{M,N} \; = \; \int dk\, d\tilde{k} \, \prod_{j=1}^{M} a^*(k_j)\, w^{(0)}_{M,N}[H_f, k, \tilde{k}] \, \prod_{j=1}^{N} a(\tilde{k}_j)$$

• Important assumption:

$$J(k) \leq \omega(k)^{-(1-\mu)/2}, \text{ for some } \mu > 0.$$

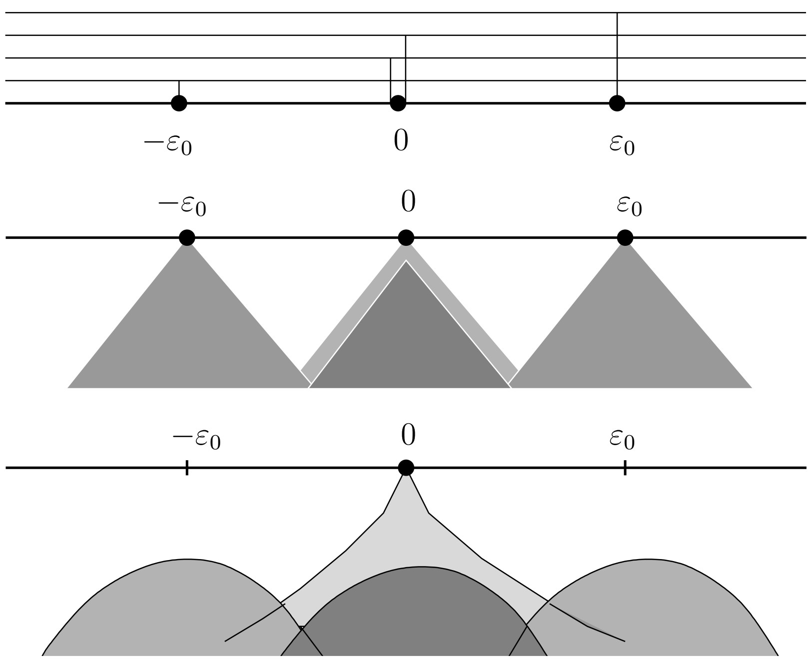
• For minimal coupling, $\mu = 0$.

• Banach space of Hamiltonians $(\mathcal{W}, \|\cdot\|_w)$,
then $\|H_{(0)} - H_f\|_w = \mathcal{O}(g)$.

• Renorm. map $\mathcal{R} : \mathcal{W} \supseteq \mathcal{D} \to \mathcal{W}$
with fixed point $H_f - z$;

• $\mathcal{R}$ is contracting on a certain
small ball in $\mathcal{W}$ about $H_f - z$;

• Iteration of $\mathcal{R}$ on $H_{(0)}$ flows to $H_f - z$
$\Rightarrow$ infrared asymptotic freedom.



TOP: Spectrum of L_0 with $H_{el} = \mathrm{diag}[\varepsilon_0/2, -\varepsilon_0/2]$.
MIDDLE: Spectrum of $L_0(i\vartheta)$, for $\vartheta > 0$.
BOTTOM: (appr.) Spectrum of $L_g(i\vartheta)$, for $\vartheta > 0$.

II Outline of the Renormalization Group

II.1 Motivation and Definitions

II.2 Full Model

The Hilbert space of the system is given by

$$\mathcal{H} \;:=\; \mathcal{H}_{el} \otimes \mathcal{F}\,, \tag{II.1}$$

where $\mathcal{H}_{el}$ is the Hilbert space of N electrons, and $\mathcal{F}$ is the photon Fock space.

$$\mathcal{H}_{el} \;:=\; \mathcal{A}_N\, L^2\Big[\,(\mathbb{R}^3 \times \mathbb{Z}_2)^N\,\Big]\,, \tag{II.2}$$

$$\mathcal{F} \;:=\; \bigoplus_{n=0}^{\infty} \mathcal{S}_n\, L^2\Big[\,(\mathbb{R}^3 \times \mathbb{Z}_2)^n\,\Big]\,. \tag{II.3}$$

The dynamics is generated by the Hamiltonian

$$H_\alpha \;:=\; \sum_{j=1}^{N} \Big[\vec{\sigma}_j \cdot \Big(-i\vec{\nabla}_{\vec{x}_j} - 2\pi^{1/2}\alpha^{3/2}\vec{A}_\kappa(\alpha\vec{x}_j)\Big)\Big]^2 \;+\; V_c(x) \;+\; H_f\,, \tag{II.4}$$

with units s.th. $\hbar = 1$, $m_{el} = 1/2$. $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ denote the Pauli matrices, $\alpha \approx 1/137$ is fine structure constant, and $\vec{A}(\vec{x})$ denotes the quantized vector potential of the transverse modes of the electromagnetic field in the Coulomb gauge, i.e.,

$$\vec{A}(\vec{x}) \;:=\; \sum_{\lambda=1,2} \int \frac{d^3k\, \kappa(|\vec{k}|/K)}{\pi\sqrt{2\omega(\vec{k})}} \Big\{\vec{\varepsilon}_\lambda(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} a^*_\lambda(\vec{k}) \;+\; \vec{\varepsilon}_\lambda(\vec{k})^* e^{i\vec{k}\cdot\vec{x}} a_\lambda(\vec{k})\Big\}\,, \tag{II.5}$$

where κ is an ultraviolet cutoff, $\omega(\vec{k}) := |\vec{k}|$, and $\vec{\varepsilon}_\lambda(\vec{k})$, $\lambda = 1, 2$, are photon polarization vectors satisfying

$$\vec{\varepsilon}_\lambda(\vec{k})^* \cdot \vec{\varepsilon}_\mu(\vec{k}) \;=\; \delta_{\lambda\mu} \,, \qquad \vec{k} \cdot \vec{\varepsilon}_\lambda(\vec{k}) \;=\; 0 \,, \qquad \text{for } \lambda, \mu = 1, 2. \tag{II.6}$$

Moreover, $a^*_\lambda(\vec{k})$, $a_\lambda(\vec{k})$ are standard creation- and annihilation operators on $\mathcal{F}$ obeying the *canonical commutation relations*

$$[a^\#_\lambda(\vec{k}_1), a^\#_\mu(\vec{k}_2)] \;=\; 0, \quad [a_\lambda(\vec{k}_1), a^*_\mu(\vec{k}_2)] \;=\; \delta_{\lambda\mu}\, \delta(\vec{k}_1 - \vec{k}_2) \,, \tag{II.7}$$

where $a^\# = a$ or a^*. The electrostatic interaction between N electrons and M nuclei is described by the Coulomb potential

$$V_c(x) \;:=\; \sum_{j=1}^{N} \sum_{m=1}^{M} \frac{-Z_m}{|\vec{x}_j - \vec{R}_m|} \;+\; \sum_{1 \le i < j \le N} \frac{1}{|\vec{x}_i - \vec{x}_j|} \,. \tag{II.8}$$

Finally, the Hamiltonian of the free quantized electromagnetic field is given by

$$H_f \;:=\; \sum_{\lambda=1,2} \int d^3k \; a^*_\lambda(\vec{k})\, \omega(\vec{k})\, a_\lambda(\vec{k}) \,. \tag{II.9}$$

II.3 Simplified Model for these Lectures

The Hilbert space we work with in these lectures is

$$\mathcal{H} \;:=\; \mathcal{H}_{el} \otimes \mathcal{F} \,, \tag{II.10}$$

where

- $\mathcal{H}_{el} := L^2(\mathbb{R}^3)$ is the Hilbert space of one spinless, nonrelativistic particle,
- $\mathcal{F} = \mathcal{F}_b[L^2(\mathbb{R}^3)]$ is the Fock space of a scalar massless field,
- the interaction is linear in creation and annihilation operators, i.e., we neglect quadratic terms,
- and we study the ground state energy rather than resonances ($\Rightarrow$ no complex dilatation necessary).

The dynamics is generated by the Hamiltonian

$$H_g \;:=\; H_0 \;+\; gW \tag{II.11}$$

$$H_0 \;:=\; H_{el} \otimes \mathbf{1} \;+\; \mathbf{1} \otimes H_f \;+\; gW \,, \tag{II.12}$$

$$H_{el} \;:=\; -\Delta + V(x) \,, \tag{II.13}$$

$$H_f \;:=\; \int d^3k \; a^*(k)\, \omega(k)\, a(k) \,. \tag{II.14}$$

Here, H_{el} is a Schrödinger operator assumed to obey

Hypothesis H- 1. *The potential* $V \in L^2 + L^\infty(\mathbb{R}^3; \mathbb{R})$ *(*$\Rightarrow V$ *is an infinitesimal perturbation of* Δ*) is such that*

$$E_0 := \inf \sigma[H_{el}] = 0 \,, \quad \ker[H_{el}] = \mathbb{C} \cdot \varphi_{el} \,, \quad \mathrm{dist}\Big\{ 0 \,,\; \sigma[H_{el}] \setminus \{0\} \Big\} = 2 \,. \tag{II.15}$$

According to our simplifying assumptions, the interaction is given by

$$W \equiv a^*(G) + a(G) := \int d^3k \, G(k) \otimes a^*(k) + G^*(k) \otimes a(k) \, , \tag{II.16}$$

where we assume the coupling functions $G(k)$ to be operators on $\mathcal{H}_{el}$, pointwise in $k \in \mathbb{R}^3$, obeying

Hypothesis H- 2.

$$J(k) := \max\left\{ \left\|(H_{el}+1)^{-1/2} \, G(k)\right\| \, , \ \left\|G(k) \, (H_{el}+1)^{-1/2}\right\| \right\} < \infty \, , \tag{II.17}$$

$$\Lambda_1 := \left(\int d^3k \, \left(1 + \omega(k)^{-1}\right) J(k)^2 \right)^{1/2} < \infty \, , \tag{II.18}$$

$$\Lambda_5 := \sup_{k \in \mathbb{R}^3} \left\{ \omega(k)^{1/2 - \mu/2} J(k) \right\} < \infty \, , \tag{II.19}$$

for some $\mu > 0$.

We remark that $\mu = 0$ in (II.5), so the model described in Subsection II.2 does not fulfill Hypothesis 2.

II.4 Relative Bounds on the Interaction

Our first task is to show that W is a relative H_0 form bounded perturbation. This justifies viewing H_g as a small deviation from H_0.

Lemma II.1.

$$\left\|(H_0+1)^{-1/2} \, W \, (H_0+1)^{-1/2}\right\| \leq 2\Lambda_1 \, . \tag{II.20}$$

Proof: Pick $\varphi, \psi \in \mathcal{H}$. Then, by Schwarz' inequality,

$$\begin{aligned} & \left|\langle \varphi \mid (H_0+1)^{-1/2} \, a(G) \, (H_0+1)^{-1/2} \, \psi \rangle\right| \qquad \text{(II.21)} \\ = & \left| \int dk \ \langle \varphi \mid (H_0+1)^{-1/2} \, G^*(k) \otimes a(k) \, (H_0+1)^{-1/2} \, \psi \rangle \right| \\ \leq & \int dk \ \left\|(G(k) \otimes 1) \, (H_0+1)^{-1/2} \, \varphi\right\| \ \left\|(1 \otimes a(k)) \, (H_0+1)^{-1/2} \, \psi\right\| \\ \leq & \ \|\varphi\| \int dk \ \left\|G(k)(H_{el}+1)^{-1/2} \otimes 1\right\| \ \left\|(1 \otimes a(k)) \, (H_0+1)^{-1/2} \, \psi\right\| \\ \leq & \ \|\varphi\| \left(\int dk \, \frac{J(k)^2}{\omega(k)} \right)^{1/2} \left(\int dk \, \omega(k) \ \left\|(1 \otimes a(k)) \, (H_0+1)^{-1/2} \, \psi\right\|^2 \right)^{1/2} \\ \leq & \ \Lambda_1 \, \|\varphi\| \ \left\|(1 \otimes H_f) \, (H_0+1)^{-1/2} \, \psi\right\| \\ \leq & \ \Lambda_1 \, \|\varphi\| \, \|\psi\| \, . \ \square \end{aligned}$$

II.5 The Feshbach Map and Pull-Through Formula

Let $\mathcal{H}$ be a Hilbert space, H a densely defined, closed operator on $\mathcal{H}$, $P = P^2 =: \overline{P}$, $\|P\| < \infty$ be a bounded projection, $z \in \mathbb{C}$, and define the **Feshbach operator**

$$\mathcal{F}(H - z) := P(H-z)P - PH\overline{P}(\overline{P}H\overline{P} - z)^{-1}\overline{P}HP \, . \tag{II.22}$$

Lemma II.2. *[Feshbach map]: Assume that*

$$\|(\overline{P}H\overline{P}-z)^{-1}\overline{P}\|\,,\ \|PH\overline{P}(\overline{P}H\overline{P}-z)^{-1}\|\,,\ \|(\overline{P}H\overline{P}-z)^{-1}\overline{P}HP\|\,,\ \|PHP\| \ <\ \infty\,. \tag{II.23}$$

Then

(a) $H-z$ *invertible on* $\mathcal{H}$ $\iff$ $\mathcal{F}_P(H-z)$ *invertible on* $P\mathcal{H}$*;*

(b) $H-z$ *invertible on* $\mathcal{H}$ $\implies$ $P(H-z)^{-1}P=\mathcal{F}_P(H-z)^{-1}P$*;*

(c) $(H-z)\psi=0, \psi\neq 0$ $\implies$ $P\psi\neq 0, \mathcal{F}_P(H-z)^{-1}P\psi=0.$

(d) Let $S_P := P-(\overline{P}H\overline{P}-z)^{-1}\overline{P}HP$*. Then*
$\mathcal{F}_P(H-z)\varphi=0, \varphi=P\varphi\neq 0$ $\implies$ $(H-z)\,S_P\varphi=0, S_P\varphi\neq 0.$

Proof: ./.

Another important ingredient of our analysis is the following Pull-Through formula.

Lemma II.3 (Pull-Through). *Let* $F:\mathbb{R}_{+0}\longrightarrow\mathbb{C},\ F(r)\ =\ \mathcal{O}(r=1)$*. Then* $F[H_f]$ *is defined on* $\mathcal{D}(H_f)$ *and*

$$a(k)F[H_f] \;=\; F[H_f+\omega(k)]a(k)\,, \tag{II.24}$$

$$F[H_f]a^*(k) \;=\; a^*(k)F[H_f+\omega(k)]\,. \tag{II.25}$$

Proof: Let $\psi\ =\ \prod_{j=1}^N a^*(k_j)\Omega$. Then

$$\begin{aligned} F[H_f]a^*(k)\psi \;&=\; F\left[\omega(k)+\sum_{j=1}^N\omega(k_j)\right]a^*(k)\psi \\ &=\; a^*(k)F\left[\omega(k)+\sum_{j=1}^N\omega(k_j)\right] \;=\; a^*(k)F[\omega(k)+H_f]\psi \end{aligned} \tag{II.26}$$

II.6 Elimination of High-Energy Degrees of Freedom

As a first application of the Feshbach map, we use it to eliminate the high-energy degrees of freedom of our spectral problem. To this end we choose the projection

$$P_0 \;:=\; \chi[H_0<1] \;=\; P_{el}\otimes\chi[H_f<1]\,, \tag{II.27}$$

where $P_{el}=|\varphi_{el}\rangle\langle\varphi_{el}|$.

Now we check the invertibility of $\overline{P}_0H_g\overline{P}_0-\lambda$ on $\mathrm{Ran}\,\overline{P}_0$, noting that $\overline{P}_0=\chi[H_0\geq 1]$. Observe that $\langle\varphi_{el}\otimes\Omega\mid H_g(\varphi_{el}\otimes\Omega)\rangle\ =\ 0$, thus $E_0(g)\leq 0$, and we may assume $\lambda\leq\frac{1}{2}$ henceforth. We construct the inverse of $\overline{P}_0H_g\overline{P}_0-\lambda$ on $\mathrm{Ran}\,\overline{P}_0$ by a Neumann series expansion,

$$\begin{aligned} &\left(\overline{P}_0H_g\overline{P}_0-\lambda\right)^{-1}\overline{P}_0 \\ &=\; \sum_{L=0}^\infty\left(\frac{\overline{P}_0}{H_0-\lambda}\right)\left\{-gW\left(\frac{\overline{P}_0}{H_0-\lambda}\right)\right\}^L \\ &=\; \sum_{L=0}^\infty\underbrace{\left(\frac{(H_0+1)^{1/2}\overline{P}_0}{H_0-\lambda}\right)}_{\|\cdot\|\leq 4}\left\{\underbrace{\left((H_0+1)^{-1/2}(-gW)(H_0+1)^{-1/2}\right)}_{\|\cdot\|\leq 2g\Lambda_1}\underbrace{\left(\frac{(H_0+1)\overline{P}_0}{H_0-\lambda}\right)}_{\|\cdot\|\leq 4}\right\}^L \\ &\qquad\underbrace{\left(\frac{\overline{P}}{(H_0+1)^{1/2}}\right)}_{\|\cdot\|\leq 2}\,. \end{aligned} \tag{II.28}$$

Thus the series is norm-convergent and

$$\left\| \left(\overline{P}_0 H_g \overline{P}_0 - \lambda\right)^{-1} \overline{P}_0 \right\| \;\leq\; 8 + \mathcal{O}(g)\,. \tag{II.29}$$

As a result, Lemma II.2 implies

Lemma II.4. *Let* $\chi_\rho := \chi[H_f < \rho]$ *and*

$$\mathcal{H}_{red} \;:=\; \mathrm{Ran}(\chi_1) \;=\; \chi[H_f < 1]\mathcal{F}\,. \tag{II.30}$$

Then $H - \lambda$ *is invertible iff* $H_{(0)}[\lambda]$ *is, where*

$$P_{el} \otimes \left(H_{(0)}[\lambda] - \lambda\right) \;:=\; \mathcal{F}_{P_0}(H_g - \lambda)\,, \tag{II.31}$$

so $H_{(0)}[\lambda]$ *is the following operator* $\in \mathcal{B}[\mathcal{H}_{red}]$,

$$\begin{aligned} \left(H_{(0)}[\lambda] - \lambda\right) \;&=\; \chi_1(H_f - \lambda + g\langle W\rangle_{el})\chi_1 \\ &\quad -\chi_1 \left\langle\, gW\,\overline{P}_0(\overline{P}_0 H_g \overline{P}_0 - \lambda)^{-1}\overline{P}_0\, gW \,\right\rangle_{el} \chi_1\,, \end{aligned} \tag{II.32}$$

where $\langle\,\cdot\,\rangle_{el} := \langle\varphi_{el}|\;\cdot\;\varphi_{el}\rangle$. *Moreover,* $H_{(0)}[\lambda]$ *can be expanded in a norm-convergent series,*

$$\begin{aligned} H_{(0)}[\lambda] \;&=\; \chi_1(H_f - \lambda) \;+\; g\chi_1\,\langle W\rangle_{el}\,\chi_1 \;-\; g^2\chi_1 \left\langle W \left(\frac{\overline{P}_0}{H_0 - \lambda}\right) W \right\rangle_{el} \chi_1 \\ &\quad + g^3\chi_1 \left\langle W \left(\frac{\overline{P}_0}{H_0 - \lambda}\right) W \left(\frac{\overline{P}_0}{H_0 - \lambda}\right) W \right\rangle_{el} \chi_1 \;-\; \ldots\,. \end{aligned} \tag{II.33}$$

II.7 Normal form of Hamiltonians

Our next goal is to write (recall $\chi_1 \equiv \chi_1(H_f) = [H_f < 1]$)

$$H_{(0)}[\lambda] \;=\; \chi_1\left(E_{(0)}[\lambda] \;+\; T_{(0)}[\lambda; H_f] \;+\; W_{(0)}[\lambda]\right)\chi_1\,, \tag{II.34}$$

where $E_{(0)}[\lambda] \in \mathbb{C}$ is an energy shift, $T_{(0)}[\lambda; H_f]$ is the new, renormalized, free (=unperturbed) Hamiltonian, and

$$W_{(0)}[\lambda] := \sum_{M+N\geq 1} W^{(0)}_{M,N}[\lambda]\,, \tag{II.35}$$

$$W^{(0)}_{M,N} := \int dk^{(\mu)} d\tilde{k}^{(N)}\; \underbrace{a^*(k^{(\mu)})}_{\prod_{j=1}^{\mu} a^*(k_j)}\; w^{(0)}_{M,N}[\lambda; H_f; k^{(\mu)}; \tilde{k}^{(N)}]\; a(\tilde{k}^{(N)}) \tag{II.36}$$

where the operators $T_{(0)}[\lambda; H_f]$, $w^{(0)}_{M,N}[\lambda; H_f; k^{(\mu)}; \tilde{k}^{(N)}]$ are defined for each $\lambda, k^{(\mu)}, \tilde{k}^{(N)}$ ptw. by the spectral theorem (functional calculus) for H_f.
Looking at $H_0 + gW - \lambda$, we see that

$$E_{(0)}[\lambda] \;=\; \Delta E_{(0)}[\lambda]\,, \tag{II.37}$$

$$T_{(0)}[\lambda; H_f] \;=\; H_f + \Delta T_{(0)}[\lambda; H_f]\,, \tag{II.38}$$

$$W_{(0)}[\lambda] \;=\; gW + \Delta W_{(0)}[\lambda]\,, \tag{II.39}$$

where $\Delta E_{(0)}[\lambda]$, $\Delta T_{(0)}[\lambda; H_f]$, and $\Delta W_{(0)}[\lambda] = \sum_{M+N\geq 1} \Delta W_{(0)}[\lambda]$ are generated from $H_{(0)}[\lambda]$ as given in the Neumann series (II.33).

To illustrate the method of deriving $\Delta E_{(0)}, \Delta T_{(0)}$, and $\Delta W_{(0)}$, we do a sample computation. In order g^2, we have

$$\left\langle W \left(\frac{\chi[H_0 \geq 1]}{H_0 - \lambda} \right) W \right\rangle_{el} = \tag{II.40}$$
$$\underbrace{\left\langle a^*(G) \left(\frac{\chi[H_0 \geq 1]}{H_0 - \lambda} \right) a^*(G) \right\rangle_{el}}_{=:X_1} + \left\langle a^*(G) \left(\frac{\chi[H_0 \geq 1]}{H_0 - \lambda} \right) a(G) \right\rangle_{el}$$
$$+ \left\langle a(G) \left(\frac{\chi[H_0 \geq 1]}{H_0 - \lambda} \right) a(G) \right\rangle_{el} + \underbrace{\left\langle a(G) \left(\frac{\chi[H_0 \geq 1]}{H_0 - \lambda} \right) a^*(G) \right\rangle_{el}}_{=:X_4} .$$

and we restrict our attention to X_1 and X_4. Due to Lemma II.3, $F[H_0]a^*(k) = a^*(k)F[H_0+\omega(k)]$. Thus

$$\begin{aligned} X_1 &= \int dk\, dk' \left\langle G(k) \otimes a^*(k) \left(\frac{\chi[H_0 \geq 1]}{H_0 - \lambda} \right) G(k') \otimes a^*(k') \right\rangle_{el} \\ &= \int dk\, dk' \left\langle G(k) \otimes a^*(k) a^*(k') \left(\frac{\chi[H_0 + \omega(k') \geq 1]}{H_0 + \omega(k') - \lambda} \right) G(k') \otimes 1 \right\rangle_{el} \\ &= \int dk\, dk'\, a^*(k) a^*(k')\, \tilde{w}_{2,0}[H_f; \lambda, k, k'] \, , \end{aligned} \tag{II.41}$$

where the integral kernel $\tilde{w}_{2,0}[r; \lambda, k, k']$ is given by

$$\tilde{w}_{2,0}[r; \lambda, k, k'] := \left\langle G(k) \left(\frac{\chi[H_{el} + r + \omega(k') \geq 1]}{H_{el} + r + \omega(k') - \lambda} \right) G(k') \right\rangle_{el} \tag{II.42}$$

Thus, we recognize X_1 as an additive contribution to $W_{2,0}^{(0)}[\lambda]$.
Similarly, additionally using the CCR, we convert X_4,

$$\begin{aligned} X_4 &= \int dk\, dk' \left\langle G(k) \otimes a(k) \left(\frac{\chi[H_0 \geq 1]}{H_0 - \lambda} \right) G(k') \otimes a^*(k') \right\rangle_{el} \\ &= \int dk\, dk' \left\langle G(k) \otimes a(k) a^*(k') \left(\frac{\chi[H_0 + \omega(k') \geq 1]}{H_0 + \omega(k') - \lambda} \right) G(k') \otimes 1 \right\rangle_{el} \\ &= \int dk\, dk' \left\langle G(k) \otimes \Big(\delta(k - k') + a^*(k') a(k) \Big) \left(\frac{\chi[H_0 + \omega(k') \geq 1]}{H_0 + \omega(k') - \lambda} \right) G(k') \otimes 1 \right\rangle_{el} \\ &= \int dk \left\langle G(k) \otimes 1 \left(\frac{\chi[H_0 + \omega(k) \geq 1]}{H_0 + \omega(k) - \lambda} \right) G(k) \otimes 1 \right\rangle_{el} \\ &\quad + \int dk\, dk' \left\langle G(k) \otimes a^*(k') \left(\frac{\chi[H_0 + \omega(k) + \omega(k') \geq 1]}{H_0 + \omega(k) + \omega(k') - \lambda} \right) G(k') \otimes a(k) \right\rangle_{el} \\ &= \Delta e_{(0)}[\lambda] \, + \, \Delta t_{(0)}[H_f; \lambda] \, + \int dk\, dk'\, a^*(k')\, \tilde{w}_{1,1}[H_f; \lambda, k, k']\, a(k) \, , \end{aligned} \tag{II.43}$$

where

$$\Delta e_{(0)}[\lambda] := \int dk \left\langle G(k) \left(\frac{\chi[H_{el} + \omega(k) \geq 1]}{H_{el} + \omega(k) - \lambda} \right) G(k) \right\rangle_{el} , \tag{II.44}$$

is a number that contributes additively to $\Delta E_{(0)}[\lambda]$,

$$\Delta t_{(0)}[r;\lambda] \;:=\; \int dk \,\Big\langle\, G(k)\,\Big(\frac{\chi[H_{el}+r+\omega(k)\geq 1]}{H_{el}+r+\omega(k)-\lambda} \;-\; \frac{\chi[H_{el}+\omega(k)\geq 1]}{H_{el}+\omega(k)-\lambda} \Big)\; G(k) \,\Big\rangle_{el} \;, \tag{II.45}$$

is a function contributing to $\Delta T_{(0)}[\lambda]$, having the property that

$$\Delta t_{(0)}[0;\lambda] \;=\; 0 \quad \text{and} \quad \|\partial_r \Delta t_{(0)}[r;\lambda]\|_\infty \;<\; \infty \,. \tag{II.46}$$

Finally

$$\tilde{w}_{1,1}[r;\lambda,k,k'] \;:=\; \Big\langle\, G(k)\,\Big(\frac{\chi[H_{el}+r+\omega(k)+\omega(k')\geq 1]}{H_{el}+r+\omega(k)+\omega(k')-\lambda} \Big)\; G(k') \,\Big\rangle_{el} \tag{II.47}$$

is an additive contribution to $\Delta W^{(0)}_{1,1}[\lambda]$.
We now appeal to the reader's imagination, that this normal-ordering –demonstrated on the examples of X_1 and X_4– can be systematically carried out for all terms in the Neumann series expansion (II.33).

II.8 Banach Space of Operators

We systematize the normal-ordering by introducing, for $\Delta := (\mu,\rho,\xi)$, where $0<\rho<1/16,\ 0<\xi<1,\ 0<\mu$,

$$\mathcal{W}'_\Delta := \mathbb{C}\oplus\mathcal{T}\oplus\Big(\bigoplus_{M+N\geq 1}\mathcal{W}_\Delta(M,N)\Big)\;, \tag{II.48}$$

where

$$\mathcal{T} := \{T:[0,1]\longrightarrow\mathbb{C} \mid \|\partial_r T\|_\infty<\infty,\ T(0)=0\}\;, \tag{II.49}$$

and, with $B_\rho := \{|k|<\rho\}$,

$$\mathcal{W}_\Delta(M,N) := \big\{ w_{M,N}:[0,1]\times B_1^M\times B_1^N \longrightarrow\mathbb{C} \mid \|w_{M,N}\|_\Delta<\infty \big\}\;, \tag{II.50}$$

$$\|w_{M,N}\|_\Delta := \max\big\{\xi^{-(M+N)}\|w_{M,N}\|^\infty_\Delta, (C_d\xi\sqrt{\rho})^{-(M+N)}\|\partial_r w_{M,N}\|_\Delta\big\} \tag{II.51}$$

$$\|w_{M,N}\|^{(\infty)}_\Delta := \sup_{[0,1]\times B_1^M\times B_1^N}\Big\{\Big|w_{M,N}[r;k^{(\mu)};\tilde{k}^{(N)}]\Big|\cdot\prod_{j=1}^M\omega(k_j)^{1/2-\mu/2}\prod_{j=1}^N\omega(\tilde{k}_j)^{1/2-\mu/2}\Big\} \tag{II.52}$$

$$\|w_{M,N}\|^{(1)}_\Delta := \sup_{[0,1]}\Big\{\int\Big|w_{M,N}[r;k^{(\mu)};\tilde{k}^{(N)}]\Big|\cdot\prod_{j=1}^M\frac{d^3k_j}{\omega(k_j)^{3/2+\mu/2}}\;\prod_{j=1}^N\frac{d^3\tilde{k}_j}{\omega(\tilde{k}_j)^{3/2+\mu/2}}\Big\} \tag{II.53}$$

Writing $\underline{W} := (\mathcal{W}_{M,N})_{M+N\geq 1}$, we have

$$\mathcal{W}'_\Delta := \Big\{ (E, T, \underline{W}) \in \mathbb{C} \oplus \mathcal{T} \oplus \bigoplus_{M+N\geq 1} \mathcal{W}_\Delta(M+N) \ \Big|$$

$$\|(E, T, \underline{W})\|'_\Delta := \max\Big\{ |E|,\ \|\partial_r T\|_\infty,\ \sup_{M+N\geq 1} \|W_{M+N}\|_\Delta \Big\} < \infty \Big\} . \qquad \text{(II.54)}$$

To every element $(E, T, \underline{W}) \in \mathcal{W}'_\Delta$ corresponds an operator $H \in \mathcal{B}(\mathcal{H}_{red})$ (recall $\mathcal{H}_{red} = \mathrm{Ran}\ \chi[H_f < 1]$), $\mathcal{W}'_\Delta \longrightarrow \mathcal{B}(\mathcal{H}_{red})$, in **normal form,**

$$H \ = \ \chi_1 (E + T[H_f] + W) \chi_1 \ , \quad W \ = \ \sum_{M+N\geq 1} W_{M,N} \qquad \text{(II.55)}$$

$$W_{M,N} \ = \ \int dk^{(M)} d\tilde{k}^{(N)} a^*(k^{(\mu)})\, w_{M,N}[H_f; k^{(\mu)}; \tilde{k}^{(N)}]\, a(\tilde{k}^{(N)}) \, . \qquad \text{(II.56)}$$

We identify $H \equiv (E, T, \underline{W})$. Let $D_{1/2} := \{ z \in \mathbb{C} \mid |z| < 1/2 \}$.

$$\mathcal{W}_\Delta := \left\{ H[\cdot] : D_{1/2} \longrightarrow \mathcal{W}'_\Delta \ \middle|\ H[z] \text{ analytic, } \|H[\cdot]\|_\Delta := \sup_{z\in D_{1/2}} \|H[z]\|'_\Delta < \infty \right\} . \qquad \text{(II.57)}$$

From (II.37)–(II.39) we conclude that

$$H_{(0)}[\cdot] \in \mathcal{W}_\Delta \qquad \text{(II.58)}$$

and that

$$\|H_{(0)}[z] - H_f\|'_\Delta \ = \ \|(\Delta E_{(0)}[z],\ \Delta T_{(0)}[z, r] - r\ ,\ \Delta W_{(0)}[z])\|'_\Delta \ \leq \ \mathcal{O}(g) \, . \qquad \text{(II.59)}$$

Defining, for $\frac{1}{16} > \delta, \varepsilon > 0$, a polydisc in $\mathcal{W}_\Delta$,

$$\mathcal{B}(\delta, \varepsilon) := \{ (E, T, \underline{W})[\cdot] \in \mathcal{W}_\Delta \mid \|\partial_r T[z] - 1\| - \infty \leq \delta,\ E[z]| \leq \varepsilon,\ \|\underline{W}[z]\|_\Delta \leq \varepsilon \} \ , \qquad \text{(II.60)}$$

(II.59) may be expressed as

$$H_{(0)}[z] \ \in \ \mathcal{B}(cg, cg) \, . \qquad \text{(II.61)}$$

II.9 The Renormalization Map $\mathcal{R}_\rho$

$$\text{Let } 0 < \delta,\ 0 < \varepsilon < 1,\ \rho \leq \frac{1}{16} \text{ and } (E, T, \underline{W}) \in \mathcal{B}(\delta, \varepsilon) \, . \qquad \text{(II.62)}$$

Denote

$$\chi_\rho := \chi_\rho[H_f] \ \equiv \ \chi[H_f < \rho] \, , \quad \overline{\chi}_\rho \ = \ 1 - \chi_\rho \, , \qquad \text{(II.63)}$$

and assume that $|z - E[z]| \leq \rho/2$.

Lemma II.5.

$$\left\| \frac{(H_f + \rho)}{T[H_f; z] + E[z] - z} \right\| \leq 8 \, . \qquad \text{(II.64)}$$

Proof: Since $|\partial_r T - 1| \leq \delta \leq \frac{1}{16}$, $T[0; z] = 0$

$$\Longrightarrow |T[z; r] + E[z] - z| \geq (1 - \delta) \cdot r - \frac{\rho}{2} \geq \left(1 - \delta - \frac{1}{2} \right) \left(\frac{r + \rho}{2} \right) . \quad \square \qquad \text{(II.65)}$$

Lemma II.6. *If* $\|\underline{W}\|'_\Delta \leq \varepsilon$ *then*

$$\big\|(H_f+\rho)^{-1/2}\chi_1[H_f]W_{M,N}\chi_1[H_f](H_f+\rho)^{-1/2}\big\| \leq \varepsilon\cdot\varepsilon^{M+N}\cdot\big(C_d\Gamma(1+\mu)\big)^{M+N}\cdot\rho^{-\delta_{M,0}}\cdot\rho^{-\delta_{N,0}}(M!)^{-1/2}(N!)^{-1/2}\,. \tag{II.66}$$

Proof: Similar to Lemma II.1.

Lemma II.7. $\overline{\chi}_\rho H[z]\overline{\chi}_\rho - z$ *is invertible on Ran* $\overline{\chi}_\rho$.

Proof: By Lemma II.6,

$$\big\|(H_f+\rho)^{-1/2}\chi_1 W\chi_1(H_f+\rho)^{-1/2}\big\| \;\leq\; \big(1+\mathcal{O}(\xi)\big)\left(\frac{\varepsilon}{\rho^{1/2}}\right)\,. \tag{II.67}$$

Hence, Lemma II.5 implies the norm-convergence of the Neumann series,

$$\begin{aligned} &\big\|(\overline{\chi}_\rho H\overline{\chi}_\rho - z)^{-1}\overline{\chi}_\rho\big\| \\ &= \sum_{L=0}^\infty \left\|\left(\frac{\overline{\chi}_\rho}{(H_f+\rho)^{1/2}}\right)\left[\left(\frac{\overline{\chi}_\rho(H_f)\cdot(H_f+\rho)^{1/2}}{T[H_f;z]+E[z]-z}\right)\cdot\big((H_f+\rho)^{1/2}\chi_1(-W)\chi_1(H_f+\rho)^{1/2}\big)\right]^L \left(\frac{\overline{\chi}_\rho\cdot(H_f+\rho)^{1/2}}{T[H_f;z]+E[z]-z}\right)\right\| \\ &\leq \sum_{L=0}^\infty \frac{8}{2\rho}\cdot\left[\frac{8\varepsilon(1+\mathcal{O}(\xi))}{\rho^{1/2}}\right]^L \;\leq\; \frac{4}{\rho}\big(1+\mathcal{O}(\varepsilon\rho^{-1/2})\big)\;.\;\square \end{aligned} \tag{II.68}$$

Lemma II.7 tells us that we may apply $\mathcal{F}_{\chi_\rho}$ to $H[z]-z$:

$$\widetilde{H}[z]-z := \mathcal{F}_{\chi_\rho}[H[z]-z]\,. \tag{II.69}$$

We then apply a normal-ordering procedure as in Section II.7 to obtain

$$\widetilde{H}[z]-z := \chi_\rho\left(\widetilde{E}[z]-z+\widetilde{T}(z,H_f)+\widetilde{W}[z]\right)\chi_\rho\,, \tag{II.70}$$

$$\widetilde{W}[z] \;=\; \sum_{M+N\geq 1}\widetilde{W}_{M,N}[z]\,, \tag{II.71}$$

$$\widetilde{W}_{M,N}[z] \;=\; \int dk^{(\mu)}d\tilde{k}^{(N)}\; a^*\left(k^{(\mu)}\right)\tilde{w}_{M,N}\left[z;H_F;k^{(\mu)};\tilde{k}^{(N)}\right]a\left(\tilde{k}^{(N)}\right)\,, \tag{II.72}$$

with

$$\begin{aligned} &\left|\widetilde{W}_{M,N}\left[z;r;k^{(\mu)};\tilde{k}^{(N)}\right]\right| \\ &\leq \varepsilon\left(1+\mathcal{O}(\varepsilon/\rho^{1/2})\right)\cdot\xi^{M,N}\cdot\textstyle\prod_{j=1}^M\omega(k_j)^{-1/2+\mu/2}\prod_{j=1}^N\omega(\tilde{k}_j)^{-1/2+\mu/2}\,, \end{aligned} \tag{II.73}$$

using $\|w_{M,N}\|_\Delta^{(\infty)} \leq \varepsilon$. To get back to $\mathcal{H}_{red} = \mathrm{Ran}\;\chi_1$ from $\mathrm{Ran}\;\chi_\rho$, we rescale the photon momenta, $k\to\rho k$, by means of a unitary Γ_ρ, so

$$\frac{1}{\rho}\Gamma_\rho\,H_f\,\Gamma_\rho^* \;=\; H_f \quad\Longrightarrow\quad \Gamma_\rho\,\chi[H_f<\rho]\,\Gamma_\rho^* \;=\; \chi[\rho H_f<\rho] \;=\; \chi[H_f<1]\,. \tag{II.74}$$

One easily checks that

$$W_{(1)} := \frac{1}{\rho}\Gamma_\rho \widetilde{W} \Gamma_\rho^* = \sum_{M+N\geq 1} W_{M,N}^{(1)} ,$$

with coupling functions

$$w_{M+N}^{(1)}\left[z; r, k^{(\mu)}, \tilde{k}^{(N)}\right] := \rho^{3/2(M+N)-1} \tilde{w}_{M+N}\left[z; \rho H_f, \rho k^{(\mu)}, \rho\tilde{k}^{(N)}\right] . \tag{II.75}$$

The key point about assuming $\mu > 0$ is that

$$\begin{aligned} &\left| w_{M+N}^{(1)}\left[z; r, k^{(\mu)}, \tilde{k}^{(N)}\right]\right| \\ &\leq \varepsilon \cdot \left[1 + \mathcal{O}\left(\varepsilon\rho^{-1/2}\right)\right] \cdot \xi^{M+N} \cdot \rho^{3/2(M+N)-1} \cdot \prod_{j=1}^M \omega(\rho k_j)^{-1/2+\mu/2} \prod_{j=1}^N \omega(\rho\tilde{k}_j)^{-1/2+\mu/2} \\ &= \varepsilon\left[1 + \mathcal{O}\left(\varepsilon\rho^{-1/2}\right)\right] \cdot \xi^{M+N} \cdot \underbrace{\rho^{\left(1+\frac{\mu}{2}\right)(M+N)-1}}_{\leq \rho^{\mu/2},\ \text{since}\ \underline{M+N\geq 1}} \prod_{j=1}^M \omega(k_j)^{-1/2+\mu/2} \omega(\tilde{k}_j)^{-1/2+\mu/2} \end{aligned} \tag{II.76}$$

$$\Longrightarrow \left\| w_{M+N}^{(1)}[z] \right\|_\Delta^{(\infty)} \leq \varepsilon \cdot \rho^{\mu/2} \cdot \underbrace{\left[1 + \mathcal{O}\left(\varepsilon\rho^{-1/2}\right)\right].}_{\leq 2} \tag{II.77}$$

We now define the renormalization map

$$\mathcal{R}_\rho : \mathcal{W}_\Delta \longrightarrow \mathcal{W}_\Delta , \tag{II.78}$$

by

$$H[z] \mapsto \mathcal{R}_\rho(H)[z] := \frac{1}{\rho}\Gamma_\rho \mathcal{F}_{\chi_\rho}\left(H\left[Z^{-1}(z)\right] - Z^{-1}(z)\right) \Gamma_\rho^* - z , \tag{II.79}$$

where $Z : \{|z - E[z]| < \rho/2\} \longrightarrow D_{1/2}$, $Z(\zeta) := \frac{1}{\rho}(\rho - E[\zeta])$ is a suitable rescaling of the spectral parameter, z, to project out an unstable direction of the flow generated by $\mathcal{R}$. Observe that $\mathcal{R}_\rho$ has the isospectral property:

$$H\left[Z^{-1}(z)\right] - Z^{-1}(z) \text{ invertible on } \mathcal{H}_{red} \Leftrightarrow \mathcal{R}_\rho(H)[z] - z \text{ invertible on } \mathcal{H}_{red} . \tag{II.80}$$

The main fact about $\mathcal{R}_\rho$ is its contraction property:

Theorem II.8. *$\mathcal{R}_\rho$ is defined on $\mathcal{B}(\delta, ve)$, for $0 < \delta,\ \varepsilon \leq 1/16$, and, for $\rho^\mu < 1/2$,*

$$\mathcal{R}\rho : \mathcal{B}(\delta, \varepsilon) \longrightarrow \mathcal{B}\left(\delta + \frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right) . \tag{II.81}$$

To apply $\mathcal{R}_\rho$ iteratively, we define $H_{(n)}[z] := \mathcal{R}^n(H_{(0)})[z]$. According to Theorem II.8, we have

$$H_{(0)} \in \mathcal{B}(cg, cg) \Longrightarrow H_{(1)} \in \mathcal{B}\left(cg + \frac{cg}{2}, \frac{cg}{2}\right) \Longrightarrow \ldots \Longrightarrow H_{(n)} \in \mathcal{B}\left(cg \sum_{k=0}^n 2^{-k}, cg2^{-n}\right) . \tag{II.82}$$

Since we have chosen an appropriate topology, $\{z_*\} := \bigcap_{n=1}^\infty \left(Z_1^{-1} \circ \ldots \circ Z_n^{-1}\right)(D_{1/2})$ defines a number in $\mathbb{R}$ which turns out to be the perturbed eigenvalue sought for. Indeed, $H_{(n)}$ converges in the sense of (II.82), and the limit contains no interaction term anymore,

$$H_{(n)} \overset{\|\cdot\|_\Delta}{\longrightarrow} H_{(\infty)}[z_*] = E_{(\infty)}[z_*] + T_{(\infty)}(H_f; [z_*]) . \tag{II.83}$$

This implies, in particular, that the vacuum vector is an eigenvector of $H_{(\infty)}[z_*]$,

$$H_{(\infty)}[z_*]\,\Omega \;=\; E_{(\infty)}[z_*]\Omega\,. \tag{II.84}$$

We can now recover the original eigenvector of $H_g - z_*$ by successively applying the operator S_{χ_ρ} from Lemma II.2(d).

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