

**VECTOR STOCHASTIC INTEGRALS
AND THE FUNDAMENTAL THEOREMS
OF ASSET PRICING**

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Abstract. This paper deals with the foundations of the stochastic mathematical finance, and it has three main purposes:

1. We present a self-contained construction of the *vector stochastic integral* $H \bullet X$ with respect to a d -dimensional semimartingale $X = (X_t^1, \dots, X_t^d)$. This notion is more general than the *componentwise stochastic integral* $\sum_{i=1}^d H^i \bullet X^i$.

2. We show that the vector stochastic integrals are important in the mathematical finance. To be more precise, the notion of the componentwise stochastic integral is insufficient in the *First* and the *Second Fundamental Theorems of Asset Pricing*.

3. We prove the Second Fundamental Theorem of Asset Pricing in the general setting, i.e. in the continuous-time case for a general multidimensional semimartingale. The proof is based on the martingale techniques and, in particular, on the properties of the vector stochastic integral.

Key words and phrases. Vector stochastic integral, componentwise stochastic integral, σ -martingales, processes with independent increments, arbitrage, completeness, representation of local martingales, First Fundamental Theorem of Asset Pricing, Second Fundamental Theorem of Asset Pricing.

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Contents

1. Introduction	2
2. Notations and Definitions	11
3. Construction of the Vector Stochastic Integral	15
4. Properties of the Vector Stochastic Integral	22
5. Sigma-Martingales	33
6. Vector Stochastic Integrals in the First Fundamental Theorem of Asset Pricing ..	40
7. Second Fundamental Theorem of Asset Pricing	43
References	48
Index of Notations	51
Index of Terms	53

1 Introduction

1. Vector stochastic integrals. The notion of the stochastic integral was introduced by K. Itô [18] who constructed the stochastic integral with respect to a Brownian motion. H. Kunita and S. Watanabe [28] extended the notion of the stochastic integral to the square integrable martingales. C. Doléans-Dade and P.-A. Meyer [12] constructed the stochastic integral with respect to a continuous semimartingale. Further developments (see the paper [32] by P.-A. Meyer) were related to the case, where the integrand H is a predictable (locally) bounded process and the integrator X is a semimartingale (not necessarily continuous).

J. Jacod [19] constructed the stochastic integral with respect to a semimartingale for unbounded predictable integrands that satisfy some integrability conditions. The space of integrands considered by J. Jacod is in a sense the most general one, and the stochastic integral cannot be constructed for a larger class of integrands. This construction of the vector stochastic integral is also cited in the book [11; Ch. VIII, §3] by C. Dellacherie and P.-A. Meyer.

The method of J. Jacod is based on the characterization of jumps of a semimartingale. Another approach, which leads to the equivalent definition of the stochastic integral, was proposed by C.S. Chou, P.-A. Meyer and C. Stricker [7].

In all the papers mentioned above, the integrator X is a one-dimensional process. The generalization for a multidimensional semimartingale $X = (X^1, \dots, X^d)$ seems to be obvious. The space of integrands is taken as the set of processes $H = (H^1, \dots, H^d)$ such that, for each $i = 1, \dots, d$, H^i is integrable with respect to X^i . The stochastic integral is defined as the sum $\sum_{i=1}^d H^i \bullet X^i$ of one-dimensional stochastic integrals. In what follows, we will call this notion the *componentwise stochastic integral*.

However, the notion of the componentwise stochastic integral is found to be insufficient in some areas of the stochastic analysis. This was noticed by L. Galtchouk [15] who showed that the space of componentwise stochastic integrals (with respect to a fixed multidimensional semimartingale X) is not necessarily closed in the appropriate topology. We cite the corresponding example in Section 6 of this paper (see Example 6.4).

In order to obtain a closed space of stochastic integrals, one should generalize the notion of the componentwise stochastic integral. Such a generalization was proposed by J. Jacod [20]. He introduced the *vector stochastic integral* with respect to a multidimensional semimartingale. This construction yields a closed space of stochastic integrals as shown by J. Mémin [30].

The vector stochastic integral in [20] is constructed in the implicit form as opposed to the above mentioned papers, where the stochastic integral was constructed first for "simple" integrands, and then extended to general integrands through a limit procedure. The explicit approach to the vector stochastic integration can be found in the book [36; Ch. VII, §1a] by A.N. Shiryaev. However, the construction of the integral in [36] is given with no proofs. In this paper, we take the same approach to the vector stochastic integration as in [36] and provide complete proofs (see Sections 3, 4).

Note that the vector stochastic integral $H \bullet X$ of a d -dimensional process H with respect to a d -dimensional semimartingale X is a one-dimensional process. The word "vector" reflects the fact that both H and X are vector-valued processes. We hope that this will not lead to a confusion with the *vector-valued stochastic integral* $(H^1 \bullet X^1, \dots, H^d \bullet X^d)$.

The construction of the vector stochastic integral is a bit complicated for two reasons:

- I. It takes into account the "interference" of different components of a multidimensional process X .
- II. Even in the one-dimensional case, this construction differs from most of the usual constructions since the space of integrands H includes the processes that are not locally bounded.

However, the notion of the vector stochastic integral is necessary for many problems both in the mathematical finance and in the "pure" stochastic analysis. The reason is as follows. Unlike simpler constructions, this notion yields a closed space of stochastic integrals. In particular, the closedness of the space of integrals is the key point in the proofs of the *First* and the *Second Fundamental Theorems of Asset Pricing*.

The description of the vector stochastic integral can be summarized as follows. If the componentwise stochastic integral of H with respect to X exists, then the vector stochastic integral also exists and the integrals coincide. However, it may happen that the componentwise stochastic integral does not exist, while the vector stochastic integral is well defined (see Section 6). Moreover, the space of the vector stochastic integrals (with respect to a fixed semimartingale X) is closed in the appropriate topology (see Section 4).

It is worth noting that the theory of stochastic integration is still in progress. One of the modern trends deals with the construction of stochastic integrals with respect to processes, which are not semimartingales. A typical example of such a process is a fractional Brownian motion. The construction of a stochastic integral with respect to a fractional Brownian motion is described, for example, in the paper [37] by A.N. Shiryaev. Note that if X is not a semimartingale, then the space of X -integrable processes cannot include *all* the locally bounded predictable processes. This is a consequence of the Bichteler-Dellacherie-Mokobodski theorem (see [2]).

Another way to extend stochastic integration is to consider infinite-dimensional processes X . If X takes values in a Hilbert space, then the corresponding construction is similar to that in the finite-dimensional case (see the book [31] by M. Métivier). However, the construction of a stochastic integral with respect to a Banach-valued process X is

still far from being complete (see the paper [3] by T. Björk, G.Di. Masi, Yu.M. Kabanov and W. Runggaldier).

2. The First Fundamental Theorem of Asset Pricing. In the simplest setting, this theorem asserts that a model of a financial market is arbitrage-free if and only if there exists an equivalent martingale measure. Thus, the theorem establishes the equivalence between two objects of a completely different nature: the notion of the arbitrage arises from practice (informally, arbitrage means an opportunity to make money from nothing), while the existence of an equivalent martingale measure is a purely mathematical notion.

It is reasonable to assume that real financial markets satisfy the no-arbitrage condition (however, this is not completely true; see [38]). Then the First Fundamental Theorem of Asset Pricing opens the way for the martingale methods to be used in the mathematical finance.

The First Fundamental Theorem of Asset Pricing has different formulations in accordance with the level of generality.

Let us first consider the discrete-time case. Let $X = (X_n^1, \dots, X_n^d)_{0 \leq n \leq N}$ be an adapted process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{0 \leq n \leq N}, \mathbf{P})$. The process X is interpreted as a discounted price process of d assets on a securities market (so that X_n^i is the discounted price of the asset i at the time n). The σ -field \mathcal{F}_n is interpreted as the “information on the market up to the time n ”. In what follows, we will say that $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{0 \leq n \leq N}, \mathbf{P}; X)$ is a *model* of a financial market.

Definition 1.1. A *self-financing strategy* π is a pair (x, H) , where $x \in \mathbb{R}$ (x is interpreted as the initial capital of the strategy π) and $H = (H_n^1, \dots, H_n^d)_{0 \leq n \leq N}$ is a (\mathcal{F}_n) -predictable process, i.e., for any $n = 1, \dots, N$, H_n is \mathcal{F}_{n-1} -measurable and H_0 is \mathcal{F}_0 -measurable. In what follows, we will omit the words “self-financing” and simply say *strategy*.

The discounted *capital process* of a strategy $\pi = (x, H)$ is given by

$$V_n^\pi = x + \sum_{i=1}^d \sum_{k=1}^n H_k^i (X_k^i - X_{k-1}^i), \quad n = 0, \dots, N.$$

For the financial interpretation of the above definition, see the book [36; Ch. V, §1a] by A.N. Shiryaev.

Definition 1.2. A strategy $\pi = (x, H)$ realizes *arbitrage* if

- i) $x = 0$;
- ii) $V_N^\pi \geq 0$ \mathbf{P} -a.s.;
- iii) $\mathbf{P}\{V_N^\pi > 0\} > 0$.

A model satisfies the *No-Arbitrage* condition (or a model is *arbitrage-free*) if such a strategy does not exist. Notation: (NA).

Proposition 1.3 (I FTAP in the discrete time). *A model is arbitrage-free if and only if there exists an equivalent martingale measure, i.e. $\mathbf{M}(\mathbf{P}) \neq \emptyset$, where*

$$\mathbf{M}(\mathbf{P}) = \{\mathbf{Q} \sim \mathbf{P} : X \text{ is a } (\mathcal{F}_n, \mathbf{Q})\text{-martingale}\}.$$

Diagrammatically,

$$(NA) \iff M(\mathbf{P}) \neq \emptyset.$$

This result was proved by J.M. Harrison, S.R. Pliska [17] (for a finite Ω) and by R.C. Dalang, A. Morton, W. Willinger [8] (in the general case). Simple proofs of this result can be found in the papers: [22] by J. Jacod and A.N. Shiryaev, [25] by Yu.M. Kabanov, [26] by Yu.M. Kabanov and D.O. Kramkov, [27] by Yu.M. Kabanov and C. Stricker as well as in the book [36; Ch. V, §2e] by A.N. Shiryaev.

Let us now pass on to the continuous-time case. Let $X = (X_t^1, \dots, X_t^d)_{t \geq 0}$ be a semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$. The filtration (\mathcal{F}_t) is assumed to be right-continuous. The collection $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P}; X)$ is called a continuous-time *model* of a financial market.

Definition 1.4. A (self-financing) *strategy* π is a pair (x, H) , where $x \in \mathbb{R}$ and $H = (H_t^1, \dots, H_t^d)_{t \geq 0}$ is a X -integrable process, i.e. there exists a *vector stochastic integral* $H \bullet X$ (see Definition 3.9).

The discounted *capital process* of a strategy $\pi = (x, H)$ is given by

$$V_t^\pi = x + (H \bullet X)_t, \quad t \geq 0.$$

Definition 1.5. A strategy $\pi = (x, H)$ realizes *arbitrage* if

- i) $x = 0$;
- ii) there exists a constant a such that

$$\mathbf{P}\{\forall t \geq 0, V_t^\pi \geq a\} = 1;$$

- iii) there exists a limit

$$V_\infty^\pi := \lim_{t \rightarrow \infty} V_t^\pi \quad \mathbf{P}\text{-a.s.};$$

- iv) $V_\infty^\pi \geq 0$ \mathbf{P} -a.s.;
- v) $\mathbf{P}\{V_\infty^\pi > 0\} > 0$.

A model satisfies the *No-Arbitrage* condition (or a model is *arbitrage-free*) if such a strategy does not exist. Notation: (NA).

Remark. If one eliminates condition ii) from the above definition (the so-called *admissibility* restriction), then arbitrage can be constructed in almost all the classical models. This is done as follows. Let, for example, X be a Brownian motion started at zero (the Bachelier model). Take

$$x = 0, \quad \tau = \inf\{t \geq 0 : X_t = 1\}, \quad H_t = I(t \leq \tau).$$

Then properties i), iii), iv), v) of the above definition are satisfied for the strategy $\pi = (x, H)$. On the other hand, it is reasonable to assume that the Brownian motion is arbitrage-free. \square

It turns out that in the continuous-time setting, the condition (NA) does not guarantee the existence of an equivalent martingale measure. Moreover, it does not guarantee even the existence of an equivalent local martingale measure (see [9; Example 7.7]). Therefore, in order to obtain the First Fundamental Theorem of Asset Pricing in the continuous-time case, one should replace the condition (NA) by a stronger one. The following modifications of the no-arbitrage property were introduced by F. Delbaen and W. Schachermayer in [9].

Definition 1.6. A sequence of strategies $\pi_k = (x_k, H_k)$ realizes *free lunch with vanishing risk* if

- i) for each k , $x_k = 0$;
- ii) for each k , there exists a constant a_k such that

$$\mathbb{P}\{\forall t \geq 0, V_t^{\pi_k} \geq a_k\} = 1;$$

- iii) for each k , there exists a limit

$$V_\infty^{\pi_k} := \lim_{t \rightarrow \infty} V_t^{\pi_k} \quad \mathbb{P}\text{-a.s.};$$

- iv) for each k ,

$$V_\infty^{\pi_k} \geq -\frac{1}{k} \quad \mathbb{P}\text{-a.s.};$$

- v) there exist constants $\delta_1 > 0$, $\delta_2 > 0$ such that, for each k ,

$$\mathbb{P}\{V_\infty^{\pi_k} > \delta_1\} > \delta_2.$$

A model satisfies the *No Free Lunch with Vanishing Risk* condition if such a sequence of strategies does not exist. Notation: (NFLVR).

Definition 1.7. A sequence of strategies $\pi_k = (x_k, H_k)$ realizes *free lunch with bounded risk* if it satisfies conditions i), ii), iii) of Definition 1.6 as well as the following conditions:

- iv)' there exists a constant a such that, for each k ,

$$\mathbb{P}\{\forall t \geq 0, V_t^{\pi_k} \geq a\} = 1;$$

- v) there exist constants $\delta_1 > 0$, $\delta_2 > 0$ such that, for each k ,

$$\mathbb{P}\{V_\infty^{\pi_k} > \delta_1\} > \delta_2,$$

and, for each $\delta > 0$,

$$\mathbb{P}\{V_\infty^{\pi_k} < -\delta\} \xrightarrow{k \rightarrow \infty} 0.$$

A model satisfies the *No Free Lunch with Bounded Risk* condition if such a sequence of strategies does not exist. Notation: (NFLBR).

Proposition 1.8 (I FTAP in the continuous time). *Suppose that X is locally bounded. Then each of the conditions (NFLVR), (NFLBR) is equivalent to the existence of an equivalent local martingale measure, i.e. $\text{LM}(\mathbb{P}) \neq \emptyset$, where*

$$\text{LM}(\mathbb{P}) = \{\mathbb{Q} \sim \mathbb{P} : X \text{ is a } (\mathcal{F}_t, \mathbb{Q})\text{-local martingale}\}.$$

Diagrammatically,

$$(NA) \iff (NFLVR) \iff (NFLBR) \iff \text{LM}(\mathbb{P}) \neq \emptyset \iff \text{M}(\mathbb{P}) \neq \emptyset.$$

The proof is given in the paper [9] by F. Delbaen and W. Schachermayer.

Remark. The condition $\text{LM}(\mathbb{P}) \neq \emptyset$ does not imply that $\text{M}(\mathbb{P}) \neq \emptyset$. This can be seen from the example $X = \|B\|^{-1}$, where B is a 3-dimensional Brownian motion started at a point that is not equal to zero. For this example, $\text{LM}(\mathbb{P}) = \{\mathbb{P}\}$, while $\text{M}(\mathbb{P}) = \emptyset$. \square

The above proposition is rather general, but it contains quite an unpleasant assumption that X is locally bounded. In many models of the mathematical finance, X is not locally bounded. Such are, for example, the exponential Lévy models, i.e. the models of the form $X_t = X_0 e^{L_t}$, where L is a Lévy process. One could expect that the assumption of the local boundedness can easily be eliminated from the above proposition. The surprising thing is that without this assumption, neither of the conditions $(NFLVR)$, $(NFLBR)$ is sufficient for the existence of an equivalent local martingale measure. This was discovered by F. Delbaen and W. Schachermayer [10]. Moreover, they found the way to reformulate the First Fundamental Theorem of Asset Pricing in such a way that it also holds for processes that are not locally bounded.

Definition 1.9. A semimartingale $X = (X_t)_{t \geq 0}$ is called a σ -martingale if there exists a sequence of predictable sets $D_n \subseteq \Omega \times \mathbb{R}_+$ such that $D_n \subseteq D_{n+1}$, $\bigcup D_n = \Omega \times \mathbb{R}_+$ and, for any $n \in \mathbb{N}$, the process $X^{D_n} := I_{D_n} \bullet X$ is a uniformly integrable martingale.

A d -dimensional process $X = (X_t^1, \dots, X_t^d)_{t \geq 0}$ is called a σ -martingale if each of its components is a σ -martingale.

Another equivalent definition of this class is the following (for the proof of the equivalence, see Section 5).

Definition 1.10. A process $X = (X_t^1, \dots, X_t^d)_{t \geq 0}$ is called a σ -martingale if there exist a local martingale $M = (M_t^1, \dots, M_t^d)_{t \geq 0}$ and a process $H = (H_t^1, \dots, H_t^d)_{t \geq 0}$ such that, for each i , H^i is M^i -integrable and $X^i = X_0^i + H^i \bullet M^i$.

This class of processes was introduced by C.S. Chou [6] and M. Émery [14] under the name “semimartingales de la classe Σ_m ”. The definition they used is Definition 1.10. F. Delbaen and W. Schachermayer [10] called these processes “ σ -martingales” (they used the same definition). Definition 1.9 was proposed by T. Goll and J. Kallsen [16].

These processes are discussed in Section 5. Note that any local martingale is a σ -martingale. The reverse is not true (see Example 5.2).

Proposition 1.11 (I FTAP in the continuous time; general case). (a) *Each of the conditions $(NFLVR)$, $(NFLBR)$ is equivalent to the existence of an equivalent σ -martingale measure, i.e. $\text{SM}(\mathbb{P}) \neq \emptyset$, where*

$$\text{SM}(\mathbb{P}) = \{Q \sim \mathbb{P} : X \text{ is a } (\mathcal{F}_t, Q)\text{-}\sigma\text{-martingale}\}.$$

(b) *If the components of X are nonnegative, then each of the conditions $(NFLVR)$, $(NFLBR)$ is equivalent to the existence of an equivalent local martingale measure, i.e. $\text{LM}(\mathbb{P}) \neq \emptyset$.*

Diagrammatically,

$$(NA) \iff (NFLVR) \iff (NFLBR) \iff \text{SM}(\mathbb{P}) \neq \emptyset \iff \text{LM}(\mathbb{P}) \neq \emptyset.$$

Statement (a) was proved by F. Delbaen and W. Schachermayer in [10]. A simple proof was given by Yu.M. Kabanov in [24]. Statement (b) follows from (a) and the following result proved by J.-P. Ansel and C. Stricker [1]: a σ -martingale with nonnegative components is a local martingale.

Remarks. (i) The example showing that

$$\text{SM}(\mathbb{P}) \neq \emptyset \not\Rightarrow \text{LM}(\mathbb{P}) \neq \emptyset$$

is cited in Section 5 (see Example 5.3).

(ii) The definition of a strategy and hence, the definition of $(NFLVR)$, $(NFLBR)$ employ vector stochastic integrals. If one replaces the vector stochastic integrals by the componentwise stochastic integrals, then a smaller class of strategies and, as a result, a weaker form of $(NFLVR)$, $(NFLBR)$ are obtained. It turns out that the weakened $(NFLBR)$ property is still sufficient for the existence of an equivalent σ -martingale measure, while the weakened $(NFLVR)$ property is insufficient for the existence of such a measure. This was proved in the paper [5] by A.S. Cherny. Thus, the vector stochastic integrals are essential for the First Fundamental Theorem of Asset Pricing. A detailed discussion of this problem is given in Section 6. \square

The description of the First Fundamental Theorem of Asset Pricing can be summarized as follows. In the discrete-time case, this theorem admits a very simple and natural formulation. In the continuous-time case, the definition of arbitrage should be reformulated carefully. Furthermore, if the process X is not locally bounded, one has to consider σ -martingales. Moreover, if X is multidimensional, one should deal with the vector stochastic integrals.

3. The Second Fundamental Theorem of Asset Pricing. This theorem is a counterpart of the First Fundamental Theorem of Asset Pricing. In the simplest setting, the theorem asserts that an arbitrage-free model is complete if and only if the equivalent martingale measure is unique. Thus, the theorem establishes the equivalence between two objects of a completely different nature: the notion of the completeness arises from practice (informally, completeness means a possibility to hedge any contingent claim), while the uniqueness of an equivalent martingale measure is a purely mathematical notion.

It should be pointed out that all the models employed in the mathematical finance are arbitrage-free since the absence of arbitrage is the basic assumption of the modern theory. On the other hand, only a few of the models are complete (such is, for example, the famous Black-Scholes model), while most of the models are incomplete (such are, for example, the stochastic volatility models). If a model is complete, it is rather easy to price and hedge contingent claims. However, the completeness is not an obligatory property of a model.

The Second Fundamental Theorem of Asset Pricing has several formulations in accordance with the level of generality.

Let us first consider the discrete-time case. We assume here that \mathcal{F}_0 is \mathbb{P} -trivial and $\mathcal{F} = \mathcal{F}_N$.

Definition 1.12. A model is *complete* if for any \mathcal{F} -measurable function f , there exists a strategy π such that

$$f = V_N^\pi \quad \text{P-a.s.}$$

Definition 1.13. A process X has the *predictable representation property with respect to a measure \mathbb{Q}* if for any $(\mathcal{F}_n, \mathbb{Q})$ -local martingale $M = (M_n)_{0 \leq n \leq N}$, there exists a predictable process $H = (H_n^1, \dots, H_n^d)_{0 \leq n \leq N}$ such that

$$M_n = M_0 + \sum_{i=1}^d \sum_{k=1}^n H_k^i (X_k^i - X_{k-1}^i), \quad n = 0, \dots, N.$$

Notation: (PRP).

Proposition 1.14 (II FTAP in the discrete time). *Suppose that $\mathbb{M}(\mathbb{P}) \neq \emptyset$. Then the following conditions are equivalent:*

- (i) *the model is complete;*
- (ii) $|\mathbb{M}(\mathbb{P})| = 1$;
- (iii) *there exists $\mathbb{Q} \in \mathbb{M}(\mathbb{P})$ such that X has the predictable representation property with respect to \mathbb{Q} .*

Diagrammatically,

$$\text{completeness} \iff |\mathbb{M}(\mathbb{P})| = 1 \iff (\text{PRP}).$$

The proof of this statement can be found in the paper [22] by J. Jacod and A.N. Shiryaev as well as in the book [36; Ch. V, §4f] by A.N. Shiryaev.

Remark. It is shown in [22], [36] that if $|\mathbb{M}(\mathbb{P})| = 1$, then $\mathcal{F}_n = \mathcal{F}_n^X$ ($n = 0, \dots, N$) up to \mathbb{P} -null sets. \square

Let us now pass on to the continuous-time case. We assume that the filtration (\mathcal{F}_t) is right-continuous, \mathcal{F}_0 is \mathbb{P} -trivial and $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t$.

Definition 1.15. A model is *complete* if for each bounded \mathcal{F} -measurable function f , there exists a strategy π such that

- i) there exist constants a and b such that

$$\mathbb{P}\{\forall t \geq 0, a \leq V_t^\pi \leq b\} = 1;$$

- ii) there exists a limit

$$V_\infty^\pi := \lim_{t \rightarrow \infty} V_t^\pi \quad \text{P-a.s.};$$

- iii) $f = V_\infty^\pi$ P-a.s.

Definition 1.16. A process X has the *predictable representation property with respect to a measure \mathbb{Q}* if for any $(\mathcal{F}_t, \mathbb{Q})$ -local martingale $M = (M_t)_{t \geq 0}$, there exists an X -integrable process $H = (H_t^1, \dots, H_t^d)_{t \geq 0}$ such that $M = M_0 + H \bullet X$. Notation: (PRP).

We will now formulate the Second Fundamental Theorem of Asset Pricing in the general case as a counterpart of Proposition 1.11. The counterpart of Proposition 1.8 is not formulated because it immediately follows from Theorem 1.17. This theorem is one of the main results of this paper.

Theorem 1.17 (II FTAP in the continuous time; general case). (a) *Suppose that $\text{SM}(\mathbf{P}) \neq \emptyset$. Then the following conditions are equivalent:*

- (i) *the model is complete;*
- (ii) $|\text{SM}(\mathbf{P})| = 1$;
- (iii) *there exists $\mathbf{Q} \in \text{SM}(\mathbf{P})$ such that X has the predictable representation property with respect to \mathbf{Q} .*

(b) *Suppose that the components of X are nonnegative and $\text{LM}(\mathbf{P}) \neq \emptyset$. Then the following conditions are equivalent:*

- (i) *the model is complete;*
- (ii) $|\text{LM}(\mathbf{P})| = 1$;
- (iii) *there exists $\mathbf{Q} \in \text{LM}(\mathbf{P})$ such that X has the predictable representation property with respect to \mathbf{Q} .*

Diagrammatically,

$$\text{completeness} \iff |\text{SM}(\mathbf{P})| = 1 \iff (\text{PRP}).$$

This form of the Second Fundamental Theorem of Asset Pricing can hardly be found in the literature. The most complicated implication

$$|\text{SM}(\mathbf{P})| = 1 \implies \text{completeness} \tag{1.1}$$

follows from the paper [10; Theorem 5.14] by F. Delbaen and W. Schachermayer, but the proof here is based on different (simpler) arguments.

The following implication is a consequence of the results obtained by H.P. Ansel and C. Stricker [1], J. Jacod [19; (11.2)]:

$$|\text{LM}(\mathbf{P})| = 1 \implies \text{completeness}. \tag{1.2}$$

The σ -martingales are not considered in these papers. Implication (1.1) can be obtained from (1.2) by using the associativity property of stochastic integrals. However, we give a complete proof of (1.1), which is similar to the proof of (1.2) in [19]. In order to treat σ -martingales, we apply some properties of the vector stochastic integrals.

Remarks. (i) The use of σ -martingales is essential for the Second Fundamental Theorem of Asset Pricing in the general case (when X has unbounded jumps). We present in Section 5 an example of a model, for which $|\text{SM}(\mathbf{P})| = 1$ (and thus, the model is complete), whereas $\text{LM}(\mathbf{P}) = \emptyset$ (see Example 5.3).

(ii) If the vector stochastic integrals in Definition 1.4 are replaced by the component-wise stochastic integrals, then Theorem 1.17 is no longer true (see Example 7.2). The importance of the vector stochastic integrals for the problems concerning completeness was stressed in the papers: [4] by M. Chatelain and C. Stricker, [15] by L. Galtchouk, [23] by R. Jarrow and D. Madan.

(iii) If condition i) of Definition 1.15 is eliminated, then Theorem 1.17 is no longer true (see Example 7.3).

(iv) In the continuous-time case, the condition $|\text{SM}(\mathbf{P})| = 1$ does not imply that $\mathcal{F}_t = \mathcal{F}_t^X$ (see Example 7.4).

(v) A model can be complete, whereas $\text{SM}(\mathbf{P}) = \emptyset$ (see Example 7.5). \square

The above description of the Second Fundamental Theorem of Asset Pricing can be summarized as follows. In the discrete-time case, this theorem admits a very simple formulation. In the continuous-time case, the definition of completeness should be reformulated carefully. Furthermore, if X is not locally bounded, one has to consider σ -martingales. Moreover, if X is multidimensional, one should deal with the vector stochastic integrals.

4. The structure of the paper. Section 2 contains some well-known facts from the stochastic calculus.

In Section 3, we construct the vector stochastic integral successively: for local martingales, for finite-variation processes and for semimartingales.

Section 4 contains some basic properties of the vector stochastic integral: linearity in H and in X , associativity (“ $K \bullet (H \bullet X) = (KH) \bullet X$ ”), closedness of the space of stochastic integrals, stability of the stochastic integral under the change of measure and filtration as well as the property “[$H \bullet X, K \bullet Y$] = $HK \bullet [X, Y]$ ”.

In Section 5, we prove that various definitions of the σ -martingale are equivalent. We cite Émery’s example of a σ -martingale that is not a local martingale. We also cite the basic properties of the σ -martingales as well as their description in terms of the semimartingale characteristics. Moreover, we prove that any local martingale with independent increments is in fact a martingale; any σ -martingale that is a Lévy process is in fact a martingale.

In Section 6, we cite Galtchouk’s example, which shows that the space of componentwise stochastic integrals with respect to a multidimensional semimartingale may not be closed in the Émery topology. We also describe the relationship of different No Free Lunch conditions stated for the vector stochastic integrals and the componentwise stochastic integrals.

In Section 7, we prove the Second Fundamental Theorem of Asset Pricing (Theorem 1.17). We also provide several related (counter-)examples.

2 Notations and Definitions

In this section, we introduce some (standard) notations and cite some well-known definitions and facts from the martingale theory. These definitions and statements can be found in many textbooks on the stochastic analysis (see, for example, [19], [21], [29]). We have included this section for the convenience of the reader.

Throughout this section, we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$. We assume that (\mathcal{F}_t) is right-continuous.

1. Processes and σ -fields. We recall that the *predictable σ -field* \mathcal{P} on $\Omega \times \mathbb{R}_+$ is generated by the left-continuous (\mathcal{F}_t) -adapted processes and the *optional σ -field* \mathcal{O} is generated by the càdlàg (i.e. right-continuous with left-hand limits) (\mathcal{F}_t) -adapted

processes. A process $H = (H_t^1, \dots, H_t^d)_{t \geq 0}$ is called *predictable* if it is measurable with respect to \mathcal{P} (as a map from $\Omega \times \mathbb{R}_+$ to \mathbb{R}^d); a process H is called *optional* if it is measurable with respect to \mathcal{O} .

Proposition 2.1. *The σ -field \mathcal{P} is generated by any of the following collections of sets:*

- (i) $B \times \{0\}$, where $B \in \mathcal{F}_0$, and $B \times (s, t]$, where $B \in \mathcal{F}_s$;
- (ii) $B \times \{0\}$, where $B \in \mathcal{F}_0$, and $\llbracket 0, \tau \rrbracket$, where τ is a stopping time.

For the proof, see [21; Ch. I, (2.2)].

Definition 2.2. A process H is *locally bounded* if there exists an increasing sequence $(\tau_n)_{n=1}^\infty$ of stopping times such that $\tau_n \rightarrow \infty$ a.s. and, for each $n \in \mathbb{N}$, the stopped process H^{τ_n} is bounded.

Definition 2.3. A process A is a process of *locally integrable variation* if there exists an increasing sequence $(\tau_n)_{n=1}^\infty$ of stopping times such that $\tau_n \rightarrow \infty$ a.s. and, for each $n \in \mathbb{N}$, $\mathbf{E}(\text{Var } A)_{\tau_n} < \infty$. The class of one-dimensional (resp: d -dimensional) adapted processes of locally integrable variation is denoted by \mathcal{A}_{loc} (resp: $\mathcal{A}_{\text{loc}}^d$).

Definition 2.4. A sequence of processes $(X_n)_{n=1}^\infty$ converges to a process X *in probability uniformly on compact intervals* if for any $t \geq 0$, $\sup_{s \leq t} |(X_n)_s - X_s| \xrightarrow[n \rightarrow \infty]{\mathbf{P}} 0$.

Notation: $X_n \xrightarrow{\text{u.p.}} X$.

Remarks. (i) All the processes from the spaces \mathcal{A}_{loc} , $\mathcal{A}_{\text{loc}}^d$ (and from the spaces \mathcal{M}_{loc} , $\mathcal{M}_{\text{loc}}^d$, \mathcal{S} , \mathcal{S}^d , \mathcal{S}_p , \mathcal{S}_p^d , \mathcal{V}^+ , \mathcal{V} , \mathcal{V}^d defined below) are supposed to be càdlàg, i.e. right-continuous with left-hand limits. \square

(ii) By the equality of two processes $X = Y$ we will mean the equality up to indistinguishability (i.e. $\mathbf{P}\{\forall t \geq 0, X_t = Y_t\} = 1$). \square

2. Local martingales. Recall that a process M is called a *local martingale* if there exists an increasing sequence $(\tau_n)_{n=1}^\infty$ of stopping times such that $\tau_n \rightarrow \infty$ a.s. and, for each $n \in \mathbb{N}$, the stopped process M^{τ_n} is a uniformly integrable martingale. The sequence (τ_n) is called a *localizing sequence* for M . The class of one-dimensional (resp: d -dimensional) local martingales will be denoted by \mathcal{M}_{loc} (resp: $\mathcal{M}_{\text{loc}}^d$).

Proposition 2.5. *If $M \in \mathcal{M}_{\text{loc}}$, then $[M]^{1/2} \in \mathcal{A}_{\text{loc}}$.*

For the proof, see [21; Ch. I, (4.55)].

The set

$$\mathcal{H}^1 = \left\{ M \in \mathcal{M}_{\text{loc}} : \mathbf{E} \sup_{t \geq 0} |M_t| < \infty \right\}$$

is a Banach space with respect to the norm $\|M\|_{\mathcal{H}^1} = \mathbf{E} \sup_{t \geq 0} |M_t|$. Any $M \in \mathcal{H}^1$ is uniformly integrable. In particular, almost surely there exists a limit $M_\infty = \lim_{t \rightarrow \infty} M_t$.

We will also consider the space

$$\mathcal{H}^\infty = \left\{ M \in \mathcal{M}_{\text{loc}} : \sup_{t \geq 0} |M_t| \in L^\infty \right\}$$

equipped with the norm $\|M\|_{\mathcal{H}^\infty} = \left\| \sup_{t \geq 0} |M_t| \right\|_{L^\infty}$.

Proposition 2.6 (Davis inequalities). *There exist constants $0 < c < C$ such that, for any $M \in \mathcal{H}^1$ with $M_0 = 0$,*

$$c\mathbf{E}[M]_\infty^{1/2} \leq \mathbf{E} \sup_{t \geq 0} |M_t| \leq C\mathbf{E}[M]_\infty^{1/2}.$$

For the proof, see [29; Ch. I, §9].

Propositions 2.5 and 2.6 yield

Corollary 2.7. *Let $M \in \mathcal{M}_{\text{loc}}^d$. Then there exists an increasing sequence $(\tau_n)_{n=1}^\infty$ of stopping times such that $\tau_n \rightarrow \infty$ a.s. and, for each $n \in \mathbb{N}$, $M^{\tau_n} \in \mathcal{H}^1$.*

Proposition 2.8. *The dual space of \mathcal{H}^1 can be identified with the space BMO of martingales with bounded mean oscillation (for the definition of this space, see [19; (2.7)]); any $N \in \text{BMO}$ corresponds to the following functional on \mathcal{H}^1 :*

$$\mathcal{H}^1 \ni M \longmapsto \mathbf{E}(M_\infty N_\infty).$$

Moreover, $\text{BMO} \subset \mathcal{H}_{\text{loc}}^\infty$, i.e. for any $N \in \text{BMO}$, there exists an increasing sequence $(\tau_n)_{n=1}^\infty$ of stopping times such that $\tau_n \rightarrow \infty$ a.s. and, for each $n \in \mathbb{N}$, $N^{\tau_n} \in \mathcal{H}^\infty$.

For the proof of this proposition, see [19; Ch. II, (2.35), (2.38)].

Definition 2.9. Two local martingales M and N are said to be *orthogonal* if $M_0 N_0 = 0$ and $MN \in \mathcal{M}_{\text{loc}}$. This will be denoted as $M \perp N$.

Definition 2.10. A linear subspace $\mathcal{L} \subseteq \mathcal{H}^1$ is called a *stable subspace* if it is closed in \mathcal{H}^1 and, for any stopping time τ , $A \in \mathcal{F}_0$ and $M \in \mathcal{L}$, we have $I_A M^\tau \in \mathcal{L}$.

Proposition 2.11. *Let $N \in \text{BMO}$ and \mathcal{L} be a stable subspace of \mathcal{H}^1 . The following conditions are equivalent:*

- (i) $\mathbf{E}(M_\infty N_\infty) = 0$ for each $M \in \mathcal{L}$;
- (ii) $M \perp N$ for each $M \in \mathcal{L}$.

For the proof, see [19; Ch. IV, (4.7)].

3. Semimartingales. We recall that a d -dimensional process X is called a *semimartingale* if it admits a decomposition $X = X_0 + A + M$, where $A \in \mathcal{V}^d$, $M \in \mathcal{M}_{\text{loc}}^d$ and $A_0 = M_0 = 0$. The space of one-dimensional (resp: d -dimensional) semimartingales will be denoted by \mathcal{S} (resp: \mathcal{S}^d).

A semimartingale X is called *special* if there exists a decomposition $X = X_0 + A + M$ into a *predictable* finite-variation process A and a local martingale M ($A_0 = M_0 = 0$). Such a decomposition (if it exists) is unique (see [21; Ch. I, §4c]). It is called the *canonical decomposition* of X . The space of one-dimensional (resp: d -dimensional) special semimartingales will be denoted by \mathcal{S}_p (resp: \mathcal{S}_p^d).

Proposition 2.12. *If $X \in \mathcal{S}^d$ is continuous, then X is special and its canonical decomposition consists of continuous A and M .*

For the proof, see [21; Ch. I, (4.24)].

Proposition 2.13. *If $X \in \mathcal{S}^d$ has bounded jumps (i.e. $\|\Delta X\| \leq a$ for some $a \in \mathbb{R}$), then X is special.*

For the proof, see [21; Ch. I, (4.24)].

4. Quadratic covariation. A sequence $\Delta^n = \{0 = \tau_0^n < \tau_1^n < \dots\}$ is a *Riemannian sequence* if τ_m^n are stopping times, for any $n \in \mathbb{N}$, $\tau_m^n \xrightarrow[m \rightarrow \infty]{\text{a.s.}} \infty$ and

$$\forall t \geq 0, \sup_m (\tau_{m+1}^n \wedge t - \tau_m^n \wedge t) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

Proposition 2.14. *If $X, Y \in \mathcal{S}$, then there exists a process $[X, Y] \in \mathcal{V}$ such that, for any Riemannian sequence $\Delta^n = \{0 = \tau_0^n < \tau_1^n < \dots\}$, the processes*

$$S_t^n = \sum_{m=0}^{\infty} (X_{\tau_{m+1}^n \wedge t} - X_{\tau_m^n \wedge t})(Y_{\tau_{m+1}^n \wedge t} - Y_{\tau_m^n \wedge t}), \quad t \geq 0$$

converge to $[X, Y]$ in probability uniformly on compact intervals.

For the proof, see [21; Ch. I, (4.47)].

Definition 2.15. The process $[X, Y]$ is called the *quadratic covariation* of X and Y . The process $[X, X]$ is called the *quadratic variation* of X and will further be denoted as $[X]$.

Proposition 2.16. *Let $X, Y \in \mathcal{S}$ and X^c, Y^c denote the continuous martingale parts of X, Y , respectively (for the definition, see [21; Ch. I, (4.18)]). Then*

$$[X, Y] = \sum_{s \leq \cdot} \Delta X_s \Delta Y_s + [X^c, Y^c].$$

For the proof, see [21; Ch. I, (4.52)].

5. The monotone class lemma. A family \mathfrak{M} of subsets of Ω is called a *monotone class* if the following conditions are satisfied:

- i) $\emptyset, \Omega \in \mathfrak{M}$;
- ii) if $A, B \in \mathfrak{M}$ and $A \subseteq B$, then $B \setminus A \in \mathfrak{M}$;
- iii) if $A, B \in \mathfrak{M}$ and $A \cap B = \emptyset$, then $A \cup B \in \mathfrak{M}$;
- iv) if $(A_n)_{n=1}^{\infty} \in \mathfrak{M}$ and $A_n \subseteq A_{n+1}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathfrak{M}$.

Proposition 2.17 (Monotone class lemma). *Suppose that \mathfrak{A} is a family of subsets of Ω that is closed under finite intersections (i.e. for $A, B \in \mathfrak{A}$, we have $A \cap B \in \mathfrak{A}$). Then the minimal monotone class that contains all the sets from \mathfrak{A} coincides with the σ -field generated by \mathfrak{A} .*

For the proof, see [33; §I.4].

3 Construction of the Vector Stochastic Integral

1. Integral with respect to a local martingale. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ be a filtered probability space and let $M \in \mathcal{M}_{\text{loc}}^d(\mathbf{P})$. Then there exist $C \in \mathcal{V}^+$ and optional processes π^{ij} ($i, j = 1, \dots, d$) such that

$$[M^i, M^j] = \int_0^\cdot \pi_s^{ij} dC_s. \quad (3.1)$$

Evidently, one can choose (π^{ij}) in such a way that $\pi_t^{ij}(\omega) = \pi_t^{ji}(\omega)$ for all i, j, ω, t .

Let us take a dense subset $(\lambda_k)_{k=1}^\infty$ in \mathbb{R}^d and consider the sets

$$D_k = \left\{ (\omega, t) : \sum_{i,j=1}^d \lambda_k^i \pi_t^{ij}(\omega) \lambda_k^j \geq 0 \right\},$$

$$D = \left\{ (\omega, t) : \forall \lambda \in \mathbb{R}^d, \sum_{i,j=1}^d \lambda^i \pi_t^{ij}(\omega) \lambda^j \geq 0 \right\}.$$

Obviously, each D_k is optional and $D = \bigcap_{k=1}^\infty D_k$. Hence, the processes $\tilde{\pi}^{ij} = \pi^{ij} I_D$ are optional. In view of the equality

$$[\langle \lambda_k, M \rangle] = \int_0^\cdot \left(\sum_{i,j=1}^d \lambda_k^i \pi_s^{ij} \lambda_k^j \right) dC_s,$$

we have

$$\int_0^\infty I_{D_k^c} dC_s = 0, \quad \int_0^\infty I_{D^c} dC_s = 0.$$

Therefore, equality (3.1) remains valid if π^{ij} is replaced by $\tilde{\pi}^{ij}$. Thus, the processes π^{ij} satisfying (3.1) can be chosen in such a way that, for each (ω, t) , the matrix $(\pi_t^{ij}(\omega))$ is symmetric and positively definite. In what follows, we will always choose such a version of π^{ij} . In particular, for any $\lambda \in \mathbb{R}^d$, $\omega \in \Omega$, $t \geq 0$, we have

$$\sum_{i,j=1}^d \lambda^i \pi_t^{ij}(\omega) \lambda^j = \langle \pi_t(\omega) \lambda, \lambda \rangle \leq \|\lambda\|^2 \text{tr } \pi_t(\omega) = \|\lambda\|^2 \sum_{i=1}^d \pi_t^{ii}(\omega). \quad (3.2)$$

Set

$$\dot{L}^1(M) = \left\{ H = (H^1, \dots, H^d) : H \text{ is predictable and } \mathbb{E} \left(\int_0^\infty \left(\sum_{i,j=1}^d H_s^i \pi_s^{ij} H_s^j \right) dC_s \right)^{1/2} < \infty \right\}.$$

Let $L^1(M)$ be the space of the equivalence classes of elements from $\dot{L}^1(M)$ under the equivalence relation

$$H \sim K \iff \int_0^\infty \left(\sum_{i,j=1}^d (H_s^i - K_s^i) \pi_s^{ij} (H_s^j - K_s^j) \right) dC_s = 0 \quad \text{a.s.}$$

Definition 3.1. A *simple integrand* is a process H of the form

$$H_t = h_0 I(t = 0) + \sum_{k=1}^m h_k I(\tau_k < t \leq \tau_{k+1}), \quad t \geq 0, \quad (3.3)$$

where $0 \leq \tau_1 \leq \dots \leq \tau_{m+1}$ are stopping times and each h_k is a bounded d -dimensional \mathcal{F}_{τ_k} -measurable random variable.

Lemma 3.2. (a) *The function*

$$\|H\|_{L^1(M)} = \mathbb{E} \left(\int_0^\infty \left(\sum_{i,j=1}^d H_s^i \pi_s^{ij} H_s^j \right) dC_s \right)^{1/2}$$

is a norm on $L^1(M)$.

(b) *The set of simple integrands that belong to $L^1(M)$ is dense in $L^1(M)$.*

Proof. (a) One should check only the triangle inequality for $\|\cdot\|_{L^1(M)}$. As $(\pi_t^{ij}(\omega))$ is symmetric and positively definite, we have

$$\sum_{i,j=1}^d (\mu^i + \nu^i) \pi_t^{ij}(\omega) (\mu^j + \nu^j) \leq \left(\left(\sum_{i,j=1}^d \mu^i \pi_t^{ij}(\omega) \mu^j \right)^{1/2} + \left(\sum_{i,j=1}^d \nu^i \pi_t^{ij}(\omega) \nu^j \right)^{1/2} \right)^2$$

for any $\mu, \nu \in \mathbb{R}^d$, $\omega \in \Omega$, $t \in \mathbb{R}_+$. Hence,

$$\begin{aligned} & \mathbb{E} \left(\int_0^\infty \left(\sum_{i,j=1}^d (H_s^i + K_s^i) \pi_s^{ij} (H_s^j + K_s^j) \right) dC_s \right)^{1/2} \\ & \leq \mathbb{E} \left(\int_0^\infty \left(\left(\sum_{i,j=1}^d H_s^i \pi_s^{ij} H_s^j \right)^{1/2} + \left(\sum_{i,j=1}^d K_s^i \pi_s^{ij} K_s^j \right)^{1/2} \right)^2 dC_s \right)^{1/2} \\ & \leq \mathbb{E} \left(\int_0^\infty \left(\sum_{i,j=1}^d H_s^i \pi_s^{ij} H_s^j \right) dC_s \right)^{1/2} + \mathbb{E} \left(\int_0^\infty \left(\sum_{i,j=1}^d K_s^i \pi_s^{ij} K_s^j \right) dC_s \right)^{1/2} \end{aligned}$$

for any $H, K \in L^1(M)$.

(b) Fix $H \in L^1(M)$. Since $[M^i]^{1/2} \in \mathcal{A}_{\text{loc}}$ for $i = 1, \dots, d$ (see Proposition 2.5), there exists an increasing sequence $(\tau_n)_{n=1}^\infty$ of stopping times such that $\tau_n \rightarrow \infty$ a.s. and

$$\mathbb{E} \left(\int_0^{\tau_n} \pi_s^{ii} dC_s \right)^{1/2} < \infty, \quad i = 1, \dots, d, \quad n \in \mathbb{N}.$$

Set $(H_n)_t = H_t I(t \leq \tau_n)$. Then $H_n \xrightarrow[n \rightarrow \infty]{L^1(M)} H$. Therefore, we can suppose from the outset that

$$\mathbb{E} \left(\int_0^\infty \pi_s^{ii} dC_s \right)^{1/2} < \infty, \quad i = 1, \dots, d. \quad (3.4)$$

Let us fix $\lambda \in \mathbb{R}^d$. In view of (3.2) and (3.4), any process of the form λI_D , where $D \in \mathcal{P}$, belongs to $L^1(M)$. Set

$$\mathfrak{M} = \{D \in \mathcal{P} : \lambda I_D \text{ can be approximated by simple integrands in } L^1(M)\}.$$

It is easy to verify that \mathfrak{M} is a monotone class. Furthermore, \mathfrak{M} contains all the sets $B \times \{0\}$, where $B \in \mathcal{F}_0$, and $B \times (s, t]$, where $B \in \mathcal{F}_s$. By the monotone class lemma combined with Proposition 2.1, $\mathfrak{M} = \mathcal{P}$.

Any bounded predictable H can be uniformly approximated by the finite sums of the form $\sum_k \lambda_k I_{D_k}$, where $D_k \in \mathcal{P}$. Hence, any bounded predictable H can be approximated by the simple integrands in $L^1(M)$.

Finally, for an arbitrary $H \in L^1(M)$,

$$H_n = HI(\|H\| \leq n) \xrightarrow[n \rightarrow \infty]{L^1(M)} H,$$

and, from the above result, H can be approximated by the simple integrands in $L^1(M)$. \square

Remark. Obviously, the space $L^1(M)$ and the norm $\|\cdot\|_{L^1(M)}$ do not depend on the choice of π^{ij} and C that satisfy (3.1). \square

For a simple integrand (3.3), we define the integral $H \bullet M$ as

$$(H \bullet M)_t = \sum_{i=1}^d \sum_{k=1}^m h_k^i (M_{t \wedge \tau_{k+1}}^i - M_{t \wedge \tau_k}^i), \quad t \geq 0.$$

Then it is easy to verify that $H \bullet M \in \mathcal{M}_{\text{loc}}$ and

$$[H \bullet M] = \int_0^\cdot \left(\sum_{i,j=1}^d H_s^i \pi_s^{ij} H_s^j \right) dC_s. \quad (3.5)$$

Let $H \in L^1(M)$. Then there exists a sequence (H_n) of simple integrands that converges to H in $L^1(M)$. Equality (3.5), together with the Davis inequalities, shows that the sequence $(H_n \bullet M)$ is fundamental in \mathcal{H}^1 . Hence, there exists a \mathcal{H}^1 -limit of this sequence. Obviously, it does not depend on the choice of a sequence (H_n) that converges to H .

Definition 3.3. Let $H \in L^1(M)$ and (H_n) be a sequence of simple integrands that converges to H in $L^1(M)$. Then the *vector stochastic integral* $H \bullet M$ is defined as the \mathcal{H}^1 -limit of $(H_n \bullet M)$.

Our next goal is to define the vector stochastic integral $H \bullet M$ for a larger class of integrands than $L^1(M)$. To this end, we introduce the class

$$L_{\text{loc}}^1(M) = \left\{ H = (H^1, \dots, H^d) : H \text{ is predictable and } \left(\int_0^\cdot \left(\sum_{i,j=1}^d H_s^i \pi_s^{ij} H_s^j \right) dC_s \right)^{1/2} \in \mathcal{A}_{\text{loc}} \right\},$$

where π^{ij} and C satisfy (3.1). For a one-dimensional local martingale M , we have a simpler formula

$$L_{\text{loc}}^1(M) = \left\{ H : H \text{ is predictable and } \left(\int_0^\cdot H_s^2 d[M]_s \right)^{1/2} \in \mathcal{A}_{\text{loc}} \right\}.$$

Let $H \in L^1_{\text{loc}}(M)$. Then there exists an increasing sequence $(\tau_n)_{n=1}^\infty$ of stopping times such that $\tau_n \rightarrow \infty$ a.s. and

$$\mathbb{E} \left(\int_0^{\tau_n} \left(\sum_{i,j=1}^d H_s^i \pi_s^{ij} H_s^j \right) dC_s \right)^{1/2} < \infty.$$

For each $n \in \mathbb{N}$, $HI_{\llbracket 0, \tau_n \rrbracket} \in L^1(M)$. Furthermore,

$$\left((HI_{\llbracket 0, \tau_{n+1} \rrbracket}) \bullet M \right)^{\tau_n} = (HI_{\llbracket 0, \tau_n \rrbracket}) \bullet M, \quad n \in \mathbb{N}.$$

Thus, there exists a unique process $H \bullet M$ such that

$$(H \bullet M)^{\tau_n} = (HI_{\llbracket 0, \tau_n \rrbracket}) \bullet M, \quad n \in \mathbb{N}. \quad (3.6)$$

Obviously, $H \bullet M$ does not depend on the choice of a localizing sequence $(\tau_n)_{n=1}^\infty$.

Definition 3.4. For $H \in L^1_{\text{loc}}(M)$, the process $H \bullet M$ that satisfies (3.6) is called the *vector stochastic integral* of H with respect to M .

Obviously, $H \bullet M \in \mathcal{M}_{\text{loc}}$.

Remark. The space $L^1_{\text{loc}}(M)$ is in a sense the largest class of predictable processes that can be integrated with respect to M in such a way that $H \bullet M \in \mathcal{M}_{\text{loc}}$. Indeed, for a “reasonably” defined stochastic integral, one should have the equality

$$[H \bullet M] = \int_0^\cdot \left(\sum_{i,j=1}^d H_s^i \pi_s^{ij} H_s^j \right) dC_s.$$

If $H \bullet M \in \mathcal{M}_{\text{loc}}$, then $[H \bullet M]^{1/2} \in \mathcal{A}_{\text{loc}}$ (see Proposition 2.5). This, together with the above equality, means that $H \in L^1_{\text{loc}}(M)$. \square

Lemma 3.5. *Let $M \in \mathcal{M}_{\text{loc}}^d$. The vector stochastic integral with respect to M has the following properties:*

(a) *if $H_1, H_2 \in L^1_{\text{loc}}(M)$, $\alpha_1, \alpha_2 \in \mathbb{R}$, then $\alpha_1 H_1 + \alpha_2 H_2 \in L^1_{\text{loc}}(M)$ and*

$$(\alpha_1 H_1 + \alpha_2 H_2) \bullet M = \alpha_1 (H_1 \bullet M) + \alpha_2 (H_2 \bullet M);$$

(b) *if $H \in L^1_{\text{loc}}(M)$, then $H \bullet M \in \mathcal{M}_{\text{loc}}$ and*

$$[H \bullet M] = \int_0^\cdot \left(\sum_{i,j=1}^d H_s^i \pi_s^{ij} H_s^j \right) dC_s,$$

where π^{ij} and C satisfy (3.1);

(c) *if $H \in L^1_{\text{loc}}(M)$ and τ is a stopping time, then*

$$(H \bullet M)^\tau = (H \bullet M^\tau) = (HI_{\llbracket 0, \tau \rrbracket}) \bullet M;$$

(d) *if $H \in L^1_{\text{loc}}(M)$ and $D \in \mathcal{P}$, then*

$$I_D \bullet (H \bullet M) = (HI_D) \bullet M;$$

(e) if $H \in L^1_{\text{loc}}(M)$, then

$$\Delta(H \bullet M) = \langle H, \Delta M \rangle;$$

(f) if $H \in L^1(M)$ and $H_n = HI(\|H\| \leq n)$, then

$$H_n \bullet M \xrightarrow[n \rightarrow \infty]{\mathcal{H}^1} H \bullet M.$$

Proof. Properties (a), (b), (c), (e), (f) are easily derived from the definition of the vector stochastic integral.

Let us prove (d). There exists an increasing sequence $(\tau_n)_{n=1}^\infty$ of stopping times such that $\tau_n \rightarrow \infty$ a.s. and

$$\mathbb{E} \left(\int_0^{\tau_n} \left(\sum_{i=1}^d \pi_s^{ii} \right) dC_s \right)^{1/2} < \infty, \quad \mathbb{E} \left(\int_0^{\tau_n} \left(\sum_{i,j=1}^d H_s^i \pi_s^{ij} H_s^j \right) dC_s \right)^{1/2} < \infty, \quad n \in \mathbb{N}.$$

Fix $n \in \mathbb{N}$. The set

$$\mathfrak{M} = \{ D \in \mathcal{P} : I_D \bullet (H \bullet M^{\tau_n}) = (HI_D) \bullet M^{\tau_n} \}$$

is a monotone class that, in view of (c), contains all the sets of the form $\llbracket 0, \tau \rrbracket$, where τ is a stopping time. Obviously, \mathfrak{M} contains all the sets of the form $B \times \{0\}$, where $B \in \mathcal{F}_0$. By the monotone class lemma combined with Proposition 2.1, we get $\mathfrak{M} = \mathcal{P}$. The application of property (c) completes the proof. \square

Lemma 3.6. *If $H \in L^1_{\text{loc}}(M_1) \cap L^1_{\text{loc}}(M_2)$, $\alpha_1, \alpha_2 \in \mathbb{R}$, then $H \in L^1_{\text{loc}}(\alpha_1 M_1 + \alpha_2 M_2)$ and*

$$H \bullet (\alpha_1 M_1 + \alpha_2 M_2) = \alpha_1 (H \bullet M_1) + \alpha_2 (H \bullet M_2). \quad (3.7)$$

Proof. Suppose first that H is bounded. Then the inclusion $H \in L^1_{\text{loc}}(\alpha_1 M_1 + \alpha_2 M_2)$ follows from (3.2), and we prove (3.7) by the standard scheme:

1. for $H = \lambda I_D$ (by the monotone class lemma);
2. for bounded H (through the uniform approximation).

Suppose now that $H \in L^1_{\text{loc}}(M_1) \cap L^1_{\text{loc}}(M_2)$. Set $H_n = HI(\|H\| \leq n)$. By the above reasoning, (3.7) is true for H_n . Choose $C \in \mathcal{V}^+$ and optional processes $\pi_1^{ij}, \pi_2^{ij}, \sigma^{ij}$ in such a way that

$$[\alpha_k M_k^i] = \int_0^\cdot (\pi_k^{ij})_s dC_s, \quad k = 1, 2,$$

$$[\alpha_1 M_1^i + \alpha_2 M_2^i] = \int_0^\cdot \sigma_s^{ij} dC_s.$$

Applying equality (3.7) for H_n , Lemma 3.5 (b) and the inequality $[X + Y] \leq 2[X] + 2[Y]$, we get

$$\begin{aligned} & \int_0^t I(\|H_s\| \leq n) \left(\sum_{i,j=1}^d H_s^i \sigma_s^{ij} H_s^j \right) dC_s \\ & \leq 2 \int_0^t I(\|H_s\| \leq n) \left(\sum_{i,j=1}^d H_s^i (\pi_1^{ij})_s H_s^j \right) dC_s + 2 \int_0^t I(\|H_s\| \leq n) \left(\sum_{i,j=1}^d H_s^i (\pi_2^{ij})_s H_s^j \right) dC_s \\ & \leq 2 \int_0^t \left(\sum_{i,j=1}^d H_s^i (\pi_1^{ij})_s H_s^j \right) dC_s + 2 \int_0^t \left(\sum_{i,j=1}^d H_s^i (\pi_2^{ij})_s H_s^j \right) dC_s, \quad t \geq 0, n \in \mathbb{N}. \end{aligned}$$

This leads to the inclusion $H \in L_{\text{loc}}^1(\alpha_1 M_1 + \alpha_2 M_2)$. Equality (3.7) for H is proved with the help of Lemma 3.5 (f). \square

2. Integral with respect to a process of finite variation. Let $A \in \mathcal{V}^d$. Then there exist $C \in \mathcal{V}^+$ and optional processes a^i ($i = 1, \dots, d$) such that

$$A^i = \int_0^\cdot a_s^i dC_s. \quad (3.8)$$

Let us consider the space

$$L_{\text{var}}(A) = \left\{ H = (H^1, \dots, H^d) : H \text{ is predictable and,} \right. \\ \left. \text{for any } t \geq 0, \int_0^t \left| \sum_{i=1}^d H_s^i a_s^i \right| dC_s < \infty \text{ a.s.} \right\}.$$

Obviously, $L_{\text{var}}(A)$ does not depend on the choice of a^i and C that satisfy (3.8).

Definition 3.7. For $H \in L_{\text{var}}(A)$, the *vector stochastic integral* is defined by

$$H \bullet A = \int_0^\cdot \left(\sum_{i=1}^d H_s^i a_s^i \right) dC_s.$$

Obviously, $H \bullet A \in \mathcal{V}$.

Lemma 3.8. *Let $A \in \mathcal{V}^d$. The vector stochastic integral with respect to A has the following properties:*

(a) *if $H_1, H_2 \in L_{\text{var}}(A)$, $\alpha_1, \alpha_2 \in \mathbb{R}$, then $\alpha_1 H_1 + \alpha_2 H_2 \in L_{\text{var}}(A)$ and*

$$(\alpha_1 H_1 + \alpha_2 H_2) \bullet A = \alpha_1 (H_1 \bullet A) + \alpha_2 (H_2 \bullet A);$$

(b) *if $H \in L_{\text{var}}(A_1) \cap L_{\text{var}}(A_2)$, $\alpha_1, \alpha_2 \in \mathbb{R}$, then $H \in L_{\text{var}}(\alpha_1 A_1 + \alpha_2 A_2)$ and*

$$H \bullet (\alpha_1 A_1 + \alpha_2 A_2) = \alpha_1 (H \bullet A_1) + \alpha_2 (H \bullet A_2);$$

(c) *if $H \in L_{\text{var}}(A)$, then $H \bullet A \in \mathcal{V}$ and*

$$\text{Var}(H \bullet A) = \int_0^\cdot \left| \sum_{i=1}^d H_s^i a_s^i \right| dC_s,$$

where a^i and C satisfy (3.8);

(d) *if $H \in L_{\text{var}}(A)$ and $D \in \mathcal{P}$, then*

$$I_D \bullet (H \bullet A) = (H I_D) \bullet A;$$

(e) *if $H \in L_{\text{var}}(A)$, then*

$$\Delta(H \bullet A) = \langle H, \Delta A \rangle;$$

(f) *if $H \in L_{\text{var}}(A)$ and $H_n = HI(\|H\| \leq n)$, then*

$$H_n \bullet A \xrightarrow[n \rightarrow \infty]{\text{u.p.}} H \bullet A.$$

Proof. Straightforward. \square

3. Integral with respect to a semimartingale. We have so far defined two vector stochastic integrals: the integral with respect to a local martingale (Definition 3.4) and the integral with respect to a process of finite variation (Definition 3.7). At the moment, it is not clear whether these two definitions coincide for a process X that belongs to $\mathcal{M}_{\text{loc}}^d \cap \mathcal{V}^d$. Therefore, in order to avoid ambiguities, we will write $(M)H \bullet X$ for the integral with respect to a local martingale given by Definition 3.4 and $(\text{LS})H \bullet X$ for the Lebesgue-Stieltjes integral given by Definition 3.7.

Definition 3.9. Let $X \in \mathcal{S}^d$. A process H is X -integrable if there exists a decomposition $X = A + M$ with $A \in \mathcal{V}^d$, $M \in \mathcal{M}_{\text{loc}}^d$ such that $H \in L_{\text{var}}(A) \cap L_{\text{loc}}^1(M)$. In this case, the *vector stochastic integral* is defined by

$$H \bullet X = (\text{LS})H \bullet A + (M)H \bullet M.$$

The space of X -integrable processes will be denoted by $L(X)$.

Obviously, $H \bullet X \in \mathcal{S}$.

Remarks. (i) If $H = (H^1, \dots, H^d)$ is a predictable locally bounded process, then, for any decomposition $X = A + M$ with $A \in \mathcal{V}^d$, $M \in \mathcal{M}_{\text{loc}}^d$, we have $H \in L_{\text{var}}(A) \cap L_{\text{loc}}^1(M)$ (the inclusion $H \in L_{\text{var}}(A)$ is obvious, while the inclusion $H \in L_{\text{loc}}^1(M)$ is verified with the help of (3.2)). Thus, $L(X)$ includes all the locally bounded predictable processes.

(ii) If $H \in L(X)$, then H may not belong to $L_{\text{var}}(A) \cap L_{\text{loc}}^1(M)$ for *all* the decompositions $X = A + M$ with $A \in \mathcal{V}^d$, $M \in \mathcal{M}_{\text{loc}}^d$ (see Example 5.2). \square

Let us prove the correctness of Definition 3.9.

Lemma 3.10. *Suppose that $Z \in \mathcal{V}^d \cap \mathcal{M}_{\text{loc}}^d$ and $H \in L_{\text{var}}(Z) \cap L_{\text{loc}}^1(Z)$. Then*

$$(\text{LS})H \bullet Z = (M)H \bullet Z. \quad (3.9)$$

Proof. By the localization, we can suppose from the outset that $H \in L^1(Z)$ and $E[Z^i]_{\infty}^{1/2} < \infty$ for $i = 1, \dots, d$ (see Proposition 2.5). Applying the monotone class lemma, we prove (3.9) for $H = \lambda I_D$, where $\lambda \in \mathbb{R}^d$, $D \in \mathcal{P}$.

Let H be a bounded predictable process. There exists a sequence (H_n) that tends to H uniformly, and each H_n is a finite sum of the form $\sum_k \lambda_k I_{D_k}$ with $D_k \in \mathcal{P}$ (and thus, (3.9) is true for H_n). Then

$$\begin{aligned} (\text{LS})H_n \bullet Z &\xrightarrow[n \rightarrow \infty]{\text{u.p.}} (\text{LS})H \bullet Z, \\ (M)H_n \bullet Z &\xrightarrow[n \rightarrow \infty]{\mathcal{H}^1} (M)H \bullet Z, \end{aligned}$$

and consequently,

$$(M)H_n \bullet Z \xrightarrow[n \rightarrow \infty]{\text{u.p.}} (M)H \bullet Z.$$

This proves the statement for a bounded H .

Finally, for $H \in L_{\text{var}}(Z) \cap L^1(Z)$, we use the same reasoning as above with $H_n = HI(\|H\| \leq n)$. \square

Lemmas 3.6, 3.8 (b) and 3.10 yield

Corollary 3.11 (Correctness of Definition 3.9). *Suppose that $X = A + M = A' + M'$ with $A, A' \in \mathcal{V}^d$, $M, M' \in \mathcal{M}_{\text{loc}}^d$ and*

$$H \in L_{\text{var}}(A) \cap L_{\text{loc}}^1(M) \cap L_{\text{var}}(A') \cap L_{\text{loc}}^1(M').$$

Then

$$(\text{LS})H \bullet A + (\text{M})H \bullet M = (\text{LS})H \bullet A' + (\text{M})H \bullet M'.$$

4 Properties of the Vector Stochastic Integral

1. Linearity. We will prove here two properties: “linearity in X ” and “linearity in H ”. The first one is a direct consequence of Lemmas 3.6 and 3.8 (b).

Theorem 4.1. *If $H \in L(X_1) \cap L(X_2)$, $\alpha_1, \alpha_2 \in \mathbb{R}$, then $H \in L(\alpha_1 X_1 + \alpha_2 X_2)$ and*

$$H \bullet (\alpha_1 X_1 + \alpha_2 X_2) = \alpha_1 (H \bullet X_1) + \alpha_2 (H \bullet X_2).$$

The “linearity in H ” is less trivial. In order to prove it, we will need the auxiliary lemma. This lemma is also very useful for proving other properties of the vector stochastic integral. The lemma is taken from [20].

Lemma 4.2. *Let $X \in \mathcal{S}_p^d$ and let $X = X_0 + A + M$ be the canonical decomposition of X . Suppose that $H \in L(X)$. Then*

$$H \bullet X \in \mathcal{S}_p \iff H \in L_{\text{var}}(A) \cap L_{\text{loc}}^1(M).$$

In this case, the canonical decomposition of $H \bullet X$ is given by

$$H \bullet X = H \bullet A + H \bullet M. \tag{4.1}$$

Proof. Suppose that $H \bullet X \in \mathcal{S}_p$. Set $Y = H \bullet X$. Let $Y = B + N$ be the canonical decomposition of Y . For any $n \in \mathbb{N}$, we have $I(\|H\| \leq n) \in L_{\text{var}}(B) \cap L_{\text{loc}}^1(N)$. Furthermore,

$$I(\|H\| \leq n) \bullet Y = I(\|H\| \leq n) \bullet B + I(\|H\| \leq n) \bullet N \tag{4.2}$$

is the canonical decomposition of $I(\|H\| \leq n) \bullet Y$ (note that both $I(\|H\| \leq n)$ and B are predictable). Set $H_n = HI(\|H\| \leq n)$. Then

$$H_n \bullet X = H_n \bullet A + H_n \bullet M \tag{4.3}$$

is the canonical decomposition of $H_n \bullet X$. Applying Lemmas 3.5 (d) and 3.8 (d), one can verify that

$$I(\|H\| \leq n) \bullet Y = H_n \bullet X. \tag{4.4}$$

Using (4.2), (4.3), (4.4) and the uniqueness of the canonical decomposition, we get

$$\begin{aligned} I(\|H\| \leq n) \bullet B &= H_n \bullet A, \\ I(\|H\| \leq n) \bullet N &= H_n \bullet M. \end{aligned}$$

By Lemma 3.5 (b),

$$\begin{aligned} \int_0^\cdot I(\|H_s\| \leq n) d[N]_s &= [I(\|H\| \leq n) \bullet N] = [H_n \bullet M] \\ &= \int_0^\cdot I(\|H_s\| \leq n) \left(\sum_{i,j=1}^d H_s^i \pi_s^{ij} H_s^j \right) dC_s, \end{aligned}$$

where π^{ij} and C satisfy (3.1). Consequently,

$$\int_0^\cdot \left(\sum_{i,j=1}^d H_s^i \pi_s^{ij} H_s^j \right) dC_s = [N],$$

and, applying Proposition 2.5, we deduce that $H \in L_{\text{loc}}^1(M)$.

By Lemma 3.8 (c),

$$\int_0^\cdot I(\|H_s\| \leq n) d(\text{Var } B)_s = \int_0^\cdot I(\|H_s\| \leq n) \left| \sum_{i=1}^d H_s^i a_s^i \right| dC_s,$$

where a^i , C satisfy (3.8). Consequently, for any $t \geq 0$,

$$\int_0^t \left| \sum_{i=1}^d H_s^i a_s^i \right| dC_s = (\text{Var } B)_t < \infty \quad \text{a.s.}$$

This means that $H \in L_{\text{var}}(A)$. Thus, we have proved one implication of the Lemma.

The reverse implication is obvious as well as the fact that (4.1) is the canonical decomposition of $H \bullet X$. \square

Theorem 4.3. *If $H_1, H_2 \in L(X)$, $\alpha_1, \alpha_2 \in \mathbb{R}$, then $\alpha_1 H_1 + \alpha_2 H_2 \in L(X)$ and*

$$(\alpha_1 H_1 + \alpha_2 H_2) \bullet X = \alpha_1 (H_1 \bullet X) + \alpha_2 (H_2 \bullet X).$$

Proof. Take

$$D = \{(\omega, t) : \|\Delta X_t(\omega)\| > 1 \text{ or } |\Delta(H_1 \bullet X)_t(\omega)| > 1 \text{ or } |\Delta(H_2 \bullet X)_t(\omega)| > 1\}.$$

Obviously, $D \in \mathcal{O}$ and D is a.s. discrete, i.e. for any $t \geq 0$ and almost every ω , the set $\{s : (\omega, s) \in D, s \leq t\}$ is finite. Take $Y_k = H_k \bullet X$ ($k = 1, 2$),

$$\tilde{X} = \sum_{s \leq \cdot} I_D \Delta X_s, \quad \bar{X} = X - \tilde{X}, \tag{4.5}$$

$$\tilde{Y}_k = \sum_{s \leq \cdot} I_D \Delta(Y_k)_s, \quad \bar{Y}_k = Y_k - \tilde{Y}_k, \quad k = 1, 2. \tag{4.6}$$

According to Lemmas 3.5 (e) and 3.8 (e), we get $\Delta Y_k = \langle H_k, \Delta X \rangle$ that leads to

$$\tilde{Y}_k = H_k \bullet \tilde{X}, \quad k = 1, 2 \tag{4.7}$$

(note that $H_k \in L_{\text{var}}(\tilde{X})$ since D is a.s. discrete).

We have $H_k \in L(X) \cap L(\tilde{X})$. Thanks to Theorem 4.1, $H_k \in L(\bar{X})$, and, in view of (4.7),

$$\bar{Y}_k = H_k \bullet \bar{X}, \quad k = 1, 2.$$

By Proposition 2.13, the semimartingales \bar{X} , \bar{Y}_1 and \bar{Y}_2 are special. Let $\bar{X} = \bar{X}_0 + A + M$ be the canonical decomposition of \bar{X} . According to Lemma 4.2, $H_k \in L_{\text{var}}(A) \cap L_{\text{loc}}^1(M)$. Lemmas 3.5 (a), 3.8 (a) yield that $(\alpha_1 H_1 + \alpha_2 H_2) \in L_{\text{var}}(A) \cap L_{\text{loc}}^1(M)$ and

$$(\alpha_1 H_1 + \alpha_2 H_2) \bullet \bar{X} = \alpha_1 (H_1 \bullet \bar{X}) + \alpha_2 (H_2 \bullet \bar{X}). \quad (4.8)$$

By Lemma (3.5) (a), $(\alpha_1 H_1 + \alpha_2 H_2) \in L_{\text{var}}(\tilde{X})$ and

$$(\alpha_1 H_1 + \alpha_2 H_2) \bullet \tilde{X} = \alpha_1 (H_1 \bullet \tilde{X}) + \alpha_2 (H_2 \bullet \tilde{X}). \quad (4.9)$$

Taken together, equalities (4.8), (4.9) and Theorem 4.1 yield the desired statement. \square

2. Associativity. We mean by associativity the following property: $K \bullet (H \bullet X) = (KH) \bullet X$. This property will be proved in two forms (Theorem 4.6 and Theorem 4.7 below).

Lemma 4.4. *Let $M \in \mathcal{M}_{\text{loc}}^d$ and $H \in L_{\text{loc}}^1(M)$. Let K be a one-dimensional predictable process. Then*

$$K \in L_{\text{loc}}^1(H \bullet M) \iff KH \in L_{\text{loc}}^1(M). \quad (4.10)$$

In this case,

$$K \bullet (H \bullet M) = (KH) \bullet M. \quad (4.11)$$

Proof. Statement (4.10) follows from the equality

$$\int_0^\infty K_s^2 d[H \bullet M]_s = \int_0^\infty \left(\sum_{i=1}^d K_s^2 H_s^i \pi_s^{ij} H_s^j \right) dC_s,$$

where π^{ij} and C satisfy (3.1). (This equality is a consequence of Lemma 3.5 (b)).

Equality (4.11) is proved by the standard scheme:

1. for $K = I_D$ (by Lemma 3.5 (d));
2. for bounded K (through the uniform approximation);
3. for $K \in L_{\text{loc}}^1(H \bullet M)$ (through the approximation by $KI(|K| \leq n)$). \square

Lemma 4.5. *Let $A \in \mathcal{V}^d$ and $H \in L_{\text{var}}(A)$. Let K be a one-dimensional predictable process. Then*

$$K \in L_{\text{var}}(H \bullet A) \iff KH \in L_{\text{var}}(A).$$

In this case,

$$K \bullet (H \bullet A) = (KH) \bullet A.$$

Proof. Straightforward. \square

Theorem 4.6 (First Associativity Theorem). *Let $X \in \mathcal{S}^d$ and $H \in L(X)$. Let K be a one-dimensional predictable process. Then*

$$K \in L(H \bullet X) \iff KH \in L(X). \quad (4.12)$$

In this case,

$$K \bullet (H \bullet X) = (KH) \bullet X.$$

Proof. Suppose that $K \in L(H \bullet X)$. Take

$$D = \{(\omega, t) : \|\Delta X_t(\omega)\| > 1 \text{ or } |\Delta(H \bullet X)_t(\omega)| > 1 \text{ or } |\Delta(K \bullet (H \bullet X))_t(\omega)| > 1\}.$$

Set

$$Y = H \bullet X, \quad Z = K \bullet Y$$

and define \tilde{X} , \tilde{Y} , \tilde{Z} , \bar{X} , \bar{Y} , \bar{Z} in the same way as in (4.5), (4.6). It follows from Lemmas 3.5 (e), 3.8 (e) that

$$\tilde{Y} = H \bullet \tilde{X}, \quad \tilde{Z} = K \bullet \tilde{Y}. \quad (4.13)$$

By Theorem 4.1,

$$\bar{Y} = H \bullet \bar{X}, \quad \bar{Z} = K \bullet \bar{Y}.$$

In view of Proposition 2.13, the semimartingales \bar{X} , \bar{Y} , \bar{Z} are special. Let $\bar{X} = \bar{X}_0 + A + M$ be the canonical decomposition of \bar{X} . According to Lemma 4.2, $H \in L_{\text{var}}(A) \cap L_{\text{loc}}^1(M)$ and

$$\bar{Y} = H \bullet A + H \bullet M$$

is the canonical decomposition of \bar{Y} . By Lemma 4.2, $K \in L_{\text{var}}(H \bullet A) \cap L_{\text{loc}}^1(H \bullet M)$. Applying Lemmas 4.4, 4.5, we get: $KH \in L_{\text{var}}(A) \cap L_{\text{loc}}^1(M)$ and

$$\bar{Z} = K \bullet (H \bullet A) + K \bullet (H \bullet M) = (KH) \bullet A + (KH) \bullet M = (KH) \bullet \bar{X}.$$

By (4.13), $\tilde{Z} = (KH) \bullet \tilde{X}$. Consequently, $KH \in L(X)$ and $Z = (KH) \bullet X$.

The proof of the reverse implication in (4.12) is similar. \square

The same arguments as above allow us to prove the second associativity theorem:

Theorem 4.7 (Second Associativity Theorem). *Let $X \in \mathcal{S}^d$ and let H be a d -dimensional process such that $H^i \in L(X^i)$ for each $i = 1, \dots, d$. Set $Y^i = H^i \bullet X^i$. Let K be a predictable d -dimensional process. Set $J^i = K^i H^i$, $J = (J^1, \dots, J^d)$. Then*

$$K \in L(Y) \iff J \in L(X).$$

In this case,

$$K \bullet Y = J \bullet X.$$

3. Closedness of the space of stochastic integrals. The following metric on \mathcal{S} was introduced by M. Émery in [13].

Definition 4.8. For $X, Y \in \mathcal{S}$, the *Émery distance* between X and Y equals

$$\rho(X, Y) = \sup_{|K| \leq 1} \left\{ \sum_{m=1}^{\infty} 2^{-m} \mathbf{E}(|(K \bullet (X - Y))_m| \wedge 1) \right\}$$

(the supremum is taken over the predictable K with $|K| \leq 1$). The corresponding topology is called the *semimartingale* or *Émery* topology. The convergence in this topology will be denoted as $X_n \xrightarrow{\mathcal{S}} X$.

Remarks. (i) If $M_n \xrightarrow{\mathcal{H}^1} M$, then $M_n \xrightarrow{\mathcal{S}} M$. This is a consequence of Lemma 3.5 (b) and the Davis inequalities.

(ii) Obviously, if $X_n \xrightarrow{\mathcal{S}} X$, then $X_n \xrightarrow{\text{u.p.}} X$.

(iii) The space \mathcal{S} is complete with respect to the Émery metric (for the proof, see [13; Théorème 1]). \square

The proof of the following result is borrowed from [7].

Lemma 4.9. *Suppose that a sequence (X_n) converges to X in the Émery topology with respect to a measure \mathbf{P} . Let $\mathbf{Q} \ll \mathbf{P}$. Then (X_n) converges to X in the Émery topology with respect to \mathbf{Q} .*

Proof. The function $\|X\|_{\mathcal{S}} = \rho(X, 0)$ is a quasinorm on \mathcal{S} (for the definition, see [39; Ch. I]). The space $\mathcal{S}(\mathbf{P})$ (resp: $\mathcal{S}(\mathbf{Q})$) is complete with respect to the quasinorm $\|\cdot\|_{\mathcal{S}(\mathbf{P})}$ (resp: $\|\cdot\|_{\mathcal{S}(\mathbf{Q})}$) (see Remark (iii) above). Let us consider the map

$$\varphi : \mathcal{S}(\mathbf{P}) \ni X \longmapsto X \in \mathcal{S}(\mathbf{Q}).$$

Taking into account Remark (ii) above, we deduce that the graph of φ is closed. The application of the closed graph theorem (see [39; Ch. II]) shows that φ is continuous. This yields the result. \square

Proposition 4.10. *Let $X \in \mathcal{S}^d$. Then the space*

$$\mathcal{L}(X) = \{H \bullet X : H \in L(X)\}$$

is closed in the Émery topology.

For the proof, see [30; Théorème V.4].

4. Change of measure and filtration. The vector stochastic integral remains the same under an equivalent change of measure.

Lemma 4.11. *Let $X \in \mathcal{S}^d$ and $H \in L(X)$. Set $H_n = HI(\|H\| \leq n)$. Then*

$$H_n \bullet X \xrightarrow[n \rightarrow \infty]{\mathcal{S}} H \bullet X.$$

Proof. Let $X = A + M$ be a decomposition such that $H \in L_{\text{var}}(A) \cap L_{\text{loc}}^1(M)$. There exists an increasing sequence (τ_m) of stopping times such that $\tau_m \rightarrow \infty$ a.s. and $HI_{[[0, \tau_m]]} \in L^1(M)$. By Lemma 3.5 (f),

$$(H_n I_{[[0, \tau_m]]}) \bullet M \xrightarrow[n \rightarrow \infty]{\mathcal{H}^1} (HI_{[[0, \tau_m]]}) \bullet M, \quad m \in \mathbb{N}.$$

The convergence also holds in the \acute{E} myery topology (see Remark (i) after Definition 4.8). Since $\tau_m \rightarrow \infty$ a.s., we deduce that

$$H_n \bullet M \xrightarrow[n \rightarrow \infty]{\mathcal{S}} H \bullet M.$$

For any predictable K with $|K| \leq 1$ and any $t \geq 0$, we have

$$\begin{aligned} \left(\text{Var}(K \bullet ((H - H_n) \bullet A)) \right)_t &= \int_0^t |K_s| I(\|H_s\| > n) \left| \sum_{i=1}^d H_s^i a_s^i \right| dC_s \\ &\leq \int_0^t I(\|H_s\| > n) \left| \sum_{i=1}^d H_s^i a_s^i \right| dC_s \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0, \end{aligned}$$

where a^i and C satisfy (3.8) (we used here Lemma 3.8 (c)). Hence,

$$H_n \bullet A \xrightarrow[n \rightarrow \infty]{\mathcal{S}} H \bullet A.$$

This completes the proof. \square

Lemma 4.12. *Let $X \in \mathcal{S}^d$ and (H_n) be a sequence of predictable d -dimensional processes that tends to H pointwise. Suppose that there exists $a \in \mathbb{R}$ such that $\|H_n\| \leq a$ for all $n \in \mathbb{N}$. Then*

$$H_n \bullet X \xrightarrow[n \rightarrow \infty]{\mathcal{S}} H \bullet X.$$

The proof is similar to the proof of Lemma 4.11.

Lemma 4.13. *Let $X \in \mathcal{S}^d$ and H be a predictable d -dimensional process. Set $H_n = HI(\|H\| \leq n)$ and suppose that the sequence $(H_n \bullet X)$ converges in the \acute{E} myery topology to a process Z . Then $H \in L(X)$ and $Z = H \bullet X$.*

Proof. By Proposition 4.10, there exists $K \in L(X)$ such that $Z = K \bullet X$. Set

$$J = \sum_{n=1}^{\infty} \frac{1}{n} I(n-1 \leq \|H\| < n).$$

Then J is predictable and $0 < J \leq 1$. Using Theorem 4.6 and the definition of the \acute{E} myery topology, it is easy to verify that

$$J \bullet (H_n \bullet X) \xrightarrow[n \rightarrow \infty]{\mathcal{S}} J \bullet Z = (JK) \bullet X.$$

Note that $\|JH\| \leq 1$. By Lemma 4.12 and Theorem 4.6,

$$J \bullet (H_n \bullet X) = (JH_n) \bullet X \xrightarrow[n \rightarrow \infty]{\mathcal{S}} (JH) \bullet X.$$

Thus, $(JH) \bullet X = (JK) \bullet X$.

As $J^{-1}JK(=K) \in L(X)$, Theorem 4.6 shows that $J^{-1} \in L((JK) \bullet X) = L((JH) \bullet X)$. One more application of this theorem yields that $H(=J^{-1}JH) \in L(X)$ and

$$H \bullet X = J^{-1} \bullet ((JH) \bullet X) = J^{-1}((JK) \bullet X) = K \bullet X = Z.$$

This completes the proof. \square

Remark. Lemmas 4.11 and 4.13 combined together show that $H \in L(X)$ if and only if the sequence $(H_n \bullet X)$, where $H_n = HI(\|H\| \leq n)$, converges in the Émery topology. This provides another way of defining the class $L(X)$. In the paper [7], C.S. Chou, P.-A. Meyer and C. Stricker took this approach from the outset and defined the stochastic integral as the limit of $(H_n \bullet X)$ in the Émery topology. However, they considered only the one-dimensional case. \square

In the following theorem, we use the notations $\mathcal{S}^d(\mathbb{P})$, $L(X; \mathbb{P})$, $(\mathbb{P})H \bullet X$ in order to indicate the dependence of these notions on the measure \mathbb{P} .

Theorem 4.14. *Let $X \in \mathcal{S}^d(\mathbb{P})$ and $H \in L(X; \mathbb{P})$. Let $\mathbb{Q} \ll \mathbb{P}$. Then $X \in \mathcal{S}^d(\mathbb{Q})$, $H \in L(X; \mathbb{Q})$ and*

$$(\mathbb{P})H \bullet X = (\mathbb{Q})H \bullet X. \quad (4.14)$$

Remark. The integral $(\mathbb{P})H \bullet X$ is defined up to \mathbb{P} -indistinguishability, while $(\mathbb{Q})H \bullet X$ is defined only up to \mathbb{Q} -indistinguishability. Therefore, (4.14) is understood as follows: any version of $(\mathbb{P})H \bullet X$ is a version of $(\mathbb{Q})H \bullet X$. \square

Proof of Theorem 4.14. For the implication $X \in \mathcal{S}^d(\mathbb{P}) \Rightarrow X \in \mathcal{S}^d(\mathbb{Q})$, see [21; Ch. III, (3.13)] or [29; Ch. 4, §5]. Let us first prove (4.14) for bounded H .

For $H = \lambda I_D$, where $\lambda \in \mathbb{R}^d$, $D \in \mathcal{P}$, this equality follows from the monotone class lemma. By linearity (Theorem 4.3), we extend (4.14) to the finite sums of the form $\sum_k \lambda_k I_{D_k}$. Any bounded H can be uniformly approximated by a sequence (H_n) , where each H_n has this form. By Lemma 4.12,

$$(\mathbb{P})H_n \bullet X \xrightarrow[n \rightarrow \infty]{\mathcal{S}(\mathbb{P})} (\mathbb{P})H \bullet X, \quad (4.15)$$

$$(\mathbb{Q})H_n \bullet X \xrightarrow[n \rightarrow \infty]{\mathcal{S}(\mathbb{Q})} (\mathbb{Q})H \bullet X. \quad (4.16)$$

Combining (4.15) with Lemma 4.9, we get

$$(\mathbb{P})H_n \bullet X \xrightarrow[n \rightarrow \infty]{\mathcal{S}(\mathbb{Q})} (\mathbb{P})H \bullet X.$$

As $(\mathbb{P})H_n \bullet X = (\mathbb{Q})H_n \bullet X$, we obtain $(\mathbb{P})H \bullet X = (\mathbb{Q})H \bullet X$.

Now, let $H \in L(X)$. Set $H_n = HI(\|H\| \leq n)$. By Lemma 4.11,

$$(\mathbb{P})H_n \bullet X \xrightarrow[n \rightarrow \infty]{\mathcal{S}(\mathbb{P})} (\mathbb{P})H \bullet X.$$

Hence,

$$(\mathbf{P})H_n \bullet X \xrightarrow[n \rightarrow \infty]{\mathcal{S}(\mathbf{Q})} (\mathbf{P})H \bullet X.$$

From the above result (for bounded H), $(\mathbf{P})H_n \bullet X = (\mathbf{Q})H_n \bullet X$, and therefore,

$$(\mathbf{Q})H_n \bullet X \xrightarrow[n \rightarrow \infty]{\mathcal{S}(\mathbf{Q})} (\mathbf{P})H \bullet X.$$

Applying Lemma 4.13, we get: $H \in L(X; \mathbf{Q})$ and $(\mathbf{P})H \bullet X = (\mathbf{Q})H \bullet X$. \square

The vector stochastic integral is stable not only under an equivalent change of measure, but under a change of filtration as well. The paper [20] by J. Jacod contains the following statement (we use the notations $\mathcal{S}^d(\mathcal{F})$, $L(X; \mathcal{F})$, $(\mathcal{F})H \bullet X$ to indicate that these objects depend on the filtration (\mathcal{F}_t)).

Proposition 4.15. *Let $X \in \mathcal{S}^d(\mathcal{F})$ and $H \in L(X; \mathcal{F})$. Let (\mathcal{G}_t) be a filtration such that $\mathcal{F}_t^X \subseteq \mathcal{G}_t \subseteq \mathcal{F}_t$ and H is (\mathcal{G}_t) -predictable. Then $X \in \mathcal{S}^d(\mathcal{G})$, $H \in L(X; \mathcal{G})$ and*

$$(\mathcal{F})H \bullet X = (\mathcal{G})H \bullet X.$$

Remarks. (i) The inclusion $L(X; \mathcal{F}) \subseteq L(X; \mathcal{G})$ may be strict.

(ii) If the condition $\mathcal{F}_t^X \subseteq \mathcal{G}_t \subseteq \mathcal{F}_t$ is violated, then X may not be a (\mathcal{G}_t) -semimartingale. \square

5. Quadratic covariation. We will prove here the multidimensional version of the property “ $[H \bullet X, K \bullet Y] = HK \bullet [X, Y]$ ”.

Lemma 4.16. *Let ν be a finite positive measure on $\mathcal{B}([0, 1])$. Let f, g, h be positive functions from $L^1(\nu)$ such that, for any set J of the form $(a_1, b_1] \cup \dots \cup (a_m, b_m]$ with $a_i, b_i \in \mathbb{Q}$,*

$$\int_J f d\nu \leq \left(\int_J g d\nu \right)^{1/2} \left(\int_J h d\nu \right)^{1/2}.$$

Then $f \leq \sqrt{gh}$ ν -a.e.

Proof. Suppose that $\nu\{f > \sqrt{gh}\} > 0$. Then there exist constants κ and λ such that $\nu(D) > 0$, where $D = \{f > \sqrt{\kappa\lambda}, g < \kappa, h < \lambda\}$. There exists a sequence of sets $(J_n)_{n=1}^\infty$ of the form $(a_1, b_1] \cup \dots \cup (a_m, b_m]$ with $a_i, b_i \in \mathbb{Q}$ such that $\nu(J_n \Delta D) \rightarrow 0$. Then

$$\int_{J_n} f d\nu \xrightarrow[n \rightarrow \infty]{} \int_D f d\nu, \quad \int_{J_n} g d\nu \xrightarrow[n \rightarrow \infty]{} \int_D g d\nu, \quad \int_{J_n} h d\nu \xrightarrow[n \rightarrow \infty]{} \int_D h d\nu,$$

which implies that

$$\int_D f d\nu \leq \left(\int_D g d\nu \right)^{1/2} \left(\int_D h d\nu \right)^{1/2}.$$

But this contradicts the choice of D . \square

Lemma 4.17. *Let $M \in \mathcal{M}_{\text{loc}}^d$, $N \in \mathcal{M}_{\text{loc}}^e$. Choose $C \in \mathcal{V}^+$ and optional processes π^{ij} , ρ^{ij} , σ^{ij} in such a way that $[M^i, M^j] = \pi^{ij} \bullet C$, $[M^i, N^j] = \rho^{ij} \bullet C$, $[N^i, N^j] = \sigma^{ij} \bullet C$. Then, for any optional processes $H = (H_t^1, \dots, H_t^d)_{t \geq 0}$ and $K = (K_t^1, \dots, K_t^e)_{t \geq 0}$,*

$$\left| \sum_{i,j=1}^{d,e} H^i \rho^{ij} K^j \right| \bullet C \leq \left(\left(\sum_{i,j=1}^d H^i \pi^{ij} H^j \right) \bullet C \right)^{1/2} \left(\left(\sum_{i,j=1}^e K^i \sigma^{ij} K^j \right) \bullet C \right)^{1/2}.$$

Proof. Take $\alpha \in \mathbb{R}^d$, $\beta \in \mathbb{R}^e$ and a set J of the form $(a_1, b_1] \cup \dots \cup (a_m, b_m]$. Then

$$\left(I_J \sum_{i,j=1}^{d,e} \alpha^i \rho^{ij} \beta^j \right) \bullet C = [(\alpha I_J) \bullet M, (\beta I_J) \bullet N],$$

and hence,

$$\left| I_J \sum_{i,j=1}^{d,e} \alpha^i \rho^{ij} \beta^j \right| \bullet C = \text{Var}[(\alpha I_J) \bullet M, (\beta I_J) \bullet N].$$

It follows from Proposition 2.14 that

$$\begin{aligned} \text{Var}[(\alpha I_J) \bullet M, (\beta I_J) \bullet N] &\leq [(\alpha I_J) \bullet M]^{1/2} [(\beta I_J) \bullet N]^{1/2} \\ &= \left(\left(I_J \sum_{i,j=1}^d \alpha^i \pi^{ij} \alpha^j \right) \bullet C \right)^{1/2} \left(\left(I_J \sum_{i,j=1}^e \beta^i \sigma^{ij} \beta^j \right) \bullet C \right)^{1/2}. \end{aligned}$$

Using Lemma 4.16, we conclude that

$$\left| \sum_{i,j=1}^{d,e} \alpha^i \rho_t^{ij}(\omega) \beta^j \right| \leq \left(\sum_{i,j=1}^d \alpha^i \pi_t^{ij}(\omega) \alpha^j \right)^{1/2} \left(\sum_{i,j=1}^e \beta^i \sigma_t^{ij}(\omega) \beta^j \right)^{1/2}$$

for $\mathbf{P} \otimes dC$ -a.e. (ω, t) . The application of the Cauchy inequality leads to the desired statement. \square

Lemma 4.18. *Let $M \in \mathcal{M}_{\text{loc}}^d$, $N \in \mathcal{M}_{\text{loc}}^e$, $H \in L_{\text{loc}}^1(M)$, $K \in L_{\text{loc}}^1(N)$. Choose $C \in \mathcal{V}^+$ and optional processes π^{ij} , ρ^{ij} , σ^{ij} in such a way that $[M^i, M^j] = \pi^{ij} \bullet C$, $[M^i, N^j] = \rho^{ij} \bullet C$, $[N^i, N^j] = \sigma^{ij} \bullet C$. Then $\sum_{i,j=1}^{d,e} H^i \rho^{ij} K^j \in L(C)$ and*

$$[H \bullet M, K \bullet N] = \left(\sum_{i,j=1}^{d,e} H^i \rho^{ij} K^j \right) \bullet C. \quad (4.17)$$

Proof. The inclusion $\sum_{i,j=1}^{d,e} H^i \rho^{ij} K^j \in L(C)$ follows from the previous lemma.

In order to prove (4.17), suppose first that $H \in L^1(M)$, $K \in L^1(N)$. Then there exist sequences (H_n) and (K_n) of simple integrands such that $H_n \xrightarrow[n \rightarrow \infty]{L^1(M)} H$, $K_n \xrightarrow[n \rightarrow \infty]{L^1(N)} K$. Obviously, equality (4.17) is valid for H_n and K_n .

Set $\bar{H}_n = H_n - H$, $\bar{K}_n = K_n - K$. Then

$$\begin{aligned} [H_n \bullet M, K_n \bullet N] - [H \bullet M, K \bullet N] \\ = [\bar{H}_n \bullet M, K \bullet N] + [H \bullet M, \bar{K}_n \bullet N] + [\bar{H}_n \bullet M, \bar{K}_n \bullet N]. \end{aligned} \quad (4.18)$$

We have

$$|[\overline{H}_n \bullet M, K \bullet N]| \leq [\overline{H}_n \bullet M]^{1/2} [K \bullet N]^{1/2}.$$

Since

$$[\overline{H}_n \bullet M]_{\infty}^{1/2} \xrightarrow[n \rightarrow \infty]{\text{P}} 0,$$

we have

$$[\overline{H}_n \bullet M, K \bullet N] \xrightarrow[n \rightarrow \infty]{\text{u.p.}} 0.$$

Applying the same reasoning to the other terms in the right-hand side of (4.18), we conclude that

$$[H_n \bullet M, K_n \bullet N] \xrightarrow[n \rightarrow \infty]{\text{u.p.}} [H \bullet M, K \bullet N].$$

On the other hand,

$$\begin{aligned} & \left(\sum_{i,j=1}^{d,e} H_n^i \rho^{ij} K_n^j \right) \bullet C - \left(\sum_{i,j=1}^{d,e} H^i \rho^{ij} K^j \right) \bullet C \\ &= \left(\sum_{i,j=1}^{d,e} \overline{H}_n^i \rho^{ij} K^j \right) \bullet C + \left(\sum_{i,j=1}^{d,e} H^i \rho^{ij} \overline{K}_n^j \right) \bullet C + \left(\sum_{i,j=1}^{d,e} \overline{H}_n^i \rho^{ij} \overline{K}_n^j \right) \bullet C. \end{aligned}$$

In view of the previous lemma, this sum tends to zero in probability uniformly on compact intervals. As a result, (4.17) is true for H and K .

The statement for the case $H \in L_{\text{loc}}^1(M)$, $K \in L_{\text{loc}}^1(N)$ can now easily be derived with the help of the equality $[X^\tau, Y^\tau] = [X, Y]^\tau$. \square

Remark. It follows from the above lemma that if $M \in \mathcal{M}_{\text{loc}}^d$, $H \in L_{\text{loc}}^1(M)$, then, for any $N \in \mathcal{M}_{\text{loc}}$,

$$[H \bullet M, N] = \left(\sum_{i=1}^d H^i \rho^i \right) \bullet C,$$

where ρ^i , C are chosen in such a way that $[M^i, N] = \rho^i \bullet C$. J. Jacod [20] took this property as a definition of the vector stochastic integral with respect to a local martingale. \square

Theorem 4.19. *Let $X \in \mathcal{S}^d$, $Y \in \mathcal{S}^e$, $H \in L(X)$, $K \in L(Y)$. Choose $C \in \mathcal{V}^+$ and optional processes π^{ij} , ρ^{ij} , σ^{ij} in such a way that $[X^i, X^j] = \pi^{ij} \bullet C$, $[X^i, Y^j] = \rho^{ij} \bullet C$, $[Y^i, Y^j] = \sigma^{ij} \bullet C$. Then $\sum_{i,j=1}^{d,e} H^i \rho^{ij} K^j \in L(C)$ and*

$$[H \bullet X, K \bullet Y] = \left(\sum_{i,j=1}^{d,e} H^i \rho^{ij} K^j \right) \bullet C.$$

Proof. Let $X = A + M$, $Y = B + N$ be decompositions of X and Y such that $H \in L_{\text{var}}(A) \cap L_{\text{loc}}^1(M)$, $K \in L_{\text{var}}(B) \cap L_{\text{loc}}^1(N)$. By the appropriate choice of C , we may assume that all the covariations $[A^i, B^j]$, $[M^i, B^j]$, $[A^i, N^j]$, $[M^i, N^j]$ are absolutely continuous with respect to C , and therefore, there exist processes ρ_{AB}^{ij} , ρ_{MB}^{ij} , ρ_{AN}^{ij} , ρ_{MN}^{ij} such that

$$\begin{aligned} [A^i, B^j] &= \rho_{AB}^{ij} \bullet C, & [M^i, B^j] &= \rho_{MB}^{ij} \bullet C, \\ [A^i, N^j] &= \rho_{AN}^{ij} \bullet C, & [M^i, N^j] &= \rho_{MN}^{ij} \bullet C. \end{aligned}$$

It follows from Proposition 2.16 that

$$\rho_{MB}^{ij} \bullet C = \sum_{s \leq \cdot} \Delta M_s^i \Delta B_s^j.$$

Hence,

$$\begin{aligned} \left| \sum_{i,j=1}^{d,e} H^i \rho_{MB}^{ij} K^j \right| \bullet C &= \sum_{s \leq \cdot} |\langle H_s, \Delta M_s \rangle \langle K_s, \Delta B_s \rangle| \\ &\leq \left(\sum_{s \leq \cdot} \langle H_s, \Delta M_s \rangle^2 \right)^{1/2} \left(\sum_{s \leq \cdot} \langle K_s, \Delta B_s \rangle^2 \right)^{1/2} \\ &= \left(\sum_{s \leq \cdot} (\Delta(H \bullet M)_s)^2 \right)^{1/2} \left(\sum_{s \leq \cdot} (\Delta(K \bullet B)_s)^2 \right)^{1/2}, \end{aligned}$$

and this expression is finite by Proposition 2.16. Furthermore,

$$\left(\sum_{i,j=1}^{d,e} H^i \rho_{MB}^{ij} K^j \right) \bullet C = \sum_{s \leq \cdot} \langle H_s, \Delta M_s \rangle \langle K_s, \Delta B_s \rangle = [H \bullet M, K \bullet B].$$

In a similar way we consider the combinations (A, B) , (A, N) and (M, N) (we apply the previous lemma to treat (M, N)). The desired result follows now from the equalities

$$\rho^{ij} = \rho_{AB}^{ij} + \rho_{MB}^{ij} + \rho_{AN}^{ij} + \rho_{MN}^{ij}$$

and

$$\begin{aligned} [H \bullet X, K \bullet Y] &= [H \bullet A, K \bullet B] + [H \bullet M, K \bullet B] \\ &\quad + [H \bullet A, K \bullet N] + [H \bullet M, K \bullet N]. \end{aligned} \quad \square$$

Corollary 4.20. *Let $X \in \mathcal{S}$, $Y \in \mathcal{S}$, $H \in L(X)$, $K \in L(Y)$. Then $HK \in L([X, Y])$ and*

$$[H \bullet X, K \bullet Y] = HK \bullet [X, Y].$$

Corollary 4.21. *Let $X \in \mathcal{S}^d$ and let X^c denote the continuous martingale part of X (for the definition, see [21; Ch. I, (4.27)]). If $H \in L(X)$, then $H \in L(X^c)$ and*

$$(H \bullet X)^c = H \bullet X^c.$$

Proof. Let $X = A + M$ be a decomposition of X such that $H \in L_{\text{var}}(A) \cap L_{\text{loc}}^1(M)$. Let $M = X^c + N$ be a decomposition of M into a continuous local martingale and a totally discontinuous local martingale. Then, in view of Proposition 2.16, $[(X^c)^i, N^j] = 0$, and hence, $[M^i, M^j] = [(X^c)^i, (X^c)^j] + [N^i, N^j]$. Consequently, $H \in L_{\text{loc}}^1(X^c)$ and

$$H \bullet X = H \bullet A + H \bullet X^c + H \bullet N.$$

It follows from Lemma 3.5 (e) that $H \bullet X^c$ is a continuous local martingale. Furthermore, by Theorem 4.19 combined with Proposition 2.16,

$$[H \bullet N] = \sum_{s \leq \cdot} \langle H_s, \Delta N_s \rangle^2.$$

On the other hand,

$$[H \bullet N] = \sum_{s \leq \cdot} \Delta(H \bullet N)_s^2 + [(H \bullet N)^c] = \sum_{s \leq \cdot} \langle H_s, \Delta N_s \rangle^2 + [(H \bullet N)^c].$$

Hence, $(H \bullet N)^c = 0$, which means that $H \bullet N$ is totally discontinuous. As a result, $(H \bullet X)^c = H \bullet X^c$. \square

5 Sigma-Martingales

1. Definitions and examples. The proof of the main part of the following statement is borrowed from [10].

Lemma 5.1. *Let $X \in \mathcal{S}^d$. The following conditions are equivalent:*

- (i) *for each $i = 1, \dots, d$, there exists a sequence of sets $D_n \in \mathcal{P}$ such that $D_n \subseteq D_{n+1}$, $\bigcup D_n = \Omega \times \mathbb{R}_+$ and each $I_{D_n} \bullet X^i$ is a uniformly integrable martingale;*
- (ii) *there exist $M \in \mathcal{M}_{\text{loc}}^d$ and $H = (H^1, \dots, H^d)$ such that, for each $i = 1, \dots, d$, $H^i \in L(M^i)$ and $X^i = X_0^i + H^i \bullet M^i$;*
- (iii) *there exist $N = (N^1, \dots, N^d)$ with $N^i \in \mathcal{H}^1$ and a strictly positive process K such that, for each $i = 1, \dots, d$, $K \in L(N^i)$ and $X^i = X_0^i + K \bullet N^i$.*

Proof. (i) \Rightarrow (iii) There exists a sequence of sets $D_n \in \mathcal{P}$ such that $D_n \uparrow \Omega \times \mathbb{R}_+$ and, for each $i = 1, \dots, d$, the process $M^{ni} = I_{D_n} \bullet X^i$ is a martingale. In view of Corollary 2.7, there exists a collection $(\tau_{mn})_{m,n=1}^\infty$ of stopping times such that, for each $n \in \mathbb{N}$, the sequence (τ_{mn}) is increasing in m , $\tau_{mn} \xrightarrow{m \rightarrow \infty} \infty$ a.s. and $(M^{ni})^{\tau_{mn}} \in \mathcal{H}^1$ ($m, n \in \mathbb{N}$, $i = 1, \dots, d$). We can find a sequence of scalars (α_{mn}) such that $0 < \alpha_{mn} < 2^{-(m+n)}$ and

$$\|\alpha_{mn}(M^{ni})^{\tau_{mn}}\|_{\mathcal{H}^1} \leq 2^{-(m+n)}, \quad m, n \in \mathbb{N}, i = 1, \dots, d.$$

If we take $\tau_{0n} = 0$,

$$\begin{aligned} \tilde{K} &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{mn} I(\tau_{n(m-1)} < t \leq \tau_{mn}) I_{D_n}, \\ N^i &= \tilde{K} \bullet X^i, \quad i = 1, \dots, d, \end{aligned}$$

then $0 < \tilde{K} < 1$ and $N^i \in \mathcal{H}^1$. Using Theorem 4.6, we get $X^i = X_0^i + K \bullet N^i$ ($i = 1, \dots, d$) with $K = \tilde{K}^{-1}$.

(iii) \Rightarrow (ii) This implication is obvious.

(ii) \Rightarrow (i) Using Corollary 2.7, we can find an increasing sequence $(\tau_n)_{n=1}^\infty$ of stopping times such that $\tau_n \rightarrow \infty$ a.s. and $(M^i)^{\tau_n} \in \mathcal{H}^1$ for each $i = 1, \dots, d$, $n \in \mathbb{N}$. Then the sets

$$D_n = \bigcap_{i=1}^d \{|H^i| \leq n\} \cap [0, \tau_n]$$

satisfy all the conditions of (i). \square

In the discrete-time case, the class of σ -martingales coincides with the class of local martingales (see [22], [36; Ch. II, §1c]). In the continuous-time case (even in the one-dimensional setting), the situation is different, and the class \mathcal{M}_σ can be strictly larger than \mathcal{M}_{loc} . The following example was given by M. Émery [14].

Example 5.2 (Émery). *Let τ be a random variable with $\mathbb{P}\{\tau > t\} = e^{-t}$. Let η be independent of τ with $\mathbb{P}\{\eta = +1\} = \mathbb{P}\{\eta = -1\} = 1/2$. Set*

$$X_t = \begin{cases} 0 & \text{if } t < \tau, \\ \eta/\tau & \text{if } t \geq \tau, \end{cases}$$

$\mathcal{F} = \sigma(\tau, \eta)$, $\mathcal{F}_t = \mathcal{F}_t^X$. Then $X \in \mathcal{M}_\sigma$ and $X \notin \mathcal{M}_{\text{loc}}$.

Proof. Set $H_t = 1/t$,

$$M_t = \begin{cases} 0 & \text{if } t < \tau, \\ \eta & \text{if } t \geq \tau. \end{cases}$$

Then $M \in \mathcal{M}_{\text{loc}}$, $H \in L_{\text{var}}(M) \subseteq L(M)$ and $H \bullet M = X$. Thus, $X \in \mathcal{M}_\sigma$.

Another way to verify the inclusion $X \in \mathcal{M}_\sigma$ is to consider the sets $D_n = \Omega \times \{\{0\} \cup [1/n; \infty)\}$ and verify that each $I_{D_n} \bullet X$ is a uniformly integrable martingale.

Furthermore, for any (\mathcal{F}_t) -stopping time T with $\mathbb{P}\{T > 0\} > 0$, we have: $\mathbb{E}|X_T| = \infty$. Therefore, $X \notin \mathcal{M}_{\text{loc}}$. \square

In the above example, there exists a measure $\mathbb{Q} \sim \mathbb{P}$ such that X is a local martingale (and even a martingale) with respect to this measure. F. Delbaen and W. Schachermayer constructed in [10] a σ -martingale that is not a local martingale with respect to any $\mathbb{Q} \sim \mathbb{P}$.

Example 5.3. *Let τ , η be the same as in Example 5.2 and let X be the two-dimensional process defined by*

$$X_t^1 = \begin{cases} 0 & \text{if } t < \tau, \\ \eta/\tau & \text{if } t \geq \tau, \end{cases} \quad X_t^2 = \begin{cases} -t & \text{if } t < \tau, \\ -\tau + 1 & \text{if } t \geq \tau. \end{cases}$$

Set $\mathcal{F} = \sigma(\tau, \eta)$, $\mathcal{F}_t = \mathcal{F}_t^X$. Then $X \in \mathcal{M}_\sigma^2(\mathbb{P})$ and $X \notin \mathcal{M}_{\text{loc}}^2(\mathbb{P})$. Moreover, for any $\mathbb{Q} \neq \mathbb{P}$ such that $\mathbb{Q} \sim \mathbb{P}$, we have $X \notin \mathcal{M}_\sigma^2(\mathbb{Q})$. In other words, $\text{SM}(\mathbb{P}) = \{\mathbb{P}\}$, while $\text{LM}(\mathbb{P}) = \emptyset$.

Proof. In a similar way as above, we prove that $X^1 \in \mathcal{M}_\sigma(\mathbb{P})$. The process X^2 is a $(\mathcal{F}_t, \mathbb{P})$ -martingale (it is the compensated Poisson process stopped at the time of the first jump). Thus, $X \in \mathcal{M}_\sigma^2(\mathbb{P})$.

By the above reasoning, $X^1 \notin \mathcal{M}_{\text{loc}}(\mathbb{P})$. This means that $X \notin \mathcal{M}_{\text{loc}}^2(\mathbb{P})$.

Suppose that $\mathbb{Q} \sim \mathbb{P}$ and $X \in \mathcal{M}_\sigma^2(\mathbb{Q})$. Then $X^2 \in \mathcal{M}_\sigma(\mathbb{Q})$. For any $t \geq 0$, the process X^2 stopped at the time t is bounded, and therefore, it is a martingale (see

Proposition 5.4). Thus, $\mathbb{E}_{\mathbb{Q}}X_t^2 = 0$ for any $t \geq 0$. Set $F(t) = \mathbb{Q}\{\tau \leq t\}$. We have

$$\begin{aligned} 0 &= \mathbb{E}_{\mathbb{Q}}X_t^2 = -t\mathbb{Q}\{\tau > t\} + \int_0^t (-s + 1) dF(s) \\ &= -t\mathbb{Q}\{\tau > t\} + \mathbb{Q}\{\tau \leq t\} - sF(s)\Big|_0^t + \int_0^t F(s) ds \\ &= -t + \mathbb{Q}\{\tau \leq t\} + \int_0^t F(s) ds. \end{aligned}$$

Therefore,

$$0 = -1 + F'(t) + F(t), \quad t \geq 0.$$

Hence, the function $G(s) = 1 - F(s)$ satisfies the following differential equation:

$$G'(t) = -G(t), \quad G(0) = 1.$$

Thus, $G(t) = e^{-t}$ implying that $\text{Law}(\tau|\mathbb{Q}) = \text{Law}(\tau|\mathbb{P})$.

For any $s > 0$, the process $Z_t = X_t^1 - X_{t \wedge s}^1$ is a σ -martingale. Being bounded, it is a martingale (see Proposition 5.4). Consequently, for any $0 < s < t$, we have

$$0 = \mathbb{E}_{\mathbb{Q}}Z_t = \mathbb{E}_{\mathbb{Q}}(X_t^1 I(s < \tau \leq t)) = \mathbb{E}_{\mathbb{Q}}\left(\eta \frac{I(s < \tau \leq t)}{\tau}\right).$$

Therefore, $\mathbb{E}_{\mathbb{Q}}(\eta|\tau) = 0$ \mathbb{Q} -a.s. Since $\mathbb{Q} \sim \mathbb{P}$, the random variable η takes only values ± 1 \mathbb{Q} -a.s. We get

$$\mathbb{Q}(\eta = +1|\tau) = \mathbb{Q}(\eta = -1|\tau) = \frac{1}{2} \quad \mathbb{Q}\text{-a.s.}$$

This means that η and τ are \mathbb{Q} -independent. As a result,

$$\text{Law}(\tau, \eta|\mathbb{Q}) = \text{Law}(\tau, \eta|\mathbb{P}).$$

Hence, $\mathbb{Q} = \mathbb{P}$. This completes the proof. \square

2. Properties of σ -martingales. The above examples show that a σ -martingale may not be a local martingale. However, under some additional conditions, a σ -martingale is a local martingale.

Proposition 5.4 (Ansel, Stricker). *Suppose that $X \in \mathcal{M}_{\sigma}$ and X is bounded below. Then $X \in \mathcal{M}_{\text{loc}}$.*

For the proof, see [1; Corollaire 3.5].

Corollary 5.5. *Suppose that $X \in \mathcal{M}_{\sigma}$ and X is locally bounded. Then $X \in \mathcal{M}_{\text{loc}}$. In particular, if $X \in \mathcal{M}_{\sigma}$ is continuous, then $X \in \mathcal{M}_{\text{loc}}$.*

Example 5.2 shows that a stochastic integral with respect to a local martingale may not be a local martingale. It also shows that a stochastic integral with respect to a special semimartingale may not be a special semimartingale. In other words, the classes \mathcal{M}_{loc} and \mathcal{S}_p are not stable under the stochastic integration. The next result shows that the class \mathcal{M}_{σ} is stable under the stochastic integration. This is one of the advantages of σ -martingales.

Lemma 5.6. *If $X \in \mathcal{M}_\sigma^d$ and $H \in L(X)$, then $H \bullet X \in \mathcal{M}_\sigma$.*

Proof. By Lemma 5.1, there exist $M^i \in \mathcal{H}^1$ and a predictable processes K such that, for each $i = 1, \dots, d$, $K \in L(M^i)$ and $X^i = X_0^i + K \bullet M^i$. By Theorem 4.7, $H \bullet X = J \bullet M$, where $J^i = H^i K$. Set $G = \|J\| \vee 1$, $\tilde{J} = J/G$. By Theorem 4.6, $J \bullet M = G \bullet (\tilde{J} \bullet M)$. Since \tilde{J} is bounded, $\tilde{J} \bullet M = \sum_{i=1}^d \tilde{J}^i \bullet M^i$ is a uniformly integrable martingale. As a result, $H \bullet X \in \mathcal{M}_\sigma$. \square

3. Characterization of σ -martingales. Let $X \in \mathcal{S}^d$ and let (B, C, ν) denote its characteristics with respect to a truncation function h (for the definitions, see [21; Ch. II, §2a]). It is known that there exist a predictable increasing process $A \in \mathcal{A}_{\text{loc}}$, predictable processes $b^i, c^{ij} \in L(A)$ ($i, j = 1, \dots, d$) and a transition kernel K from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ to $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that

$$B^i = b^i \bullet A, \quad C^{ij} = c^{ij} \bullet A, \quad \nu(\omega, dt, dx) = K(\omega, t, dx) dA_t(\omega). \quad (5.1)$$

Proposition 5.7. (a) *The process X belongs to $\mathcal{M}_{\text{loc}}^d$ if and only if*

$$\int_{\{\|x\|>1\}} \|x\| K(\cdot, \cdot, dx) \in L(A)$$

and, for $\mathbf{P} \otimes dA$ -a.e. (ω, t) ,

$$b(\omega, t) + \int_{\mathbb{R}} (x - h(x)) K(\omega, t, dx) = 0.$$

(b) *The process X belongs to \mathcal{M}_σ^d if and only if, for $\mathbf{P} \otimes dA$ -a.e. (ω, t) ,*

$$\int_{\{\|x\|>1\}} \|x\| K(\omega, t, dx) < \infty$$

and

$$b(\omega, t) + \int_{\mathbb{R}} (x - h(x)) K(\omega, t, dx) = 0.$$

For the proof of (a), see [21; Ch. II, (2.30)]. For the proof of (b), see [24; Lemma 3].

Remark. The paper [16] by T. Goll and J. Kallsen contains a similar characterization of martingales. \square

4. Processes with independent increments. Let \mathcal{PII} (resp: \mathcal{PII}^d) denote the class of one-dimensional (resp: d -dimensional) processes with (\mathcal{F}_t) -independent increments, i.e.

$$\begin{aligned} \mathcal{PII} = \{ & X = (X_t)_{t \geq 0} : X \text{ is } (\mathcal{F}_t)\text{-adapted and,} \\ & \text{for any } s \leq t, X_t - X_s \text{ is independent of } \mathcal{F}_s \}. \end{aligned}$$

Let \mathcal{LP} (resp: \mathcal{LP}^d) denote the class of one-dimensional (resp: d -dimensional) (\mathcal{F}_t) -Lévy processes, i.e.

$$\begin{aligned} \mathcal{LP} = \{ & X = (X_t)_{t \geq 0} : X \text{ is a } (\mathcal{F}_t)\text{-adapted Lévy process} \\ & \text{and, for any } s \leq t, X_t - X_s \text{ is independent of } \mathcal{F}_s \}. \end{aligned}$$

Let \mathcal{M} (resp: \mathcal{M}^d) denote the class of one-dimensional (resp: d -dimensional) (\mathcal{F}_t) -martingales.

The results given below lead to the following conclusions: diagrammatically,

$$\begin{aligned} \mathcal{M}_\sigma^d &\supset \mathcal{M}_{\text{loc}}^d \supset \mathcal{M}^d \\ \mathcal{M}_\sigma^d \cap \mathcal{PII}^d &\supset \mathcal{M}_{\text{loc}}^d \cap \mathcal{PII}^d = \mathcal{M}^d \cap \mathcal{PII}^d \\ \mathcal{M}_\sigma^d \cap \mathcal{LP}^d &= \mathcal{M}_{\text{loc}}^d \cap \mathcal{LP}^d = \mathcal{M}^d \cap \mathcal{LP}^d, \end{aligned}$$

where all the inclusions can be made strict by an appropriate choice of the filtered probability space.

The example of a local martingale that is not a martingale is well known: it is sufficient to consider the process $X = \|B\|^{-1}$, where B is a 3-dimensional Brownian motion started at a point that is not equal to zero (for more details, see [34; Ch. V, (2.13)]).

Theorem 5.8. *Let $X \in \mathcal{PII}^d$. Then the following conditions are equivalent:*

- (i) $X \in \mathcal{M}_{\text{loc}}^d$;
- (ii) $X \in \mathcal{M}^d$.

Proof. Suppose that $X \in \mathcal{M}_{\text{loc}}^d \cap \mathcal{PII}^d$. Take the truncation function $h(x) = xI(\|x\| \leq 1)$. By [21; Ch. II, (4.15)], there exists a deterministic version of the characteristics (B, C, ν) of the process X , and moreover, for any $t \geq 0$,

$$\begin{aligned} \mathbb{E} e^{i\langle \lambda, X_t \rangle} &= \exp \left\{ i\langle \lambda, \tilde{B}(t) \rangle - \frac{1}{2} \langle \lambda, C(t)\lambda \rangle \right. \\ &\quad \left. + \int_{[0,t] \cap J^c} \int_{\mathbb{R}^d} (e^{i\langle \lambda, x \rangle} - 1 - i\langle \lambda, x \rangle I(\|x\| \leq 1)) K(s, dx) dA(s) \right\} \\ &\times \prod_{[0,t] \cap J} \left[1 + \int_{\mathbb{R}^d} (e^{i\langle \lambda, x \rangle} - 1) K(s, dx) \Delta A(s) \right], \end{aligned} \quad (5.2)$$

where

$$\begin{aligned} \tilde{B}(t) &= B(t) - \sum_{s \leq t} \Delta B(s), \\ J &= \{t \geq 0 : \nu(\{t\} \times \mathbb{R}^d) > 0\} = \{t \geq 0 : \mathbb{P}\{\Delta X_t \neq 0\} > 0\}, \end{aligned}$$

A is an increasing function and K is a transition kernel from $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ to $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $\nu(dt, dx) = K(t, dx) dA(t)$.

Fix $t \geq 0$. By Proposition 5.7 (a),

$$\int_0^t \int_{\{\|x\| > 1\}} \|x\| K(s, dx) dA(s) < \infty.$$

Hence,

$$\sum_{s \in [0,t] \cap J} \int_{\{\|x\| > 1\}} \|x\| K(s, dx) \Delta A(s) < \infty$$

and, by the definition of the compensator,

$$\mathbb{E} \sum_{s \in [0,t] \cap J} \|\Delta X_s I(\|\Delta X_s\| > 1)\| < \infty. \quad (5.3)$$

Let $s \geq 0$. Let $(\tau_n)_{n=1}^\infty$ be a localizing sequence for X . There exists $m \in \mathbb{N}$ such that $\mathbb{P}\{\tau_m \geq s\} > 0$. The random variable $\Delta X_s^{\tau_m}$ is integrable and $\mathbb{E}\Delta X_s^{\tau_m} = 0$. Note that $\Delta X_s^{\tau_m} = \Delta X_s I(\tau_m \geq s)$ and ΔX_s is independent of $I(\tau_m \geq s)$ since $\{\tau_m \geq s\} \in \mathcal{F}_{s-}$. Hence, ΔX_s is integrable and $\mathbb{E}\Delta X_s = 0$.

Combining this with (5.3), we deduce that

$$\sum_{s \in [0, t] \cap J} \|\mathbb{E}\Delta X_s I(\|\Delta X_s\| \leq 1)\| < \infty. \quad (5.4)$$

Moreover,

$$\int_0^t \int_{\{\|x\| \leq 1\}} x^2 K(s, dx) dA(s) < \infty$$

(see [21; Ch. II, (2.13)]), and hence,

$$\sum_{s \in [0, t] \cap J} \mathbb{E}\|\Delta X_s I(\|\Delta X_s\| \leq 1)\|^2 < \infty.$$

Using this together with (5.4) and taking into account the independence of $\Delta X_s I(\|\Delta X_s\| \leq 1)$ for different s , we conclude that the sum

$$\sum_{s \in [0, t] \cap J} \Delta X_s I(\|\Delta X_s\| \leq 1)$$

converges in L^2 . Hence, it also converges in L^1 . Taking (5.3) into account, we conclude that the sum

$$Z_t := \sum_{s \in [0, t] \cap J} \Delta X_s$$

converges in L^1 . Moreover, it follows from the definition of the compensator and the independence of ΔX_s for different s that

$$\begin{aligned} \mathbb{E}e^{i\langle \lambda, Z_t \rangle} &= \prod_{s \in [0, t] \cap J} \mathbb{E}e^{i\langle \lambda, \Delta X_s \rangle} = \prod_{s \in [0, t] \cap J} [\mathbb{E}e^{i\langle \lambda, \Delta X_s \rangle} I(\Delta X_s \neq 0) + \mathbb{P}\{\Delta X_s = 0\}] \\ &= \prod_{s \in [0, t] \cap J} \left[\int_{\mathbb{R}^d} e^{i\langle \lambda, x \rangle} K(s, dx) \Delta A(s) + 1 - K(s, \mathbb{R}^d) \Delta A(s) \right] \\ &= \prod_{s \in [0, t] \cap J} \left[1 + \int_{\mathbb{R}^d} (e^{i\langle \lambda, x \rangle} - 1) K(s, dx) \Delta A(s) \right]. \end{aligned} \quad (5.5)$$

Let Y_t be an infinitely divisible random variable that is independent of Z_t and has the characteristic function

$$\mathbb{E}e^{i\langle \lambda, Y_t \rangle} = \exp \left\{ i\langle \lambda, \tilde{B}(t) \rangle - \frac{1}{2} \langle \lambda, C(t) \lambda \rangle + \int_{\mathbb{R}} (e^{i\langle \lambda, x \rangle} - 1 - i\langle \lambda, x \rangle I(\|x\| \leq 1)) \eta_t(dx) \right\},$$

where

$$\eta_t(dx) = \int_{[0, t] \cap J^c} K(s, dx) dA(s).$$

It follows from (5.2) and (5.5) that $Z_t + Y_t \stackrel{\text{law}}{=} X_t$. In view of the inequality

$$\int_{\{\|x\|>1\}} \|x\| \eta_t(dx) = \int_0^t \int_{\{\|x\|>1\}} \|x\| K(s, dx) dA(s) < \infty,$$

we conclude that $\mathbb{E}\|Y_t\| < \infty$ and

$$\varphi(t) := \mathbb{E}Y_t = \tilde{B}(t) + \int_{\{\|x\|>1\}} x \eta_t(dx) = \tilde{B}(t) + \int_{[0,t] \cap J^c} \int_{\{\|x\|>1\}} x K(s, dx) dA(s)$$

(see [35; §25]). Furthermore, $\mathbb{E}\|Z_t\| < \infty$ and $\mathbb{E}Z_t = 0$. As a result, $\mathbb{E}\|X_t\| < \infty$ and $\mathbb{E}X_t = \varphi(t)$. This, combined with the independence of increments of X , shows that $X - \varphi(\cdot)$ is a martingale. Hence, $\varphi(\cdot)$ is a local martingale. On the other hand, φ is continuous (this follows from the explicit form of φ and the definition of the set J), and therefore, $\varphi = 0$ (see [34; Ch. IV, (1.2)]). This completes the proof. \square

Example 5.9. Let $(\xi_n)_{n=1}^\infty$ be a sequence of independent random variables with

$$\mathbb{P}(\xi_n = n) = \frac{1}{2n^2}, \quad \mathbb{P}(\xi_n = -n) = \frac{1}{2n^2}, \quad \mathbb{P}(\xi_n = 0) = 1 - \frac{1}{n^2}.$$

Take $t_n = 1 - \frac{1}{n}$ and set

$$X_t = \sum_{\{n: t_n \leq t\}} \xi_n, \quad t \geq 0$$

(with probability 1, only a finite number of ξ_n differ from zero, and hence, X is well defined). Then $X \in \mathcal{M}_\sigma \cap \mathcal{PII}$, but $X \notin \mathcal{M}_{\text{loc}}$ (we consider the filtration $\mathcal{F}_t = \mathcal{F}_t^X$).

Proof. Take $D_n = \Omega \times ([0, 1 - \frac{1}{n}] \cup [1, \infty))$. Then, for any $n \in \mathbb{N}$, $I_{D_n} \bullet X$ is a uniformly integrable martingale, and hence, $X \in \mathcal{M}_\sigma$. The inclusion $X \in \mathcal{PII}$ is obvious.

Suppose that $X \in \mathcal{M}_{\text{loc}}$. Using Proposition 5.7 (a) and the same arguments as in the previous proof, we deduce that

$$\mathbb{E} \sum_{s \leq 1} |\Delta X_s I(|\Delta X_s| > 1)| < \infty.$$

On the other hand,

$$\mathbb{E} \sum_{s \leq 1} |\Delta X_s I(|\Delta X_s| > 1)| = \sum_{n=1}^{\infty} \mathbb{E}|\xi_n| = \infty.$$

Hence, $X \notin \mathcal{M}_{\text{loc}}$. \square

Theorem 5.10. Let X be a d -dimensional Lévy process with

$$\mathbb{E}e^{i\langle \lambda, X_t \rangle} = \exp \left\{ t \left[i\langle \lambda, b \rangle - \frac{1}{2} \langle \lambda, c \lambda \rangle + \int_{\mathbb{R}^d} (e^{i\langle \lambda, x \rangle} - 1 - i\langle \lambda, x \rangle I(\|x\| \leq 1)) \eta(dx) \right] \right\}.$$

Then the following conditions are equivalent:

- (i) $X \in \mathcal{M}_\sigma^d$;

- (ii) $X \in \mathcal{M}_{\text{loc}}^d$;
- (iii) $X \in \mathcal{M}^d$;
- (iv) we have

$$\int_{\{\|x\|>1\}} \|x\| \eta(dx) < \infty,$$

$$b + \int_{\{\|x\|>1\}} x \eta(dx) = 0.$$

Proof. (i) \Rightarrow (iv) The characteristics of X with respect to the truncation function $h(x) = xI(\|x\| \leq 1)$ are given by (5.1) with $A_t = t$, $b(\omega, t) = b$ and $K(\omega, t, dx) = \eta(dx)$. Now, the desired implication follows from Proposition 5.7 (b).

(iv) \Rightarrow (iii) It follows from (iv) that, for any $t \geq 0$, $\mathbb{E}\|X_t\| < \infty$ and $\mathbb{E}X_t = 0$ (see [35; §25]). This obviously yields (iii).

(iii) \Rightarrow (ii) \Rightarrow (i) These implications are obvious. \square

6 Vector Stochastic Integrals in the First Fundamental Theorem of Asset Pricing

1. Componentwise stochastic integrals. Let us compare the notion of the vector stochastic integral with the notion of the componentwise stochastic integral.

Definition 6.1. Let $X \in \mathcal{S}^d$. A d -dimensional process H is *componentwise X -integrable* if $H^i \in L(X^i)$ for each $i = 1, \dots, d$. The *componentwise stochastic integral* is defined as the sum $\sum_{i=1}^d H^i \bullet X^i$.

Remark. Any locally bounded predictable process H is componentwise X -integrable. \square

Theorem 6.2. *If H is componentwise X -integrable, then H is X -integrable and*

$$H \bullet X = \sum_{i=1}^d H^i \bullet X^i.$$

Proof. Let $X^i = A^i + M^i$ be a decomposition such that $H^i \in L_{\text{var}}(A^i) \cap L_{\text{loc}}^1(M^i)$. Set $A = (A^1, \dots, A^d)$, $M = (M^1, \dots, M^d)$. Let K_i be the d -dimensional process defined as $K_i = (0, \dots, H^i, \dots, 0)$, where H^i stands in the i -th place. Then $K_i \in L_{\text{var}}(A) \cap L_{\text{loc}}^1(M)$ for $i = 1, \dots, d$. Furthermore, $K_i \bullet X = H^i \bullet X^i$. By Theorem 4.3,

$$H \bullet X = \left(\sum_{i=1}^d K_i \right) \bullet X = \sum_{i=1}^d K_i \bullet X = \sum_{i=1}^d H^i \bullet X^i.$$

This completes the proof. \square

Thus, if the componentwise stochastic integral exists, then the vector stochastic integral also exists and the integrals coincide. The following example shows that the reverse is not true.

Example 6.3. Let $Y \in \mathcal{S}$ and K be a one-dimensional predictable process that does not belong to $L(Y)$. Set

$$X = (Y, Y), \quad H = (K, -K).$$

Then $H \in L(X)$ and $H \bullet X = 0$, while the componentwise stochastic integral of H with respect to X does not exist.

Proof. Straightforward. □

We will now turn to the example, which shows that the space of componentwise stochastic integrals with respect to a fixed multidimensional semimartingale X may not be closed in the Émery topology. This was first noticed by L. Galtchouk [15]. The example that we give is taken from [19].

Example 6.4. Let B^1, B^2 be two independent Brownian motions on some filtered probability space. Set $J_t = t$ and define the two-dimensional process X by

$$X^1 = B^1, \quad X^2 = (1 - J) \bullet B^1 + J \bullet B^2.$$

Then the space

$$\mathcal{L}_C(X) = \left\{ \sum_{i=1}^2 H^i \bullet X^i : H^i \in L(X^i) \right\}$$

is not closed in the Émery topology.

Proof. For any $\varepsilon > 0$,

$$1 - \frac{1}{J + \varepsilon} \in L(X^1), \quad \frac{1}{J + \varepsilon} \in L(X^2),$$

and, in view of Theorems 4.3, 4.6,

$$\left(1 - \frac{1}{J + \varepsilon}\right) \bullet X^1 + \frac{1}{J + \varepsilon} \bullet X^2 = \frac{\varepsilon}{J + \varepsilon} \bullet B^1 - \frac{\varepsilon}{J + \varepsilon} \bullet B^2 + B^2. \quad (6.1)$$

For each $t \geq 0$,

$$\left[\frac{\varepsilon}{J + \varepsilon} \bullet B^k \right]_t = \int_0^t \left(\frac{\varepsilon}{J_s + \varepsilon} \right)^2 ds = \int_0^t \left(\frac{\varepsilon}{1/s + \varepsilon} \right)^2 ds \xrightarrow{\varepsilon \downarrow 0} 0.$$

Hence, $Z_\varepsilon \xrightarrow[\varepsilon \downarrow 0]{\mathcal{S}} B^2$, where Z_ε denotes the process given by (6.1). Note that $Z_\varepsilon \in \mathcal{L}_C(X)$.

Suppose now that there exist processes $H^i \in L(X^i)$ ($i = 1, 2$) such that

$$H^1 \bullet X^1 + H^2 \bullet X^2 = B^2.$$

Then we obtain the equality $M^1 = M^2$, where

$$\begin{aligned} M^1 &= (H^1 + (1 - J)H^2) \bullet B^1 = K^1 \bullet B^1, \\ M^2 &= (1 - JH^2) \bullet B^2 = K^2 \bullet B^2. \end{aligned}$$

It follows from Theorem 4.19 that $[K^1 \bullet B^1, K^2 \bullet B^2] = 0$. Hence, $M^1 = M^2 = 0$. By Lemma 3.5 (b),

$$1 - JH^2(= K^2) = 0 \quad \mathbb{P} \times \mu_L\text{-a.e.},$$

where μ_L denotes the Lebesgue measure on \mathbb{R}_+ . Thus,

$$H^2 = \frac{1}{J}, \quad H^1 = 1 - \frac{1}{J} \quad \mathbb{P} \times \mu_L\text{-a.e.}$$

Consequently, $H^1 \notin L_{\text{loc}}^1(B^1)$.

The inclusion $H^1 \in L(X^1)$ means that there exists a decomposition $X^1 = A + N$ with $A \in \mathcal{V}$, $N \in \mathcal{M}_{\text{loc}}$ such that $H^1 \in L_{\text{var}}(A) \cap L_{\text{loc}}^1(N)$. According to Proposition 2.16,

$$[N]_t = [N^c]_t + \sum_{s \leq t} (\Delta N_s)^2,$$

where N^c is the *continuous martingale part* of N (see [21; Ch. I, (4.18)]). Since X^1 is continuous, we have $N^c = X^1$. Thus, $[N] = [X^1] + D$, where $D \in \mathcal{V}^+$. Now, the inclusion $H^1 \in L_{\text{loc}}^1(N)$ implies that $H^1 \in L_{\text{loc}}^1(X^1) = L_{\text{loc}}^1(B^1)$.

We arrive at a contradiction, which shows that $B^2 \notin \mathcal{L}_C(X)$. Thus, $\mathcal{L}_C(X)$ is not closed in the Émery topology. \square

Remark. The space

$$\mathcal{L}_B(X) = \{H \bullet X : H \text{ is predictable and locally bounded}\}$$

may not be closed in the Émery topology even in the one-dimensional case. For instance, this is the case if X is a Brownian motion. \square

2. The First Fundamental Theorem of Asset Pricing. Recall that, in the general case, this theorem states that each of the conditions $(NFLVR)$, $(NFLBR)$ is equivalent to the existence of an equivalent σ -martingale measure. The definition of $(NFLVR)$, $(NFLBR)$ employs the notion of a strategy (Definition 1.4), which, in turn, is based on the vector stochastic integrals. We will indicate this by the subscript V : $(NFLVR)_V$, $(NFLBR)_V$. Then the First Fundamental Theorem of Asset Pricing can be expressed as follows:

$$(NFLVR)_V \iff (NFLBR)_V \iff \text{SM}(\mathbb{P}) \neq \emptyset.$$

In the definition of a strategy, one could replace the condition that H is X -integrable by the condition that H is componentwise X -integrable. This yields a smaller class of strategies and, as a result, weaker notions $(NFLVR)_C$, $(NFLBR)_C$. These properties could even more be weakened by considering only locally bounded integrands H . This yields the properties $(NFLVR)_B$ and $(NFLBR)_B$. The implications below follow directly from the definitions:

$$\begin{array}{ccccc} (NFLBR)_B & \iff & (NFLBR)_C & \iff & (NFLBR)_V \\ \Downarrow & & \Downarrow & & \Downarrow \\ (NFLVR)_B & \iff & (NFLVR)_C & \iff & (NFLVR)_V. \end{array}$$

The question arises: which of these properties are sufficient for the existence of an equivalent σ -martingale measure? The following statement is proved in [5].

Proposition 6.5. *The properties $(NFLBR)_B$, $(NFLBR)_C$ are sufficient for the existence of an equivalent σ -martingale measure, while $(NFLVR)_B$, $(NFLVR)_C$ do not imply the existence of such a measure, i.e.*

$$\begin{array}{ccccc} (NFLBR)_B & \iff & (NFLBR)_C & \iff & (NFLBR)_V & \iff & \text{SM}(\mathbb{P}) \neq \emptyset \\ \Downarrow \Updownarrow & & \Downarrow \Updownarrow & & \Updownarrow & & \\ (NFLVR)_B & \iff & (NFLVR)_C & \iff & (NFLVR)_V & & \end{array}$$

7 Second Fundamental Theorem of Asset Pricing

1. Proof of the Theorem. Throughout this subsection, we suppose that \mathcal{F}_0 is \mathbb{P} -trivial and $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t$.

Lemma 7.1. *Let $X \in \mathcal{S}^d$. Then the space*

$$\mathcal{L}^1(X) = \{x + H \bullet X : x \in \mathbb{R}, H \in L(X) \text{ and } H \bullet X \in \mathcal{H}^1\} \quad (7.1)$$

is a stable subspace of \mathcal{H}^1 .

Proof. The convergence in \mathcal{H}^1 is stronger than the convergence in the Émery topology. Therefore, in view of Proposition 4.10, the space

$$\mathcal{L}_0^1(X) = \{H \bullet X : H \in L(X) \text{ and } H \bullet X \in \mathcal{H}^1\} = \{H \bullet X : H \in L(X)\} \cap \mathcal{H}^1 \quad (7.2)$$

is closed in \mathcal{H}^1 .

By the Hahn-Banach theorem, there exists a continuous linear functional φ on \mathcal{H}^1 such that φ equals zero on $\mathcal{L}_0^1(X)$ and φ is not equal to zero on the process $Z \equiv 1$. If $(x_n + H_n \bullet X)$ is a sequence of elements of $\mathcal{L}^1(X)$ that converges in \mathcal{H}^1 , then $\varphi(x_n + H_n \bullet X)$ also converges. We have $\varphi(x_n + H_n \bullet X) = \alpha x_n$, where $\alpha \neq 0$. Hence, the sequence (x_n) converges in \mathbb{R} and the sequence $(H_n \bullet X)$ converges in \mathcal{H}^1 . So, the closedness of $\mathcal{L}_0^1(X)$ implies the closedness of $\mathcal{L}^1(X)$.

For any $H \in L(X)$ and any stopping time τ , we have

$$x + (H \bullet X)^\tau = x + (HI_{[0, \tau]}) \bullet X.$$

Consequently, $\mathcal{L}^1(X)$ is stable. □

Proof of Theorem 1.17. We will prove only statement (a). Statement (b) follows from (a) and Proposition 5.4.

(i) \Rightarrow (ii) Suppose that the set $\text{SM}(\mathbb{P})$ contains two different measures \mathbb{Q}_1 and \mathbb{Q}_2 . Take $A \in \mathcal{F}$ and set $f = I_A$. There exist $x \in \mathbb{R}$ and $H \in L(X)$ such that $I_A = x + (H \bullet X)_\infty$ and $H \bullet X$ is uniformly bounded. The stochastic integral $H \bullet X$ is the same for \mathbb{Q}_1 and \mathbb{Q}_2 (see Theorem 4.14). It follows from Lemma 5.6 that $H \bullet X$ belongs to $\mathcal{M}_\sigma(\mathbb{Q}_1)$ and $\mathcal{M}_\sigma(\mathbb{Q}_2)$. By Proposition 5.4, $H \bullet X$ is a local martingale with respect to \mathbb{Q}_1 and \mathbb{Q}_2 . As $H \bullet X$ is uniformly bounded, we get

$$\begin{aligned} \mathbb{Q}_1(A) &= \mathbb{E}_{\mathbb{Q}_1}(x + (H \bullet X)_\infty) = x, \\ \mathbb{Q}_2(A) &= \mathbb{E}_{\mathbb{Q}_2}(x + (H \bullet X)_\infty) = x. \end{aligned}$$

Since A is arbitrary, $\mathbf{Q}_1 = \mathbf{Q}_2$. As a result, $|\mathbf{SM}(\mathbf{P})| = 1$.

(ii) \Rightarrow (iii) Suppose that $|\mathbf{SM}(\mathbf{P})| = 1$. Let \mathbf{Q} be the unique measure in $\mathbf{SM}(\mathbf{P})$. Let us prove that X has the predictable representation property with respect to \mathbf{Q} .

Suppose that $\mathcal{L}^1(X) \neq \mathcal{H}^1(\mathbf{Q})$, where $\mathcal{L}^1(X)$ is defined in (7.1). Using Proposition 2.8, Lemma 7.1 and the Hahn-Banach theorem, we get the existence of $N \in \text{BMO}(\mathbf{Q})$ such that $N \neq 0$ and $\mathbf{E}_{\mathbf{Q}}(N_{\infty}M_{\infty}) = 0$ for any $M \in \mathcal{L}^1(X)$. By Proposition 2.11, $N \perp M$ for any $M \in \mathcal{L}^1(X)$. Since $\text{BMO}(\mathbf{Q}) \subset \mathcal{H}_{\text{loc}}^{\infty}(\mathbf{Q})$, there exists a stopping time τ such that $N^{\tau} \in \mathcal{H}^{\infty}(\mathbf{Q})$ and $N^{\tau} \neq 0$. We have $N^{\tau} \perp M$ for each $M \in \mathcal{L}^1(X)$ (see [21; Ch. I, (4.13)]). One more application of Proposition 2.11 yields: $\mathbf{E}_{\mathbf{Q}}(N_{\infty}^{\tau}M_{\infty}) = 0$ for any $M \in \mathcal{L}^1(X)$.

Set

$$Z = 1 + \frac{N^{\tau}}{2\|N^{\tau}\|_{\mathcal{H}^{\infty}(\mathbf{Q})}}.$$

As $\mathcal{L}^1(X)$ contains the constant processes and \mathcal{F}_0 is \mathbf{Q} -trivial, we have $N_0 = 0$. Thus, $\mathbf{E}_{\mathbf{Q}}Z_{\infty} = 1$. Furthermore, $Z \geq 1/2$ and $\mathbf{E}_{\mathbf{Q}}(Z_{\infty}M_{\infty}) = 0$ for any $M \in \mathcal{L}_0^1(X)$, where $\mathcal{L}_0^1(X)$ is defined in (7.2).

By Lemma 5.1, X admits the representation $X^i = X_0^i + (\mathbf{Q})K \bullet Y^i$, where $Y^i \in \mathcal{H}^1(\mathbf{Q})$ ($i = 1, \dots, d$). In view of Theorem 4.7, $\mathcal{L}_0^1(X) = \mathcal{L}_0^1(Y)$. For any $p \leq q$, $A \in \mathcal{F}_p$, $i = 1, \dots, d$, the process $M_t = I_A(Y_{t \wedge q}^i - Y_{t \wedge p}^i)$ ($t \geq 0$) belongs to $\mathcal{L}_0^1(Y) = \mathcal{L}_0^1(X)$. Therefore,

$$\mathbf{E}_{\mathbf{Q}}(Z_{\infty}M_{\infty}) = \mathbf{E}_{\mathbf{Q}}(Z_{\infty}I_A(Y_q^i - Y_p^i)) = 0.$$

This means that Y is a martingale under the measure $\tilde{\mathbf{Q}}$ defined as $\tilde{\mathbf{Q}} = Z_{\infty}\mathbf{Q}$. Applying Theorem 4.14, we get

$$X^i = X_0^i + (\mathbf{Q})K \bullet Y^i = X_0^i + (\tilde{\mathbf{Q}})K \bullet Y^i, \quad i = 1, \dots, d.$$

Hence, X is a $(\mathcal{F}_t, \tilde{\mathbf{Q}})$ - σ -martingale. As $|\mathbf{SM}(\mathbf{P})| = 1$, we arrive at: $\tilde{\mathbf{Q}} = \mathbf{Q}$ and $Z_{\infty} = 1$. Consequently, $N^{\tau} = 0$ that leads to a contradiction. The contradiction shows that $\mathcal{L}^1(X) = \mathcal{H}^1(\mathbf{Q})$.

Thus, any $M \in \mathcal{H}^1(\mathbf{Q})$ admits the representation $M = M_0 + H \bullet X$. By localization (see Corollary 2.7), this result is extended to all $(\mathcal{F}_t, \mathbf{Q})$ -local martingales.

(iii) \Rightarrow (i) Let \mathbf{Q} be an element of $\mathbf{SM}(\mathbf{P})$ such that X has the predictable representation property with respect to \mathbf{Q} . Let f be a bounded \mathcal{F} -measurable function. Set $x = \mathbf{E}_{\mathbf{Q}}f$, $M_t = \mathbf{E}_{\mathbf{Q}}(f|\mathcal{F}_t)$. There exists $H \in L(X)$ such that $M = x + (\mathbf{Q})H \bullet X$. The integral $H \bullet X$ is the same under \mathbf{Q} and the original measure \mathbf{P} (see Theorem 4.14). As $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t$, we have

$$f = M_{\infty} = x + (H \bullet X)_{\infty}.$$

This proves that the model is complete. \square

Remark. The Second Fundamental Theorem of Asset Pricing in the discrete-time case (Proposition 1.14) can easily be derived from Theorem 1.17. This is done as follows. Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{0 \leq n \leq N}, \mathbf{P})$ be a filtered probability space. Define $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ by

$$\tilde{\mathcal{F}}_t = \begin{cases} \mathcal{F}_n & \text{if } n \leq t < n+1 \leq N, \\ \mathcal{F}_N & \text{if } t \geq N. \end{cases}$$

Any discrete-time process $(Z_n)_{0 \leq n \leq N}$ can be transformed into the corresponding continuous-time process $(\tilde{Z}_t)_{t \geq 0}$ as follows:

$$\tilde{Z}_t = \begin{cases} Z_n & \text{if } n \leq t < n+1 \leq N, \\ Z_N & \text{if } t \geq N. \end{cases}$$

Then \tilde{H} is $(\tilde{\mathcal{F}}_t)$ -predictable if and only if for any $n = 1, \dots, N$, H_n is \mathcal{F}_{n-1} -measurable and H_0 is \mathcal{F}_0 -measurable. In this case, $\tilde{H} \in L(\tilde{X})$ (note that \tilde{X} is piecewise constant) and

$$(\tilde{H} \bullet \tilde{X})_t = \sum_{i=1}^d \sum_{k=1}^{[t] \wedge N} H_k^i (X_k^i - X_{k-1}^i), \quad t \geq 0.$$

Thus, the discrete-time model $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{0 \leq n \leq N}, \mathbf{P}; (X_n)_{0 \leq n \leq N})$ is complete if and only if the corresponding continuous-time model $(\Omega, \mathcal{F}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \mathbf{P}; (\tilde{X}_t)_{t \geq 0})$ is complete. This yields the desired statement. \square

2. Examples and counterexamples. The notion of completeness (Definition 1.15) could be strengthened by considering only the strategies $\pi = (x, H)$, for which H is componentwise X -integrable. However, the obtained notion of *componentwise completeness* will not be equivalent to conditions (ii), (iii) of Theorem 1.17.

Example 7.2. Let B^1, B^2 be two independent Brownian motions on some filtered probability space. Set $J_t = t$ and define the two-dimensional process X by

$$X^1 = B^1, \quad X^2 = (1 - J) \bullet B^1 + J \bullet B^2.$$

Set $\mathcal{F}_t = \mathcal{F}_t^X$, $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t$. Then $\text{SM}(\mathbf{P}) = \{\mathbf{P}\}$, while the model is not componentwise complete.

Proof. For any $(\mathcal{F}_t, \mathbf{P})$ -local martingale M , there exist processes $K^1 \in L(B^1)$, $K^2 \in L(B^2)$ such that

$$M = M_0 + K^1 \bullet B^1 + K^2 \bullet B^2$$

(see [34; Ch. V, (3.5)]). Furthermore, $B^1 = X^1$ and $B^2 = H \bullet X$, where $H = (H^1, H^2)$ is given by

$$H^1 = 1 - \frac{1}{J}, \quad H^2 = \frac{1}{J}$$

(the process H is X -integrable, but not componentwise X -integrable). Applying Theorems 4.1 and 4.6, we deduce that X has the predictable representation property with respect to \mathbf{P} . By Theorem 1.17, $\text{SM}(\mathbf{P}) = \{\mathbf{P}\}$.

Now, set

$$\tau = \inf\{t \geq 0 : |B_t^2| = 1\}, \quad f = B_\tau^2.$$

Suppose that there exists a strategy $\pi = (x, H)$ that satisfies the conditions of Definition 1.15 and such that H is componentwise X -integrable. It follows from Proposition 5.4 that $H \bullet X$ is a local martingale. Being bounded, it is a martingale. This leads to the equality $H \bullet X = (B^2)^\tau$. Using the same arguments as in Example 6.4, we arrive at a contradiction. It shows that the model is not componentwise complete. \square

One could weaken the notion of completeness by eliminating condition i) of Definition 1.15. The question arises whether Theorem 1.17 remains true with this weakened notion of the completeness. The answer to this question is negative as shown by the following example.

Example 7.3. Let $\Omega = C(\mathbb{R}_+)$ and X be the coordinate process on $C(\mathbb{R}_+)$. Set $\mathcal{F}_t = \mathcal{F}_t^X$, $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t$. Let \mathbf{Q} be the distribution of the process $\psi \bullet B$ and $\tilde{\mathbf{Q}}$ be the distribution of the process $2\psi \bullet B$, where $\psi_t = I(t > 1)$ and B is a Brownian motion. Set $\mathbf{P} = \frac{1}{2}(\mathbf{Q} + \tilde{\mathbf{Q}})$. Then $|\text{SM}(\mathbf{P})| > 1$, while, for any bounded \mathcal{F} -measurable function f , there exists a strategy $\pi = (x, H)$ that satisfies conditions ii), iii) of Definition 1.15.

Proof. For any $\alpha \in (0, 1)$, the measure $\alpha\mathbf{Q} + (1 - \alpha)\tilde{\mathbf{Q}}$ belongs to $\text{SM}(\mathbf{P})$. Thus, $\text{SM}(\mathbf{P})$ is infinite.

Let us prove the second statement. Let f be a bounded \mathcal{F} -measurable function. Set

$$x = \text{esssup } f, \quad Y_t = \mathbf{E}_{\mathbf{Q}}(f|\mathcal{F}_t), \quad \tilde{Y}_t = \mathbf{E}_{\tilde{\mathbf{Q}}}(f|\mathcal{F}_t), \quad t \geq 0.$$

There exist predictable processes K, \tilde{K} such that

$$\begin{aligned} Y &= \mathbf{E}_{\mathbf{Q}}f + (\mathbf{Q})K \bullet X, \\ \tilde{Y} &= \mathbf{E}_{\tilde{\mathbf{Q}}}f + (\tilde{\mathbf{Q}})\tilde{K} \bullet X. \end{aligned}$$

There exist \mathcal{F}_1 -measurable disjoint sets A, \tilde{A} such that $\mathbf{Q}(A) = 1$, $\tilde{\mathbf{Q}}(\tilde{A}) = 1$. Set

$$\begin{aligned} \tau(\omega) &= \begin{cases} \inf\{t \geq 1 : x + X_t(\omega) - X_1(\omega) = Y_t(\omega)\} & \text{if } \omega \in A, \\ \inf\{t \geq 1 : x + X_t(\omega) - X_1(\omega) = \tilde{Y}_t(\omega)\} & \text{if } \omega \in \tilde{A}, \end{cases} \\ H_t(\omega) &= \begin{cases} 0 & \text{if } t \leq 1, \\ 1 & \text{if } 1 < t \leq \tau(\omega), \\ K_t(\omega) & \text{if } \tau(\omega) < t \text{ and } \omega \in A, \\ \tilde{K}_t(\omega) & \text{if } \tau(\omega) < t \text{ and } \omega \in \tilde{A}. \end{cases} \end{aligned}$$

Then the strategy $\pi = (x, H)$ satisfies conditions ii) and iii) of Definition 1.15. \square

Remark. Note that, for the pair (x, H) constructed in the above example, the process $x + H \bullet X$ is uniformly bounded below. This means that (x, H) is an *admissible* strategy (see [9], [36; Ch. VII, §1a]). \square

In the discrete-time case with a finite time horizon, the completeness implies that $\mathcal{F}_n = \mathcal{F}_n^X$ up to \mathbf{P} -null sets (see [22], [36; Ch. V, §4f]). In the continuous-time case, this is not true as shown by the following example.

Example 7.4. Let B be a Brownian motion. Set $X = \text{sgn } B \bullet B$, $\mathcal{F}_t = \mathcal{F}_t^B$, $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t$. Then the model is complete, while $\mathcal{F}_t \neq \mathcal{F}_t^X$.

Proof. Any (\mathcal{F}_t) -local martingale M admits the representation

$$M = M_0 + H \bullet B = M_0 + (H \text{sgn } B) \bullet X,$$

and consequently, the model is complete. On the other hand, $\mathcal{F}_t^X = \mathcal{F}_t^{|B|} \neq \mathcal{F}_t^B$ (see [34; Ch. VI, (2.2)]). \square

Remark. For a discrete-time model $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, \mathbf{P}; (X_n)_{n \geq 0})$ with an infinite time horizon, the completeness implies that $\mathcal{F}_n = \mathcal{F}_n^X$. This can be shown as follows. If this model is complete, then, by Theorem 1.17, any \mathbf{Q} -local martingale can be represented as a stochastic integral with respect to X (here, \mathbf{Q} is the unique measure in $\mathbf{SM}(\mathbf{P})$). Hence, for each $N \in \mathbb{N}$, the “truncated” model $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{0 \leq n \leq N}, \mathbf{P}; (X_n)_{0 \leq n \leq N})$ is complete. This implies that $\mathcal{F}_n = \mathcal{F}_n^X$ for $n \leq N$. As N is arbitrary, we get the result. \square

Our last example shows that a model can be complete, whereas $\mathbf{SM}(\mathbf{P})$ is empty (i.e. the model is not arbitrage-free).

Example 7.5. Let ξ be a random variable with $\mathbf{P}\{\xi = +1\} = \mathbf{P}\{\xi = -1\} = 1/2$. Set $X_n = n\xi$, $\mathcal{F}_n = \mathcal{F}_n^X$ ($n = 0, 1, 2$). Then the model is complete, while $\mathbf{SM}(\mathbf{P}) = \emptyset$.

Proof. Straightforward. \square

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Index of Notations

\mathcal{A}_{loc}	one-dimensional adapted processes of locally integrable variation	12
$\mathcal{A}_{\text{loc}}^d$	d -dimensional adapted processes of locally integrable variation	12
\mathcal{F}_n^X	natural filtration of a discrete-time process $(X_n)_{0 \leq n \leq N}$: $\mathcal{F}_n^X = \sigma(X_k; 0 \leq k \leq n)$	
\mathcal{F}_t^X	natural filtration of a continuous-time process $(X_t)_{t \geq 0}$: $\mathcal{F}_t^X = \bigcap_{\varepsilon > 0} \sigma(X_s; s \leq t + \varepsilon)$	
\mathcal{H}^1	martingales, for which $\mathbb{E}(\sup_{t \geq 0} M_t) < \infty$	12
\mathcal{H}^∞	martingales, for which $\sup_{t \geq 0} M_t \in L^\infty$	12
\mathcal{LP}	one-dimensional Lévy processes	36
\mathcal{LP}^d	d -dimensional Lévy processes	36
$L_{\text{var}}(A)$	processes integrable with respect to a finite-variation process A	20
$L_{\text{loc}}^1(M)$	processes integrable with respect to a local martingale M	17
$L(X)$	processes integrable with respect to a semimartingale X	21
$\text{LM}(\mathbb{P})$	class of equivalent local martingale measures	6
\mathcal{M}	one-dimensional martingales	
\mathcal{M}^d	d -dimensional martingales	
\mathcal{M}_{loc}	one-dimensional local martingales	12
$\mathcal{M}_{\text{loc}}^d$	d -dimensional local martingales	12
\mathcal{M}_σ	one-dimensional σ -martingales	7
\mathcal{M}_σ^d	d -dimensional σ -martingales	7
$\text{M}(\mathbb{P})$	class of equivalent martingale measures	4
(NA)	No-Arbitrage property	4
(NFLBR)	No Free Lunch with Bounded Risk property	6
(NFLVR)	No Free Lunch with Vanishing Risk property	6
\mathcal{O}	optional σ -field	12
\mathcal{P}	predictable σ -field	12
\mathcal{PII}	one-dimensional processes with independent increments	36
\mathcal{PII}^d	d -dimensional processes with independent increments	36
(PRP)	predictable representation property	9,9
\mathcal{S}	one-dimensional semimartingales	13
\mathcal{S}^d	d -dimensional semimartingales	13
\mathcal{S}_p	special one-dimensional semimartingales	13
\mathcal{S}_p^d	special d -dimensional semimartingales	13
$\text{SM}(\mathbb{P})$	class of equivalent σ -martingale measures	7
\mathcal{V}^+	increasing adapted processes	
\mathcal{V}	one-dimensional adapted processes of finite variation	
\mathcal{V}^d	d -dimensional adapted processes of finite variation	

$\text{Var } A$	the variation process of a process $A \in \mathcal{V}$: $(\text{Var } A)_t$ is the total variation of A on $[0, t]$	
V^π	the capital process of a strategy π	4,5
D^c	complement to a set D	
$H \bullet X$	vector stochastic integral with respect to a semimartingale X	21
$(\text{LS})H \bullet A$	vector stochastic integral with respect to a finite-variation process A	20
$(\text{M})H \bullet M$	vector stochastic integral with respect to a local martingale M	18
$M \perp N$	orthogonality of local martingales	13
$\langle x, y \rangle$	scalar product in \mathbb{R}^d : $\langle x, y \rangle = \sum_{i=1}^d x^i y^i$	
$[X, Y]$	quadratic covariation of semimartingales X and Y	14
$[X]$	quadratic variation of a semimartingale X	14
ΔX	the jump process of X : $\Delta X_t = X_t - X_{t-}$	
X^τ	process X stopped at the time τ : $X_t^\tau = X_{t \wedge \tau}$	
$X_n \xrightarrow{\text{u.p.}} X$	convergence in probability uniformly on compact intervals	12
$X_n \xrightarrow{\mathcal{S}} X$	convergence in the Émery topology	26
$\llbracket 0, \tau \rrbracket$	stochastic interval: $\llbracket 0, \tau \rrbracket = \{(\omega, t) : 0 \leq t \leq \tau(\omega)\}$	

Index of Terms

- Arbitrage
 - in continuous time, 5
 - in discrete time, 4
- Associativity theorem
 - first, 25
 - second, 25
- Canonical decomposition of a semimartingale, 13
- Capital of a strategy
 - in continuous time, 5
 - in discrete time, 4
- Completeness
 - in continuous time, 9
 - in discrete time, 8
- Componentwise stochastic integral, 40
- Davis inequalities, 13
- Émery
 - example, 34
 - topology, 26
- First FTAP
 - in continuous time, 6, 7
 - in discrete time, 4
- Free lunch with bounded risk, 6
- Free lunch with vanishing risk, 6
- Local martingale, 12
- Model of a financial market
 - in continuous time, 5
 - in discrete time, 4
- Monotone class lemma, 14
- Orthogonal local martingales, 13
- Predictable representation property
 - in continuous time, 9
 - in discrete time, 9
- Process
 - càdlàg, 12
 - locally bounded, 12
 - of locally integrable variation, 12
 - optional, 12
 - predictable, 12
- Quadratic covariation, 14
- Quadratic variation, 14
- σ -field
 - optional, 12
 - predictable, 12
- σ -martingale, 7
- Second FTAP
 - in continuous time, 10
 - in discrete time, 9
- Semimartingale, 13
 - special, 13
- Stable subspace, 13
- Strategy
 - in continuous time, 5
 - admissible, 5
 - in discrete time, 4
- Vector stochastic integral
 - with respect to a finite-variation process, 20
 - with respect to a local martingale, 18
 - with respect to a semimartingale, 21
- Vector-valued stochastic integral, 3